New constant of motion for coevolving voter model

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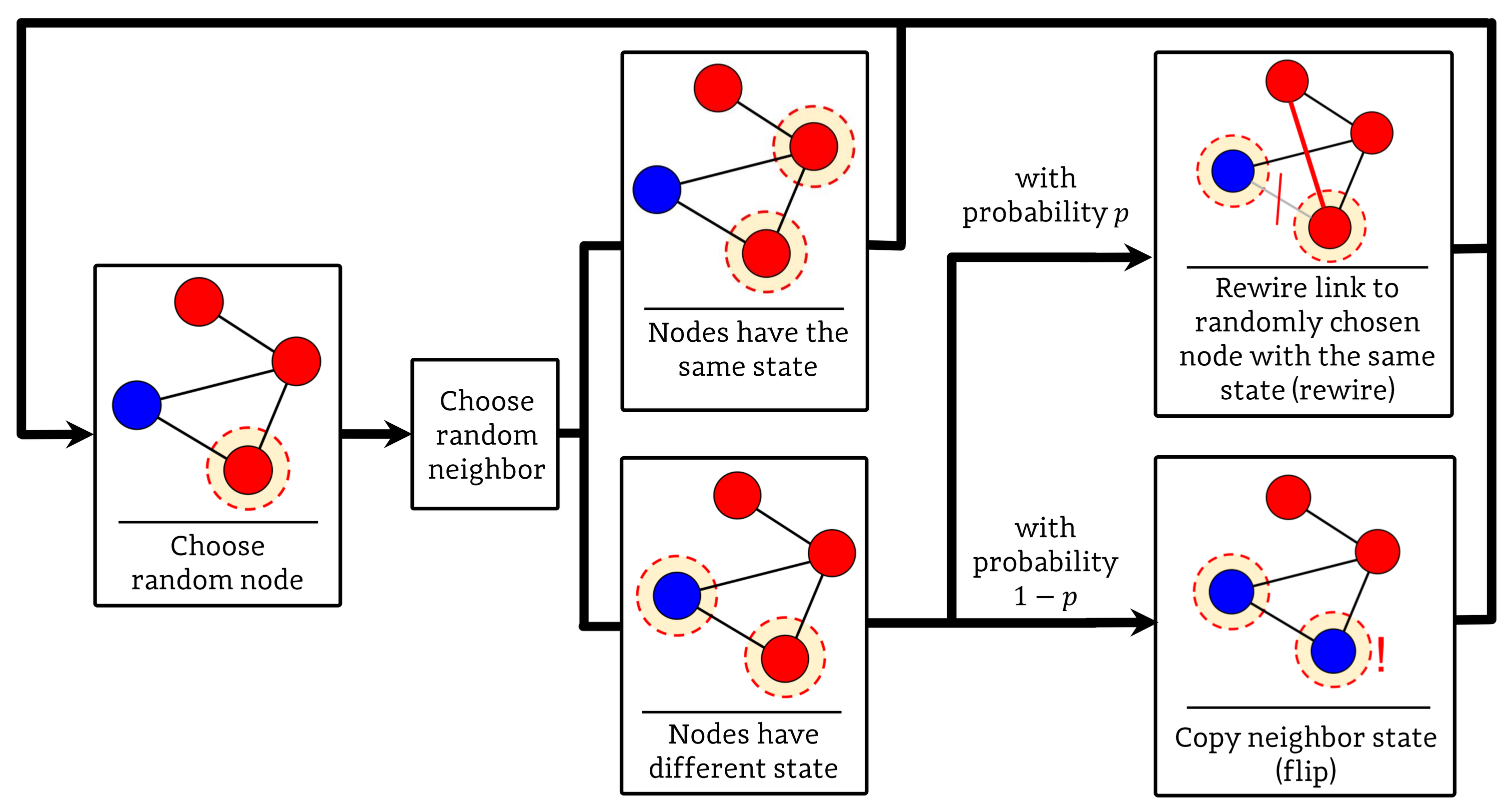
In the coevolving voter model a topology of the network changes in response to the voter dynamics on the network. In detail, nodes can change their state and links can rewire in order to connect nodes of the same state. We consider the process of reaching the final state in this model. We treat mean degree of nodes in different states as separate variables, which do not have to be equal. This allows to discuss the magnetization of nodes and the magnetization of links as potentially independent variables. Our studies shows that in the active phase mean value of magnetization of nodes and links tend to the same value. Mean field calculation indicate that these two magnetizations are coupled – their linear combination is a statistical constant of motion. Obtained results were confirmed by numerical simulations.

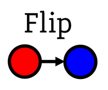
Coevolving voter model dynamics



Elementary events

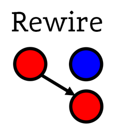
All events change numbers of given states (N_+, N_-) and numbers of links connecting given states (M_{++}, M_{--}, M_{+-}, M_{-+}). Suppose that each link between nodes is cut in the half and the half-links are classified as directed. Changes are made in the following way:

Flip



$N_+ \rightarrow N_+ - 1$
$N_- \rightarrow N_- + 1$
$M_{++} \rightarrow M_{++} - 2k_{++}$
$M_{--} \rightarrow M_{--} + 2k_{+-}$
$M_{+-} \rightarrow M_{+-} + k_{++} - k_{+-}$
$M_{-+} \rightarrow M_{-+} + k_{++} - k_{+-}$

Rewire



$M_{++} \rightarrow M_{++} + 2$
$M_{+-} \rightarrow M_{+-} - 1$
$M_{-+} \rightarrow M_{-+} - 1$

Note that the total number of links and nodes is constant.

Mean-field equations

Based on the elementary events calculation one can construct mean-field equations of motions. We assume that mean degrees of nodes $\mu_\alpha = < k_\alpha >$ possesing positive and negative spins can be different.

Total number of active links (links between nodes of different spins):

$$\frac{dM_{\alpha\beta}}{dt} = N_\alpha \Sigma_{k_\alpha} P_\alpha(k_\alpha) \Sigma_{k_{\alpha\beta}} B(k_{\alpha\beta}; k_\alpha) \cdot \frac{k_{\alpha\beta}}{k_\alpha}[(1-p)(k_{\alpha\alpha} - k_{\alpha\beta}) - p] + N_\beta \Sigma_{k_\beta} P_\beta(k_\beta) \Sigma_{k_{\beta\alpha}} B(k_{\beta\alpha}; k_\beta) \frac{k_{\beta\alpha}}{k_\beta}[(1-p)(k_{\beta\beta} - k_{\beta\alpha}) - p]$$

Spin numbers N_α:

$$\frac{dN_\alpha}{dt} = -\frac{1-p}{\frac{1}{N_\alpha}} \Sigma_{k_\alpha} P_\alpha(k_\alpha) \Sigma_{k_{\alpha\beta}} B(k_{\alpha\beta}; k_\alpha) \left(\frac{k_{\alpha\beta}}{k_\alpha}\right) + \frac{1-p}{\frac{1}{N_\beta}} \Sigma_{k_\beta} P_\beta(k_\beta) \Sigma_{k_{\beta\alpha}} B(k_{\beta\alpha}; k_\beta) \left(\frac{k_{\beta\alpha}}{k_\beta}\right)$$

We assume that the probability distribution $B(k_{\alpha\beta}; k_\alpha)$ is binomial.

New variables

Density of active links

$$\rho = \frac{2M_{+-}}{N\mu} = \frac{2M_{-+}}{N\mu}$$

Links magnetization

$$m = \frac{(M_{++} - M_{--})}{N\mu}$$

Nodes magnetization

$$n = \frac{(N_+ - N_-)}{n}$$

For the new variables equation of motions are following:

$$\frac{d\rho}{dt} = \frac{2(1-p)\rho}{1-m^2}\left[1 - m^2 - 2\rho - \frac{2}{\mu}(1-mn) + \frac{2\rho(1+m^2-2nm)}{\mu(1-m^2)}\right] - \frac{2p\rho(1-mn)}{\mu(1-m^2)}$$

$$\frac{dn}{dt} = \frac{2(1-p)\rho(m-n)}{1-m^2}$$

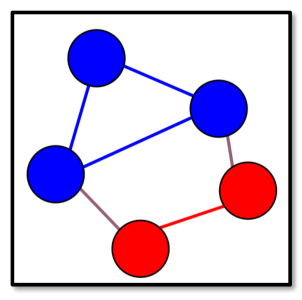
$$\frac{dm}{dt} = \frac{2p\rho(n-m)}{\mu(1-m^2)}$$

Example:

$$n = \frac{3-2}{5}$$

$$m = \frac{6-2}{12}$$

$$\rho = \frac{2 \cdot 2}{12}$$

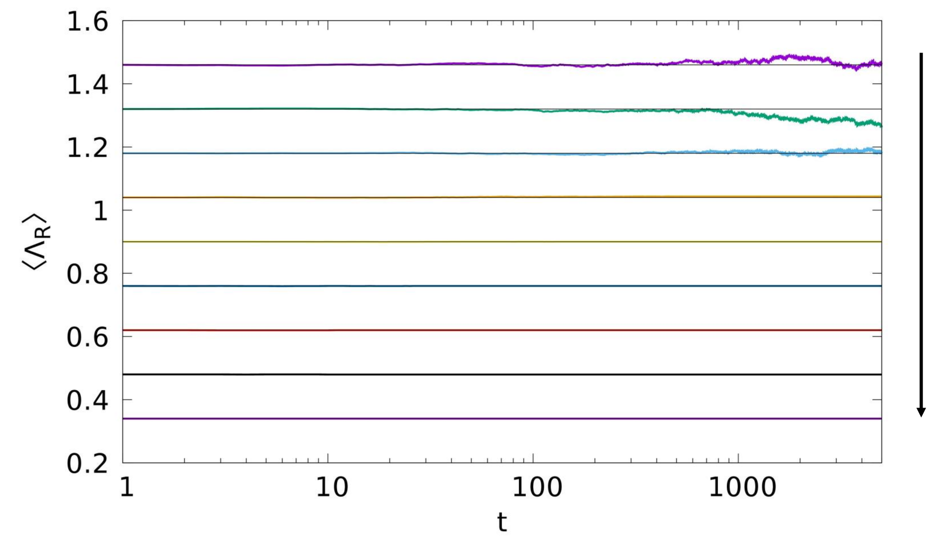


Constant of motion

From equations of motion by eliminating time one can get:

$$\Lambda = (1-p)\mu m + pn$$

where Λ is a **time independent constant** that results from initial conditions m_0 and n_0.



Time evolution of value Λ for different values of parameter p, for initial condition $n_0 = 0.2$, $m_0 = 0.4$. Value of Λ depends on initial condition.
Figure show results of numerical simulation.

$p = 1$ – no state change - $n = const$
$p = 0$ – no rewiring – $m = const$

Fixed points

As a consequence of constant of motion Λ one can reduced number of variables to two.
Differential equation have a line of fixed points $(m, \rho^* = 0)$ and a fixed point which depends on initial states:

$$\rho^* = \frac{\mu^2(1-m_0^2)(1-p)^2 + 2\mu(1-m_0 n_0)(1-p)p + (1-n_0^2)p^2}{2(\mu-1)(1-p)(\mu(1-p)+p)^2}$$

$$m^* = \frac{\Lambda}{\mu(1-p)+p}$$

After some algebra one can get:

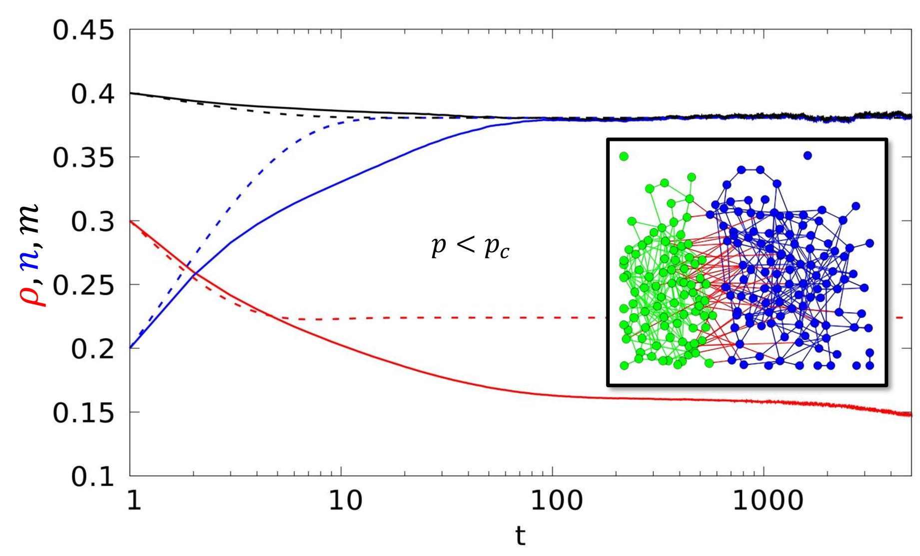
$$\rho^* = (1-m^{*2})\frac{(1-p)(\mu-1)-1}{2(1-p)(\mu-1)}$$

It is the same result as was observed in numerical simulation by Vazquez et al[1].
The stability analysis for the fixed point shows that initial conditions have no impact on whether the fixed point is stable or not. It depends only on value of mean degree in the network:

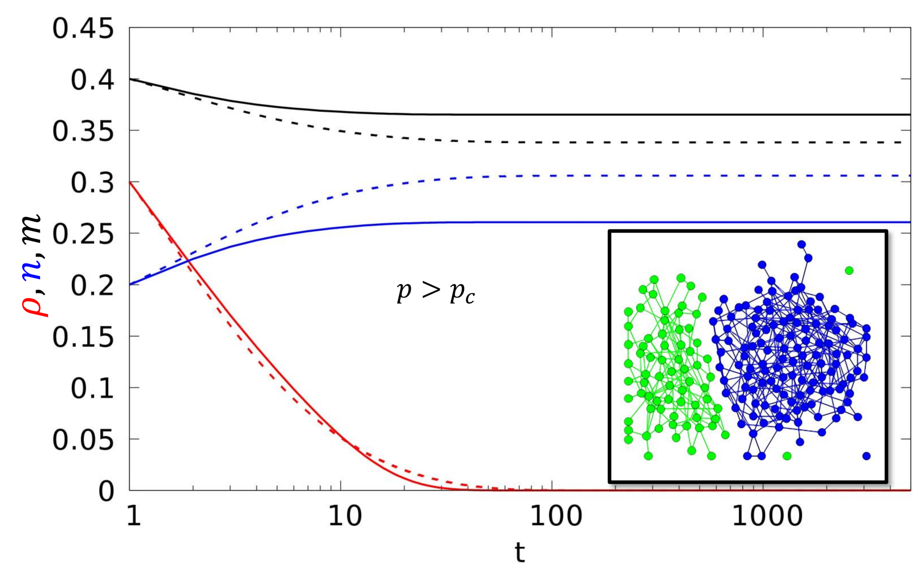
$$p_c(\mu) = \frac{\mu-2}{\mu-1}$$

[1] F. Vazquez, V.M. Eguıĺuz, and M.S. Miguel, "Generic absorbing transition in coevolution dynamics," Physical Review Letters 100 (2008).

Numerical results



Density of active links ρ, site magnetization n, **links magnetization m** in dependence of time. Data presented as solid lines are obtained from numerical simulation. Dashed lines present solution (by 4th order Runge-Kutta method) of differential equations.



Our mean field calculation agrees qualitatively with results of numerical simulations.

Our mean field calculations indicate, that the links magnetization m is coupled to the site magnetization n. Namely, a linear combination of these quantities forms a statistical constant of motion Λ. Further, the difference $m - n$ decreases in time; if the asymptotic phase is active, the mean magnetization of bonds equals to mean magnetization of sides $n = m$ and in such a case the mean degrees of nodes do not depend on the value of the internal node variable. If there are no active bonds, i.e. $\rho = 0$, the time evolution is frozen and in such a case the mean magnetization of bonds can be different from the mean magnetization of sides and the mean degrees of nodes can depend on nodes spins. Numerical calculations confirm this picture in a short time scale. Asymptotic behavior in a finite system is influenced by random fluctuations, which drive the system to an absorbing state where $\rho = 0$. The fluctuations influence each kind of magnetizations in a different way. In the case where only the rewiring takes place ($p = 1$), the site magnetization is exactly constant. On the other hand, if rewiring is absent ($p = 0$), the bond magnetization is constant only in the average. Hence the deviations of the solution from the mean field behavior is stronger for small values of p. The constant of motion Λ complements the description in the full range of the probability p between these two extrema.

"Coupling of bond- and site-ordering in the coevolving voter model" J. Toruniewska, K. Kułakowski, K. Suchecki, J. A. Hołyst, Phys. Rev. E 96, 2017

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