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Diffusion and entropy production for multi-networks with fitness factors

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Outline

A **diffusion process** at a special class of multi-networks consisting of **weakly coupled networks** is analytically solved by an appropriate **separation of time scales** and by reducing the system dynamics to a Markov chain for aggregated variables. A presence of fitness factors describing attractiveness of individual nodes is taken into account. In the case of system of two coupled networks an equation **analogous to the First Ficks Law** with an additional driving force and a corresponding diffusion constant are found. The entropy production is a sum of entropy changes resulting from a network heterogeneity and the entropy of the Markov chain. Our approach can be also used for **hierarchical networks** where several different time scales are present.

Diffusion

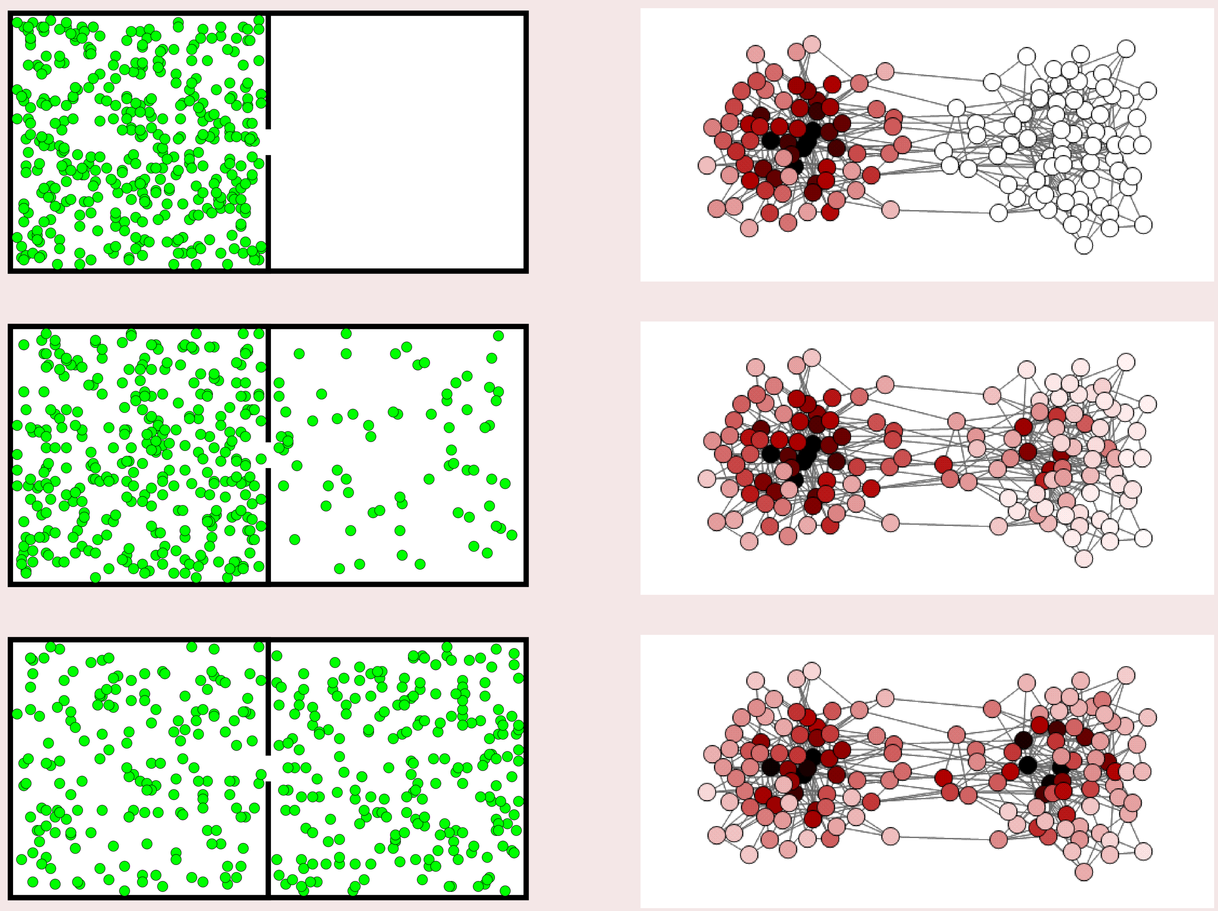
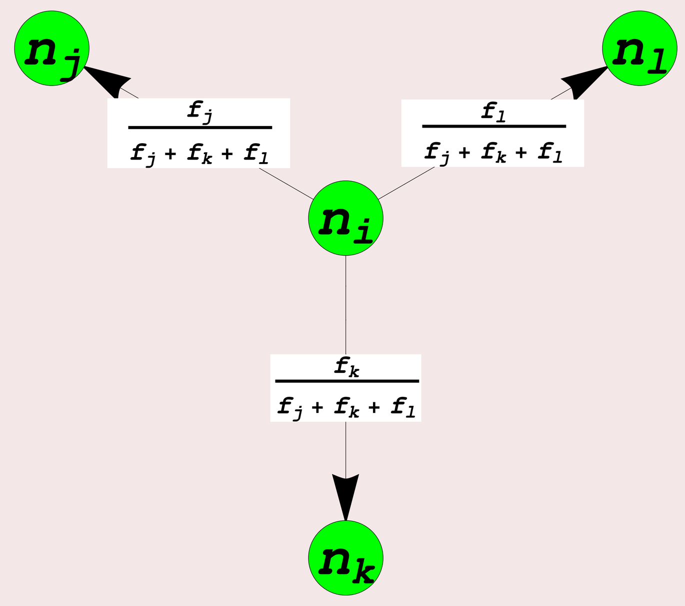


FIGURE 1: *Classical diffusion vs. diffusion on coupled networks.*

Diffusion on connected networks as Markov Chain



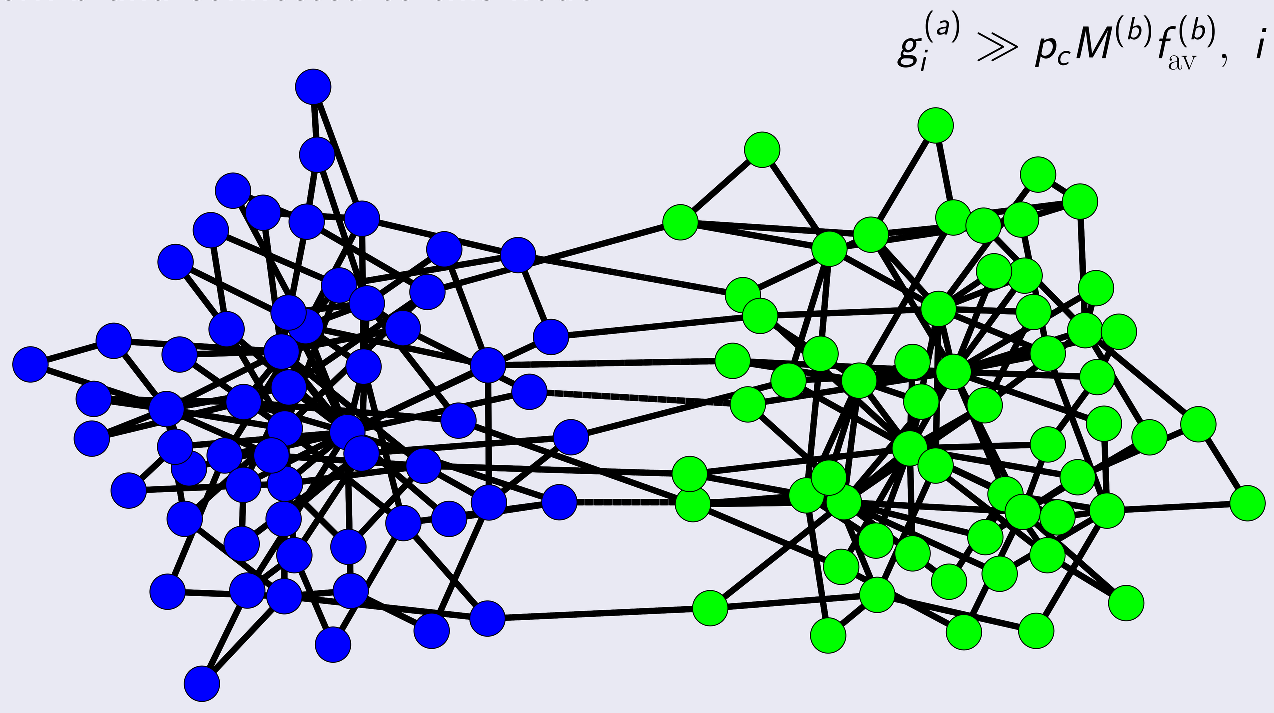
$$n_i(t+1) = f_i \sum_{k=1}^{M} \frac{A_{ik}}{g_k} n_k(t), \qquad g_k = \sum_l A_{lk} f_l \qquad \mu_i = \frac{f_i g_i}{\sum_{k=1}^{M} f_k g_k},$$

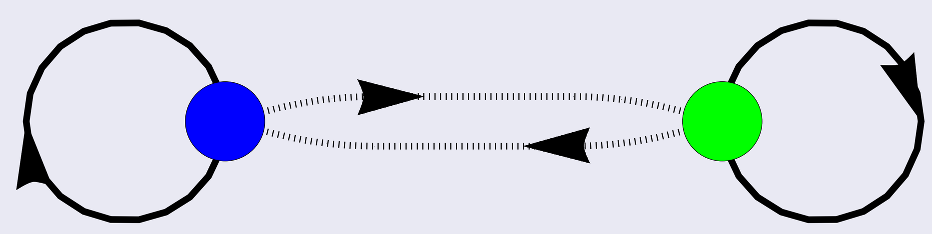
$$S_\infty = -\frac{(f_i g_i \log(f_i g_i))_{\mathrm{av}}}{(f_i g_i)_{\mathrm{av}}} + \log(f_i g_i)_{\mathrm{av}} + \log M.$$

Separation of time scales

Let us assume that a flow between a node i belonging to a network $\mathcal{A}^{(a)}$ to all nodes in this network is much larger than a flow from this node to nodes in the network $\mathcal{A}^{(b)}$ where $a,b = 1,2$ and $a \neq b$. Such a situation takes place when an attractiveness of a node neighbourhood in the network a is much larger than an attractiveness coming from nodes belonging to the network b and connected to this node

$$g_i^{(a)} \gg p_c M^{(b)} f_{\mathrm{av}}^{(b)}, \; i = 1, 2, \ldots, M^{(a)},$$





Our assumption implies

$$\boxed{n_i^{(a)}(t) = N^{(a)}(t)\mu_i^{(a)}.}$$

where $N^{(a)}(t)$ are the total number of particles in the a-th network at time t and $\mu_i^{(a)}$ are equilibrium distributions of density of particles in the non-connected networks.

Results for two networks

$$N^{(1)}(t+1) = (1-\alpha^{(21)})N^{(1)}(t) + \alpha^{(12)}N^{(2)}(t), \qquad (1)$$

where $\alpha^{(a\,b)}$ are constants dependent on $A^{(1)}$, $A^{(2)}$, p_c, $\mathbf{f}^{(1)}$, $\mathbf{f}^{(2)}$ as follows

$$\alpha^{(b\,a)} = p_c M^{(b)} \frac{f_{\mathrm{av}}^{(a)} f_{\mathrm{av}}^{(b)}}{(f_i^{(a)} g_i^{(a)})_{\mathrm{av}}}$$

here a, $b = 1, 2$ and $a \neq b$ and a corresponding equation for $N^{(2)}(t+1)$ follows directly from Eq. (1) by appropriate symmetry relations.

The parameters $\alpha^{(12)}$ and $\alpha^{(21)}$ describe integrated transition probabilities between the networks at the macroscopic level (see Figure above) and can be further simplified when there is no correlation between a node degree and a node fitness factor $\mathbf{f}^{(a)}$ e.g. $(f_i^{(a)} g_i^{(a)})_{\mathrm{av}} = f_{\mathrm{av}}^{(a)} g_{\mathrm{av}}^{(a)}$. Then

$$\alpha^{(b\,a)} = \frac{f_{\mathrm{av}}^{(b)}}{f_{\mathrm{av}}^{(a)}} \frac{p_c M^{(b)}}{k_{\mathrm{av}}^{(a)}},$$

where a, $b = 1, 2$, $a \neq b$ and $\alpha^{(12)}$, $\alpha^{(21)} \ll 1$.

Near the equilibrium state variations of $N^{(1)}(t)$ are sufficiently slow and then it is possible to write Eq. (1) in a form analogous of the First Ficks's Law with a diffusion constant D and in the presence of an additional internetwork driving force F

$$\dot{N}^{(1)}(t) = -\left(N^{(1)}(t) - N^{(2)}(t)\right) D + FN,$$

where

$$D = \frac{\alpha^{(12)} + \alpha^{(21)}}{2}, \qquad F = \frac{\alpha^{(21)} - \alpha^{(12)}}{2}.$$

Using the approximation of separation of the time scales the total entropy of the system and production of the entropy can be written as

$$S_{\mathrm{total}}(t) = \left[N^{(1)}(t)S_\infty^{(1)} + N^{(2)}(t)S_\infty^{(2)}\right] + \left[-N^{(1)}(t)\log N^{(1)}(t) - N^{(2)}(t)\log N^{(2)}(t) + N\log N\right],$$

$$\sigma(t) = \dot{N}^{(1)}(t)\left(\log N^{(2)}(t) - \log N^{(1)}(t) + S_\infty^{(1)} - S_\infty^{(2)}\right),$$

where $S_\infty^{(a)}$, $a = 1, 2$, are equilibrium entropies per particle.

If you want to read...

G. Siudem, J. A. Hołyst, *Diffusion and entropy production for multi-networks with fitness factors*, arXiv:1303.2650 [nlin.CD], (2013).

Networks of networks

It is easy to show that our approach is valid also for any system of m weakly coupled networks i.e. for networks of networks or for multiply networks. Then instead of equation from the left frame one gets

$$N^{(a)}(t+1) = \sum_{l=1}^{m} \alpha^{(a\,l)} N^{(l)}(t),$$

where $\alpha^{(a\,b)}$ describes a strength of flow from a network $\mathcal{A}^{(b)}$ to $\mathcal{A}^{(a)}$ for $a \neq b$.

$$\alpha^{(a\,b)} = p_c^{(a\,b)} M^{(a)} \frac{f_{\mathrm{av}}^{(a)} f_{\mathrm{av}}^{(b)}}{(f_i^{(b)} g_i^{(b)})_{\mathrm{av}}}, \; \alpha^{(a\,a)} = 1 - \sum_{r \neq a} \alpha^{(r\,a)}.$$

Then correspondent equation changes to

$$\dot{N}^{(a)}(t) = \sum_{l \neq a} [-D^{(l\,a)}(N^{(a)}(t) - N^{(l)}(t)) + F^{(l\,a)}(N^{(a)}(t) + N^{(l)}(t))],$$

where

$$D^{(a\,b)} = \frac{\alpha^{(a\,b)} + \alpha^{(b\,a)}}{2}, \qquad F^{(a\,b)} = \frac{\alpha^{(b\,a)} - \alpha^{(a\,b)}}{2}.$$

Hierarchical network

The equivalence between lower and higher level dynamics exist when parameters $\hat{f}_a$, $\hat{g}_a$ and $\hat{A}_{ab}$, a, $b = 1, \ldots, m$, for higher level ($\hat{X}$ means parameter X for the higher level) are appropriately defined. This can be done in several ways e.g.

$$\hat{f}_a = M^{(a)} f_{\mathrm{av}}^{(a)}, \; \hat{A}_{ab} = p_c^{(a\,b)}, \; \text{for} \;\; a \neq b,$$

$$\hat{A}_{aa} = \frac{(f_i^{(a)} g_i^{(a)})_{\mathrm{av}}}{M^{(a)}\left(f_{\mathrm{av}}^{(a)}\right)^2}\left(1 - \sum_{r \neq a} \alpha^{(r\,a)}\right).$$

Diffusion on the networks with hierarchical structure - different **diffusion constant** and **entropy production** for every level of hierarchy.

Acknowledgements

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