Blowout bifurcations in a model for chaos in spin-wave dynamics

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Abstract

Blowout bifurcations are investigated numerically in a model for chaos in the coincidence regime of high-power ferromagnetic resonance, based on interactions between the uniform mode and two pairs of parametric spin waves. This model possesses two orthogonal invariant manifolds corresponding to the excitation of only one spin-wave pair above the first-order Suhl instability threshold. Marginal synchronization of the amplitudes of spin-wave pairs, the exchange of stability of the invariant manifolds, as well as both supercritical and subcritical blowout bifurcations are observed as the system parameters are varied, with the accompanying on-off intermittency, attractor bubbling and intermingled basins of attraction.

05.45-a, 75.30Ds, 76.50+g

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I. INTRODUCTION

Systems which possess chaotic attractors contained within invariant manifolds whose dimension is less than the dimension of the phase space can exhibit the blowout bifurcation [1,2]. This bifurcation consists in the loss of the stability of the above-mentioned attractor with respect to perturbations transverse to the invariant manifold above a critical value of a bifurcation parameter. Associated with this bifurcation are such phenomena as on-off intermittency [3-5], attractor bubbling [2], and riddled basins of attraction [1,2,6-8]. There are many examples of these phenomena in chaotic maps [9-11] and electric circuits [12,13], as well as experimental observations and models from other branches of physics, as, e.g., spin wave dynamics [14–16], liquid crystals [17], plasma physics [18,19], lasers [20,21], mechanical [22], and even ecological systems [23,24].

In this paper we show that many phenomena connected with the blowout bifurcation occur in a model for spin-wave chaos in high-power ferromagnetic resonance, and that their variety is much richer than has been known so far [14–16]. The spin-wave chaos has its roots in nonlinear interactions between the uniform precession of magnetization and pairs of spin waves excited to high amplitudes by the strong rf magnetic field [25]. We consider the model of such interacting spin wave pairs as a dynamical system possessing chaotic attractors contained within invariant manifolds, corresponding to the excitation of separate spin wave pairs, and discuss systematically the possible blowout bifurcations from these manifolds. Treating each spin wave pair as a chaotic oscillator, we show that the model under study is a physical realization of a particular class of models of coupled oscillators, exhibiting rich variety of blowout bifurcations and chaotic synchronization phenomena [26]. Numerical simulations suggest that two kinds of global chaotic attractors appear in our model: first, with only one spin-wave pair excited, and another, with more, but rather distinct, spin-wave pairs. We argue that this result can be important for the understanding of the origin of low-dimensional spin-wave chaos observed experimentally [27,28].
II. BLOWOUT BIFURCATION AND RELATED PHENOMENA

The typical blowout scenarios are the supercritical (nonhysteretic) and the subcritical (hysteretic). [1]. The supercritical scenario occurs if, below the blowout, the global stable attractor of the system is that within the invariant manifold. Above the blowout a new attractor is formed which encompasses the attractor within the invariant manifold, and on-off intermittency occurs [3–5]: the phase trajectory spends long time intervals close to the invariant manifold (laminar phases) and occasionally departs from it (bursts). Below the blowout, addition of noise causes a premature occurrence of intermittent bursts called attractor bubbling [2]. The subcritical scenario occurs if below the blowout there is another stable attractor distant from that within the invariant manifold. Above the blowout only the former attractor is stable. Below the blowout both attractors are stable, but the basin of the attractor within the invariant manifold is riddled [6–8], i.e., in any neighborhood of a point belonging to this basin there is a positive measure set of points belonging to the basin of the other, distant attractor. In this case, addition of noise destabilizes the attractor within the invariant manifold, and the phase trajectory inevitably escapes from its neighborhood to the distant stable attractor [2]. In the subcritical scenario it is also possible that there are two or more attractors contained within different invariant manifolds of the system. Below the blowout these attractors are stable and their basins are mutually riddled (intermingled). If two or more of these manifolds lose transverse stability at the same point, above the blowout two-state (or multi-state) on-off intermittency between the unstable manifolds is observed [29–32]. At the border between the subcritical and supercritical scenarios there is the exchange-of-stability scenario [26], where below the blowout the global attractor is within the invariant manifold, but at the blowout point a new stable, distant attractor is born.

Various kinds of the blowout bifurcation occur in systems of mutually coupled chaotic oscillators with orthogonal invariant manifolds [26]. Such systems are described by equations

\[ \dot{\mathbf{v}}_1 = \mathbf{f}(\mathbf{v}_1, \mathbf{u}, p_1), \quad \dot{\mathbf{u}} = \mathbf{g}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}), \quad \dot{\mathbf{v}}_2 = \mathbf{f}(\mathbf{v}_2, \mathbf{u}, p_2), \]

where \( \mathbf{x}_1 = (\mathbf{v}_1, \mathbf{u}), \mathbf{x}_2 = (\mathbf{v}_2, \mathbf{u}) \) are
chaotic systems coupled by a common set of variables $u$, $p_1$, $p_2$ are parameters, and the function $g$ is symmetric with respect to exchange of the sets of variables $v_1$ and $v_2$. It is assumed that equations for $v_i$, $i = 1, 2$ are invariant with respect to scaling transformations $v_i \to \alpha v_i$, $\alpha = \text{const}$, i.e., that $f$ is linear with respect to $v_i$. Since $x_i$ are chaotic systems, two orthogonal invariant manifolds $v_1 \equiv 0$, $v_2 \equiv 0$ exist, containing the chaotic attractors. For $\Delta p = p_1 - p_2 = 0$ the systems $x_1$, $x_2$ show sized marginal synchronization [33], i.e., $v_1 = Av_2$, where $A = \text{const}$ depends on the initial conditions but not on the time. This marginal synchronization method using a common set of variables $u$ is a generalization of the replacement method of chaotic synchronization in unidirectionally coupled drive-response systems [34] to the case of mutual coupling; various kinds of blowout bifurcations are connected with the loss of stability of this synchronization. For small $\Delta p \neq 0$ only one invariant manifold is stable, depending on the sign of $\Delta p$, and the chaotic attractor inside it is a global attractor. When $\Delta p$ crosses zero, the stability of the manifolds, and thus the location of the global attractor, suddenly changes. This is an example of the exchange-of-stability blowout bifurcation. When $\Delta p$ is varied further away from zero, also the subcritical and supercritical blowout bifurcations can appear [26].

In Ref. [26] the systems with orthogonal invariant manifolds were realized by coupling generic chaotic oscillators, e.g., the Lorenz systems. Although conceptually simple, such systems have to be specially designed for experimental purposes. The aim of this paper is to show that a certain model for chaotic spin-wave dynamics belongs to the class of coupled oscillators discussed above. The single chaotic oscillator in this model consists of a spin-wave pair interacting with the uniform precession mode (the latter playing a role of the common coupling variable between various oscillators), and the attractors within the invariant manifolds correspond to the excitation of separate spin-wave pairs. Apart from being a model for an experimentally realizable system from solid state physics, the model we study has an interesting property that and all kinds of the blowout bifurcation and related phenomena, typical of systems coupled oscillators with orthogonal invariant manifolds, can be observed in its dynamics. This is in contrast with the cases studied in Ref. [26], where
usually only the exchange-of-stability and either only supercritical or subcritical blowout bifurcations were found in each kind of coupled oscillators.

III. THE MODEL

In this paper we consider a model for high-power ferromagnetic resonance in the coincidence regime, i.e., when the uniform mode with frequency $\omega_0$ is excited by the rf magnetic field with frequency $\omega \approx \omega_0$ and amplitude $h_T$, perpendicular to the dc magnetic field. If $h_T$ exceeds the first-order Suhl instability threshold $h_T^{thr}$ the uniform mode decays into pairs of spin waves with opposite wave vectors and frequencies $\omega_k \approx \omega/2$ [25]. As $h_T$ is further increased, periodic and chaotic oscillations of magnetization are observed as a result of nonlinear interactions between the uniform mode and the spin-wave pairs [16,27,28,35]. These oscillations can be explained using a model with a low number of interacting spin-wave pairs [15]; the applicability of low-dimensional models is supported by the low correlation dimension of chaotic attractors measured in experiments [27,28]. In this paper we consider a model with two spin-wave pairs $k = 1, 2$. In a dimensionless form, equations of motion for the slowly varying in time parts of the complex amplitudes of the uniform mode $a_0$ and of the spin-wave pairs $a_k$ can be written as [15]

$$
\begin{align*}
\dot{a}_0 &= \left| \eta_0 + i\Delta \omega_0 \right| \left| \eta_1 + i\Delta \omega_1 \right| \varepsilon - (\eta_0 + i\Delta \omega_0) a_0 \\
&\quad - iV_{01}a_1^2 - iV_{02}a_2^2 + \xi_{th}, \\
\dot{a}_1 &= - \left( \eta_1 + i\Delta \omega_1 \right) a_1 - iV_{01}a_1^*a_0 + \xi_{th}, \\
\dot{a}_2 &= - \left( \eta_2 + i\Delta \omega_2 \right) a_2 - iV_{02}a_2^*a_0 + \xi_{th},
\end{align*}
$$

(1)

where $\varepsilon = h_T/h_T^{thr}$, $\eta_0, \eta_k$ are phenomenological damping constants, $\Delta \omega_0 = \omega_0 - \omega$, $\Delta \omega_k = \omega_k - \omega/2$ is frequency detuning which originates from the discrete character of the spin-wave frequency spectrum due to boundary conditions in real samples, $V_{0k}$ is a coefficient of three-magnon interactions between the uniform mode and the spin wave pair $k$, $\xi_{th}$ denotes the level of thermal excitation of spin waves, and the dot denotes derivative with respect to the
rescaled time $t' = \eta_1 t$. In Eq. (1) all damping constants, detunings, and spin-wave amplitudes are normalized to $\eta_1$ and $V_{0k}$ to $V_{01}$ (thus $V_{01} = 1$ can be always assumed). Numerical simulations of the model (1) were performed using a fourth-and-fifth-order Runge-Kutta method with control of the integration step size.

Eq. (1) has a structure of the systems of mutually coupled oscillators introduced in Sec. II, with $v_1 \equiv a_1$, $v_2 \equiv a_2$, and $u \equiv a_0$. For a wide range of the parameters, chaos can be observed in Eq. (1) even with only one spin-wave pair [15]. Thus the amplitude of the uniform mode $a_0$ plays a role of the coupling variable, and the possible chaotic attractors within invariant manifolds correspond to the excitation of only one spin-wave pair. The stability of the two invariant manifolds $a_k = 0$, $k = 1, 2$, is determined by corresponding transverse Lyapunov exponents [1] $\lambda^T_k = \lim_{t \to -\infty} \ln \left( |\zeta_k(t)| / |\zeta_k(0)| \right)$, where $\zeta_k(t)$ is a small perturbation in the direction perpendicular to the invariant manifold $a_k = 0$ (i.e., $\zeta_1$ is in the direction of $a_2$ and vice versa). The time evolution of the perturbation $\zeta_1(t)$ can be obtained directly from Eq. (1) by substituting $a_2$ with $\zeta_1$ in the equation for $a_2$ and setting $a_2 = 0$ in the equation for $a_0$; the case of $\zeta_2$ is symmetric. In particular, the invariant manifold $a_k = 0$ is stable if $\lambda^T_k < 0$ and unstable otherwise, and the blowout bifurcation from this manifold corresponds to the change of the sign of the exponent from negative to positive.

IV. NUMERICAL RESULTS AND DISCUSSION

The model (1) shows chaotic dynamics for a wide range of parameters. In Fig. 1 typical time series for $|a_0|$ (proportional to absorption measured in experiments) and the chaotic attractor for the model (1) are shown. If both spin-wave pairs are identical, marginal synchronization between them occurs (Fig. 1(b)), as reported previously [38].

In the class of coupled systems under study, the marginal synchronization is characterized by $\lambda^T_1 = \lambda^T_2 = 0$, and small deviations of the parameters of spin-wave pairs destabilize the synchronized state by changing the signs of the two transverse Lyapunov exponents in the opposite way (Fig. 2(a)) [26]. Thus, e.g., for $\Delta \omega_2 \neq \Delta \omega_1$ only one invariant manifold is
stable and contains the global attractor, which corresponds to the excitation of only one spin-wave pair, and the stability of the two manifolds changes suddenly via the exchange-of-stability blowout bifurcation at $Δω_2 = Δω_1$ (Fig. 2(a)). If $Δω_2$ is further changed, also the stable invariant manifold can lose transverse stability. Since it contained the global attractor, the stability is lost via the supercritical blowout bifurcation (Fig. 2(a)), and a higher-dimensional attractor with both spin-wave pairs excited is formed. In this case, the amplitude of one spin-wave pair shows on-off intermittency just above the blowout (Fig. 2(b)), where the corresponding transverse Lyapunov exponent is small and positive, and decays to zero below the blowout (Fig. 2(c)), where the exponent is negative; addition of thermal noise leads to the appearance of bursts below the blowout via the attractor bubbling (Fig. 2(d)). The other spin-wave pair is strongly excited all the time, since the corresponding transverse Lyapunov exponent is large and positive. On-off intermittency in spin-wave chaos was observed in experiments [14,16] and numerical simulations [15]. Note that certain, not infinitesimally small difference between the parameters of the spin-wave pairs is necessary for the higher-dimensional attractor to appear.

For certain parameters in Eq. (1) both $λ^T_1$ and $λ^T_2$ can be small and positive. Then, although both spin-wave pairs are excited, the invariant manifolds are only weakly unstable and the phase trajectory spends long time intervals close to each of them. This results in intermittent bursting of amplitudes of both spin-wave pairs, and most often only one of the two pairs is strongly excited (Fig. 3). Hence the system remains effectively low-dimensional, and the energy from the rf field flows via the uniform mode only to one spin-wave pair at a given time. This is two-state on-off intermittency [29,32] between the orthogonal invariant manifolds. Since the manifolds need not be symmetric (i.e., the parameters of the spin-wave pairs can be different) and thus their transverse Lyapunov exponents need not be equal, the mean duration of laminar phases in the time series of individual spin-wave pairs can differ significantly.

It can also happen that both $λ^T_1$ and $λ^T_2$ are negative (Fig. 4(a)). Then the two attractors within orthogonal invariant manifolds are stable and their basins are intermingled. This can
be seen in Fig. 4(b–d) which show that, in any scale, initial conditions leading to different attractors are located arbitrarily close to each other. In this case, only one spin-wave pair is excited, depending on the initial conditions. As, e.g., $\varepsilon$ is varied, one of the stable manifolds can lose transverse stability (Fig. 4(a)). Since the other manifold, and the chaotic attractor within it, usually remain transversely stable, the former invariant manifold undergoes the subcritical blowout bifurcation. From Fig. 4(a) it can be seen that it is difficult to identify a single bifurcation point; in contrast, the invariant manifold seems to lose stability over a range of the parameter $\varepsilon$, where the corresponding transverse Lyapunov exponent shows irregular oscillations around zero. Such ”diffuse” subcritical blowout bifurcations are often found in systems where the dynamics within the invariant manifold depends on the variables orthogonal to it [36], which is the case for both invariant manifolds in Eq. (1). Note that, in contrast with the case mentioned in Sec. II, although below the blowout the basins of attraction were intermingled, above the blowout the two-state on-off intermittency does not occur, because only one invariant manifold is transversely unstable and the other one remains unstable (i.e., the two invariant manifolds do not lose stability at the same value of the control parameter).

The possibility of experimental observation of intermingled basins of attraction in spin-wave chaos is hampered by thermal noise, which destabilizes both invariant manifolds. As a result, the phase trajectory switches between the two manifolds in a process which can be called two-state attractor bubbling (Fig. 5), and amplitudes of both spin-wave pairs show intermittent bursting. The two-state attractor bubbling resembles two-state on-off intermittency (and in systems, where this kind of on-off intermittency appears, can be treated as its predecessor in the presence of noise). It should be again noted that, since most often only one spin-wave pair is strongly excited, the system remains effectively low-dimensional, and the energy from the rf field flows via the uniform mode only to one spin-wave pair at a given time.
V. CONCLUSIONS

In this paper we have shown that a certain model for spin-wave chaos belongs to the class of mutually coupled chaotic oscillators with orthogonal invariant manifolds. As usually in this class of systems, marginal synchronization of spin-wave amplitudes and phenomena related to the blowout bifurcation were observed, e.g., for the first time in spin-wave chaos, riddled basins of attraction and two-state on-off intermittency. However, a distinct feature of Eq. (1) is that it is a model for an experimentally realizable system from solid state physics, and that it supports all kinds of blowout bifurcations and related phenomena, reported in Ref. [26] in different systems of coupled oscillators.

Direct experimental observation of the phenomena related to the blowout bifurcations from the chaotic attractors with one spin wave pair is difficult. It requires measuring of the amplitudes of individual spin-wave pairs, which in some cases can be achieved using Brillouin light scattering [37]. However, indirect observation is possible. For example, on-off intermittency in spin-wave chaos was identified as a kind of chaos-chaos intermittency in the time series of absorption. [14,16]. The riddled basins of attraction can lead to multistability of the chaotic attractors; such multistability was in fact observed experimentally in the coincidence regime [35], but its possible relationship to the phenomenon of riddled basins was not pointed out.

Apart from the first-order Suhl instability, in high-power ferromagnetic resonance other kinds of spin-wave instability occur. For example, in the parallel pumping regime, where the rf field is parallel to the dc field, the instability consists in direct excitation of spin wave pairs by the rf field, and chaos appears due to direct nonlinear interactions between these pairs [39]. It can be shown that models for spin-wave chaos in these cases belong to the same class of coupled oscillators, with orthogonal invariant manifolds corresponding to the excitation of separate spin wave pairs. As a result, e.g., in a model for chaos in the parallel pumping marginal synchronization of spin-wave amplitudes was observed [40]. Thus the results from the model 1 can be generalized to the models for chaos in other spin-wave instabilities, where
a similar variety of blowout bifurcations and related phenomena can be expected.

Our simulations reveal that, for a wide range of the parameters, the system (1) remains low-dimensional, i.e., only one spin-wave pair is excited (always, or since the rf field energy flows only to one pair at a given time). For higher-dimensional chaos to occur the different excited spin-wave pairs would have to be distinct. This suggests that in experiments with spin-wave chaos separate spin-wave pairs rather than broad packets of spin waves with similar damping and detuning are excited. This is in agreement with experimental observations of attractors with low correlation dimension in the coincidence regime [27,28], as well as with numerical simulations of spin-wave chaos using a model with a large number of similar interacting spin-wave pairs [38].

The origin of low-dimensional chaos in experimental systems remains an intriguing question since, in principle, a degenerate spin-wave manifold $\omega_k \approx \omega/2$ with infinitely many spin-wave pairs can be excited far above the first-order Suhl instability threshold. Then, because of a large number of degrees of freedom in the system, chaotic attractors with a high correlation dimension should occur. Hence, further investigation is necessary to establish a link between the results of this paper and those of the experiments. This requires simulations of more complex models for spin-wave chaos, with a large number of interacting spin-wave pairs, as in Ref. [38], and the systematic analysis of the stability and blowout bifurcations from the attractors in the many invariant manifolds, corresponding to the excitation of separate pairs. Using such a procedure, it should be possible to find the number of excited pairs in these complex models, and to verify if for a broad range of the model parameters only separate, distinct spin-wave pairs are excited, which can reduce the dimension of the attractors.

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REFERENCES


Fig. 1. (a) Time series of $|a_0|$; (b) linear dependence between chaotic variables $|a_1|$, $|a_2|$ proving marginal synchronization; (c,d) two projections of the chaotic attractor for the system (1) with $\varepsilon = 2.0$, $V_{02} = 1.0$, $\eta_0 = 1.0$, $\Delta\omega_0 = -1.5$, $\eta_1 = \eta_2 = 1.0$ $\Delta\omega_1 = \Delta\omega_2 = 3.0$, $\xi_{therm} = 0$.

Fig. 2. (a) The transverse Lyapunov exponents $\lambda_1^T$ (circles) and $\lambda_2^T$ (dots) vs. $\Delta\omega_2$ for the system (1) with $\varepsilon = 3.0$ and other parameters as in Fig. 1; (b-d) time series of $|a_1|$ for (b) $\Delta\omega_2 = 3.066$, $\xi_{therm} = 0$ (on-off intermittency), (c) $\Delta\omega_2 = 3.064$, $\xi_{therm} = 0.0$ (below the blowout), (d) $\Delta\omega_2 = 3.064$, $\xi_{therm} = 10^{-10}$ (attractor bubbling), other parameters as in (a).

Fig. 3. Time series of (a) $|a_1|$, (b) $|a_2|$ for the system (1) with $\varepsilon = 2.0616$, $\eta_2 = 1.01$, $\Delta\omega_2 = 3.01$, and other parameters as in Fig. 1; the transverse Lyapunov exponents are $\lambda_1^T = 0.003153$, $\lambda_2^T = 0.00440$.

Fig. 4. (a) The transverse Lyapunov exponents $\lambda_1^T$ (circles) and $\lambda_2^T$ (dots) vs. $\varepsilon$ for the system (1) with $\eta_2 = 1.01$, $\Delta\omega_2 = 3.01$, and other parameters as in Fig. 1; (b-d) consecutive magnifications of the section of the intermingled basins of attraction of the chaotic attractors within the orthogonal invariant manifolds $a_1 = 0$ (black points) and $a_2 = 0$ (white points) with the manifold of initial conditions $a_0(0) = 2$, $\Im a_1(0) = \Im a_2(0) = 0$, for the system (1) with $\varepsilon = 2.0494$ (denoted by arrow in (a)) and other parameters as in (a).

Fig. 5. Time series of (a,b) $|a_1|$, (c,d) $|a_2|$ for the system (1) with $\xi_{th} = 0$ (a,c) and the noise level $\xi_{th} = 10^{-10}$ (b,d), and with other parameters as in Fig. 4(b).
Fig. 1. A. Goska and A. Krawiecki, Blowout bifurcations in a model for chaos in spin-wave dynamics.
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Fig. 3. A. Goska and A. Krawiecki,
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Fig. 5. A. Goska and A. Krawiecki,
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