Social Networks & Recommendation Systems

IX. Random walks.

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Before classes

Reminder

Before the class, please remind the Markov chain theory

Suggestes readings:

- A. Iwanik, J.K. Misiewicz, *Wykłady z procesów stochastycznych z zadaniami: Część pierwsza – Procesy Markowa*, Oficyna Wydawnicza Uniwersytetu Zielonogórskiego, (2009).
- J.R. Norris, *Markov Chains*, Cambridge University Press, (1997).
- following slides...

Markov Chains – reminder

Source

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Stochastic process $\{X_t,\ t \in \mathbb{T}\}$ is a function such as $(t, \omega) \to X_t(\omega)$, $\omega \in \Omega$, $t \in \mathbb{T}$, where $X_t(\omega)$ is random variable for every fixed $t \in \mathbb{T}$.

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Definition 2.

Set of the possible values of the stochastic process $\{X_t,\ t \in \mathbb{T}\}$ $\mathcal{S} = \{X_t(\omega) : t \in \mathbb{T},\ \omega \in \Omega\}$ we will name **state space** of the process or its **phase space**.

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We will consider at most countable phase spaces.

Markov property

Stochastic process $\{X_n, n \in \mathbb{N}\}$ with values from $\mathcal{S}_m = \{1, ..., m\}$, $m \in \mathbb{N}$ is called **Markov chain** or **Markov process**, when for any $n \in \mathbb{N}$ and $k_0, k_1, ..., k_n \in \mathcal{S}_m$ the following property is satisfied

$$\mathbb{P}\left(X_n = k_n | X_{n-1} = k_{n-1}, ..., X_0 = k_0\right) = \mathbb{P}\left(X_n = k_n | X_{n-1} = k_{n-1}\right).$$

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Definition 3.

Markov chain $\{X_n,\, n \in \mathbb{N}\}$ is **time-homogeneous** or **stationary**, if for any states $i, j \in \mathcal{S}_m$, $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ the following equality holds

$$\mathbb{P}\left(X_n = j | X_{n-1} = i\right) = \mathbb{P}\left(X_1 = j | X_0 = i\right).$$

Markov Chains – reminder

Markov operator

Probabilities $p_{ji} := \mathbb{P}\left(X_1 = j | X_0 = i\right)$, for $i, j \in \mathcal{S}_m$, $m \in \mathbb{N}$, we will name **transition probabilities**, and the corresponding matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \dots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \dots & p_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ p_{m1} & p_{m2} & p_{m3} & \dots & p_{mm} \end{bmatrix},$$

a **transition matrix**.

Attention!

A different notation was adopted here from that contained in the cited textbooks for stochastic processes, where the authors noted the probability of transition in one step as $\tilde{p}_{ij} := \mathbb{P}\left(X_1 = j | X_0 = i\right)$, which meant that in their approach the transition matrix in one step is a transposition, according to our definition $\tilde{P}^T = P$.

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From the application point of view, it is very important to ask how the Markov process works when iterated multiple times. This question is answered by the following statement.

Markov Chains – reminder

Theorem 1.

If the Markov process (with operator) P is iterated n-times, then it corresponds to a Markov process with a transition matrix equal to $Q = P^n$ in one step, where $q_{ij}(n) = [P^n]_{ij}$, is the one-step transition probability for a process that is n -times iteration of the output process.

Markov Chains – reminder

Definition 4.

Matrix $P \in \mathbb{M}^{m \times m}([0,1])$, $m \in \mathbb{N}$, $P = [p_{ij}]_{i,\, j=1,\, \dots,\, m}$ which satisfies the following conditions

- $p_{ij} \geqslant 0$ for any $i, j \in \mathcal{S}_m$,
- $\sum\limits_{i=1}^{m} p_{ij} = 1$,

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Attention

It can be seen that the adopted definition of the P matrix causes that we call the stochastic matrix a column stochastic matrix (i.e. its columns sum up to 1). Row-based stochastic matrices are appropriate with the transposed definition of P. A matrix that is both column and row stochastic is called **a double-stochastic matrix**.

The second convenient way to represent the Markov process, after the transition matrix, is the (one step) transition diagram.

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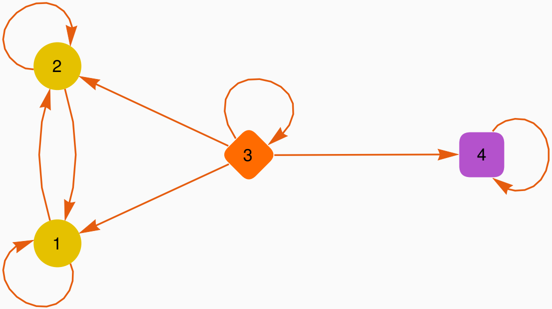
Definition 5.

Transition diagram of Markov chain $\{X_n,\ n \in \mathbb{N}\}$ is directed graph, in which vertices are the elements of the state space and edges connects two states if the corresponding transition probability is non-zero.

Transition diagram



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- State $i \in \mathcal{S}_m$ is **reachable** (accessible) from state $j \in \mathcal{S}_m$ if there exists $n \in \mathbb{N}$, such that $p_{ij}(n) > 0$. This will be denoted by $j \to i$. One state is reachable for the other as long as it is possible to move from one to the other during the iteration.

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- States $i, j \in \mathcal{S}_m$ are **communicating** if $j \to i$ and $i \to j$.

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Observation

Communication is an equivalence relation (symmetric, reflexive and transitive), so it divides all states of the Markov process' state into abstraction classes – classes of states communicating with each other.

Definition 6.

Markov chain is **irreducible**, when all of its states are communicating. Irreducibility means that all states belong to the same class of abstractions with respect to the communication relation.

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Definition 7.

Period of state $i \in \mathcal{S}_m$ is

$$o(i) = GCD(n \in \mathbb{N} : p_{ii}(n) > 0),$$

State i is **periodic** when $o(i) > 1$ and **non-periodic** when $o(i) = 1$.

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Definition 8.

Irreducible Markov chain is **periodic** if all of its states are periodic with the same period $d > 1$. Otherwise we name the chain **non-periodic**.

Theorem 2. (justification of Definition 8.)

For irrecducible Markov chain every state has the same period.

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Definition 9.
Initial distribution $\pi_i = \mathbb{P}(X_0 = i)$, $i \in \mathcal{S}_m$ for Markov chain $\{X_n,\, n \in \mathbb{N}\}$ is **stationary** or **invariant** when

$$\pi = P\pi.$$

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Stationary distribution is thus the fixed point under P.

Definition 10.
Homogeneous Markov chain $\{X_n,\ n \in \mathbb{N}\}$ is **ergodic**, if for every $i \in \mathcal{S}_m$ exist and do not depend on j the following limits

$$q_i = \lim_{n\to\infty} p_{ij}(n) > 0, \quad \text{and} \quad \sum_{i=1}^{m} q_i = 1.$$

Distribution $\mathbf{q} = (q_i)_{i=1,\ldots,m}$ we will name **ergodic**.

Proposition

Each ergodic distribution for a certain Markov chain is also a stationary distribution of this chain. Prove it!

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Attention

The implication in the proposition in general cannot be reversed. The reverse need not be true: the Markov chain may have more than one stationary distribution (when?), however, according to definition 10, the ergodic distribution is defined uniquely. Based on Theorem 1, each Markov chain has at least one stationary distribution, but the existence of an ergodic distribution for any process is not guaranteed.

Theorem 3

For any Markov chain with finite number od states exists at least one stationary distribution[1]. All stationary distributions (as a vectors from the space $\mathbb{R}^m$) belongs to the subspace spread by eigenvectors of matrix P corresponding to eigenvalue 1.

[1]This is, in fact, property of any stochastic matrix, for which 1 is eigenvalue.

[2]If the states would be countably many, then it should be assumed that each state is positive recurrence, i.e. that the probability of returning to each state in a finite time is greater than 0.

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Theory 4.

Let us consider non-periodic and irreducible Markov chain with finite number of states[2]. Then this chain has exactly one stationary distribution π. Moreover, distribution π is ergodic distribution of this process.

---

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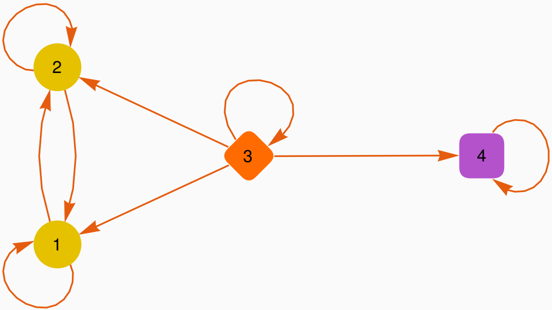
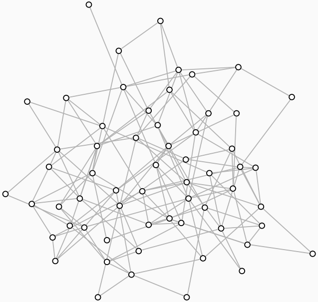
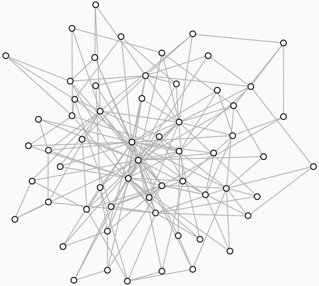
Lecture

Random walks on graphs



Random walks on networks





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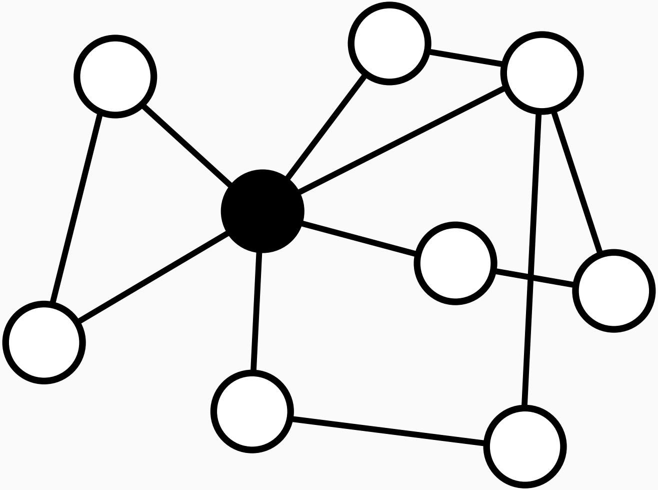

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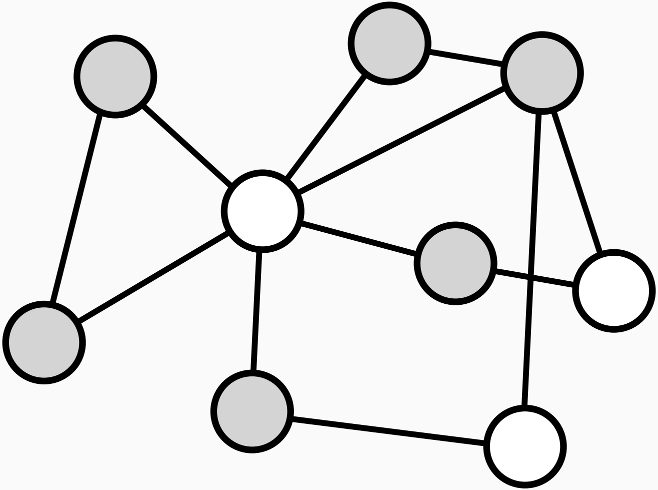

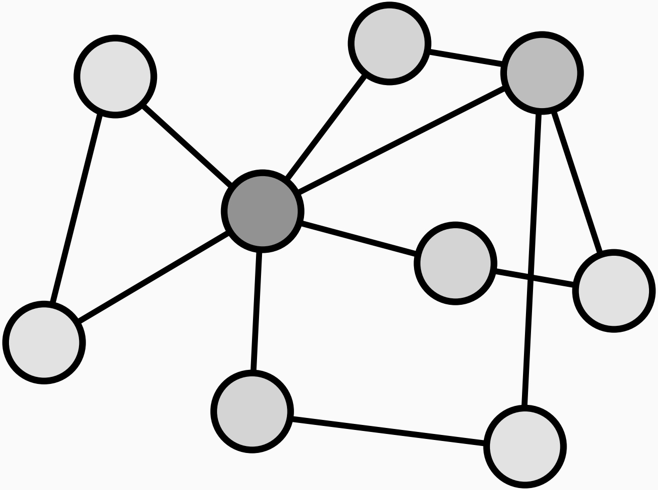

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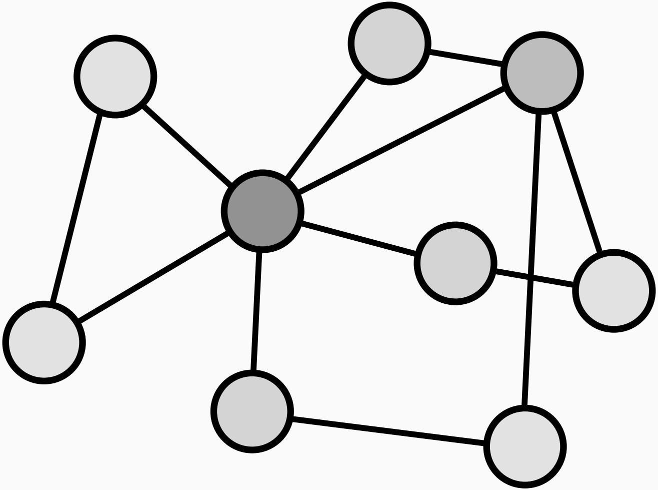
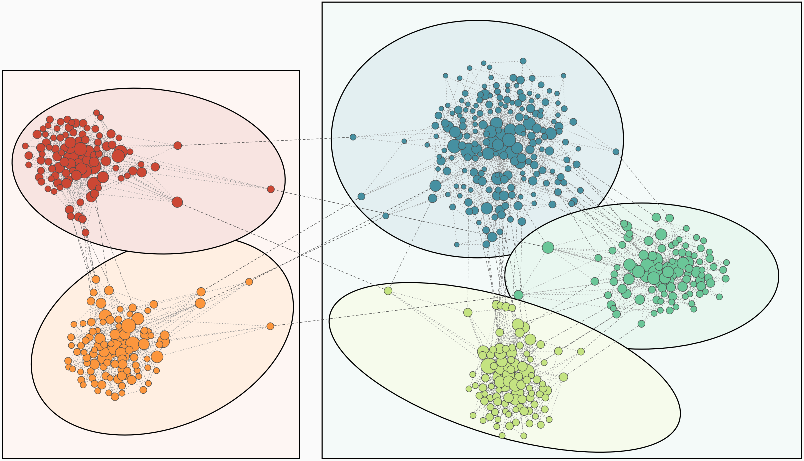


During project we will connect the above with our intuitions from Markov chains theory.

Summary

Homework

Read Fortunato, D. Hric, Phys. Rep., **659** , 1, (2016).



Thank you for your attention!

European Funds
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