Exotic Fixed Points of the Belief Propagation Algorithm

7th Symposium on Physics in Economics and Social Sciences

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Outline

What is BP algorithm?

Applications of the BP Algorithm

Our results

Conclusions

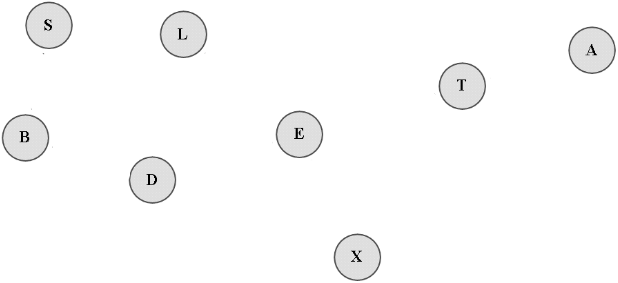

Figure: Set of the Random Variables.

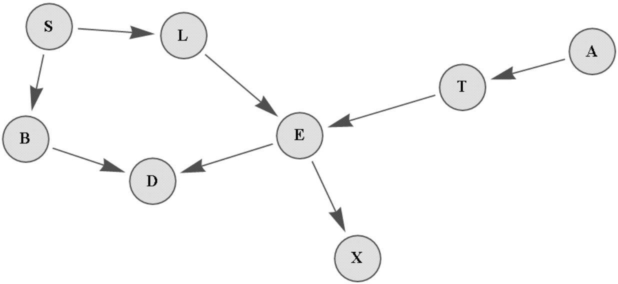

Figure: Set of the Random Variables with some dependences.

Setting of the problem

We are looking for the marginal probabilities

$$\mathbb{P}(x_N) = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_{N-1}} \mathbb{P}(x_1, x_2, \ldots, x_N) =$$

$$= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_{N-1}} \prod_{i=1}^{N} \mathbb{P}(x_i|\mathrm{Par}(x_i)).$$

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Number of summands grows exponentially with the size of the graph!

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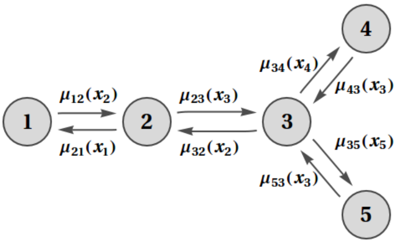

Figure: Visualisation of the impact passing messages on beliefs.

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Why beliefs are so important?

If the algorithm converges beliefs become marginal probabilities!

Dynamics of suggestions

BP dynamic

$$\mu_{ij}^{1} = F([\mu_{ij}^{0}]_{i,\, j=1,\, ...,\, N}),$$
$$\mu_{ij}^{n} = F([\mu_{i}^{n-1}j]_{i,\, j=1,\, ...,\, N}).$$

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Scheme of the BP approach

$$[\mu_{ij}^0] \xrightarrow{F} [\mu_{ij}^1] \xrightarrow{F} [\mu_{ij}^2] \xrightarrow{F} \dots \xrightarrow{F} [\mu_{ij}^{fix}] \Longrightarrow b_i = \mathrm{marginal\, probilities}$$

BP algorithm

Convergence on trees

Pearl proved in 1982, that BP algorithm on trees always convergence to unique fixed point.

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What if *our* graph is not a tree?

In general it is still open problem.

„If we ignore the existence of loops and permit the nodes to continue communicating with each other as if the network were singly connected, messages may circulate indefinitely around these loops, and the process may not converge to a stable equilibrium“

Judea Perl cited in J. S. Yedidia, W. T. Freeman, Y. Weiss, *Understanding Belief Propagation and Its Generalizations*, Exploring Artificial Intelligence in the New Millennium, **8**, 239-236, (2003).

Applications of the BP Algorithm

- **Inference problems** (Bayesian networks), [J. Pearl, Proc. of the Second National Conference on Artificial Intelligence, 133-136, (1982).]
- **Ising model** (Random Markov Field), [S. Dorogovtsev, et al., Rev. Mod. Phys., 80, 1275-1335, (2008).]
- **Decoding of error-correcting codes** (Tanner graphs), [B. J. Frey, et al., Adv. in Neural Inform. Proces. Systems, 10, (1998).]
- **low-level computer vision and AI problems and more...**, [J. S. Yedidia, et. al., Exploring Artificial Intelligence in the New Millennium, 8, 239-236, (2003).]

Asia example

Let us consider "ASIA" example, proposed by Laurtizen and Spiegelhalter [S. L. Lauritzen, D. J. Spiegelhater, Journal of the Royal Statistical Society Series B, 50, 157-224, (1988).]

- A Recent Trip to Asia (A) increases the chances of tuberculosis (T).
- Smoking (S) is a risk factor for both lung cancer (L) and bronchitis (B).
- X-ray result (X) can detect presence of either (E) tuberculosis or lung cancer, but cannot distinguish between them.
- Shortness of breath (D) may be caused by bronchitis (B), or either (E) tuberculosis or lung cancer.

Ilustracja

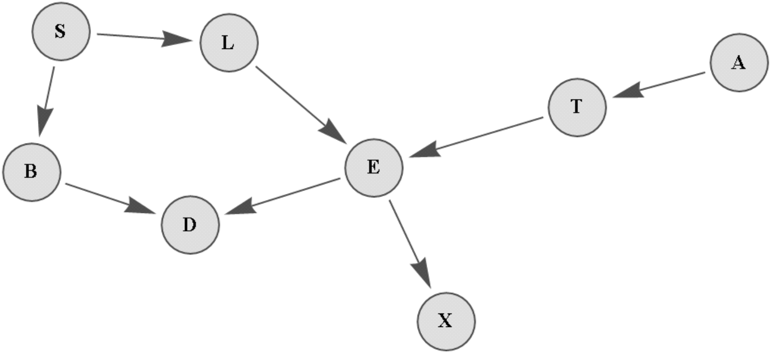

Figure: The fictional "ASIA" Bayesian Network.

Ising Model

Assumptions

- $\sigma_i = \pm 1$, $i = 1, \dots N^2$,
- Our graph - toral $N \times N$ lattice with periodic boundary conditions,

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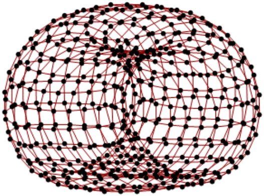

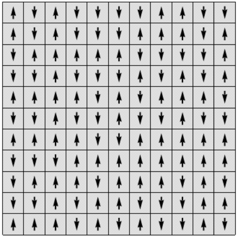

Ising Model

Hamiltonian Function

$$U(\sigma) = -\sum_{i\in G}\sum_{j\in\mathcal{N}_i} J\sigma_i\sigma_j - \sum_{i\in G} h_i\sigma_i$$

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Interpretation

For fixed external field h, coupling energy J and temperature β BP finds magnetisations, i.e. expected values in **the marginal probabilities**.

Beliefs and Messages

For our case beliefs are given as

$$b_{ij}(\sigma_i,\, \sigma_j) = k \exp\left[\beta\left(h_i\sigma_i + h_j\sigma_j + J\sigma_i\sigma_j + \sigma_i \sum_{\substack{n\in\mathcal{N}_i \\ n\neq j}} \mu_{in} + \sigma_j \sum_{\substack{n\in\mathcal{N}_j \\ n\neq i}} \mu_{jn}\right)\right],$$

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Marginalisation condition

$$b_i(\sigma_i) = \sum_{\sigma_j} b_{ij}(\sigma_i, \sigma_j).$$

Update function for the BP Algorithm for the Ising Model

$$F([\mu_{ij}]_{i,\, j=1,\, \ldots,\, N}) = \frac{1}{2\beta} \ln \frac{\cosh\left[\beta\left(h_j + J + \sum\limits_{\substack{n \in \mathcal{N}_j \\ n \neq i}} \mu_{jn}\right)\right]}{\cosh\left[\beta\left(h_j - J + \sum\limits_{\substack{n \in \mathcal{N}_j \\ n \neq i}} \mu_{jn}\right)\right]}.$$

Numerical results

We observed only two scenarios of the dynamics of BP

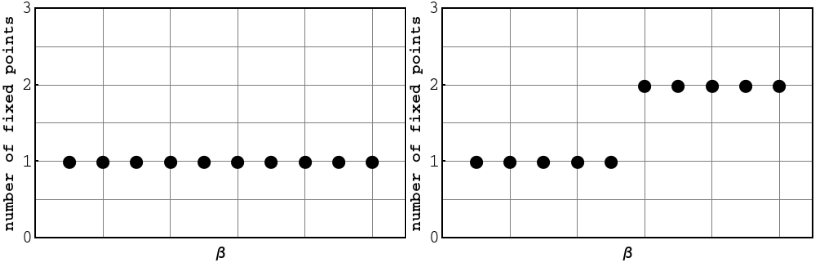

Figure: Two typical results of simulations - number of stable fixed points as function of β.

Exotic Fixed Points

Theorem (GS and GŚ, in preparation)

For every J, if $\tanh(J) > 1/3$, then one can choose $n(J)$ such that for every $k \geq n(J)$ there is a stationary point of H^0 on $\mathcal{T}_{4k}$ with values of one-beliefs $b_i(1)$ both greater and smaller than $1/2$, depending on i.

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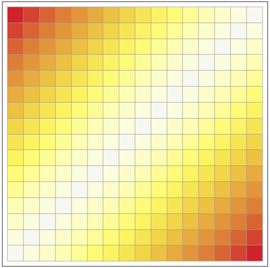

Open questions

- Stability of Exotic Fixed Points?
- Classification of periodic orbits of BP.
- Full Description of the bifurcation in the BP algorithm.

Conclusions

- Belief Propagation Algorithm, proposed by Judea Pearl, is a useful tool in the problem of inference in different mathematical structures (e.g. random Markov fields, Bayesian networks).
- Wide range of important problems in statistical physics, computer science and engineering can be solved with BP.
- In that case flow of knowledge from physics to social science runs in the opposite direction.
- We found exotic fixed points of the BP algorithm but, there are still a lot of open questions about its dynamics. $\ddot\smile$

Thank you for your attention!

This presentation you can find at
`http://if.pw.edu.pl/~siudem/research/FENS14.pdf`