# Social Networks & Recommendation Systems

## V. Static random graphs.

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European Funds
Knowledge Education Development

Warsaw University
of Technology

European Union
European Social Fund

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# Before classes

# Reminder

- Bernoulli distribution $\mathbb{P}(X=k) = \binom{N}{k} p^k (1-p)^{N-k}$,
- Piosson distribution $\mathbb{P}(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$,
- convergence of one to another (in what sense?)
- handshake lemma $\sum_i k_i = 2E$ $(\langle k \rangle = 2\frac{E}{N}.)$.

**Question:**

Why does the 2 factor appear in the formula?

# Lecture

# A reminder of what complex network science is

$$\text{complex networks} = \text{data} + \text{metrics} + \text{models} + \dots$$

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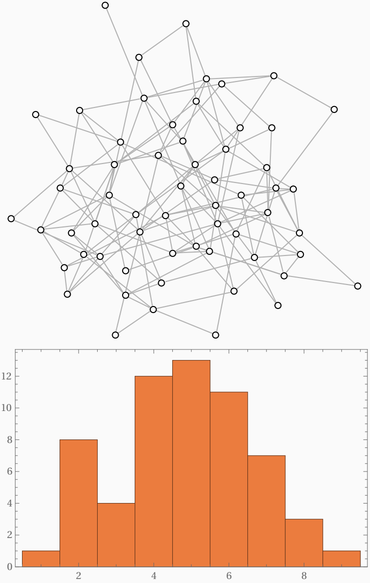
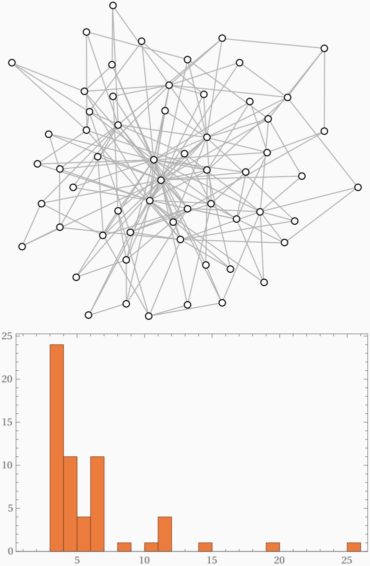
- give a chance to build good statistics while analyzing the dynamics on networks (classes 11 and 12),
- give us an insight into the mechanisms behind real processes (see next classes and BA networks),
- it is a very beautiful field of applied mathematics.

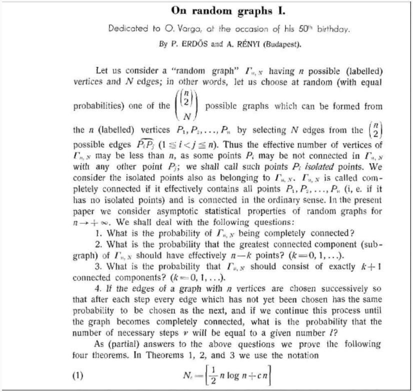
# Attention! ER graphs are *normal*




And really poissonian, as we will see in a moment...

# Erdős-Rényi model



**On random graphs I.**

Dedicated to O. Varga, at the occasion of his $50^{\text{th}}$ birthday.

By P. ERDŐS and A. RÉNYI (Budapest).

Let us consider a "random graph" $\Gamma_{n,N}$ having $n$ possible (labelled) vertices and $N$ edges; in other words, let us choose at random (with equal probabilities) one of the $\binom{\binom{n}{2}}{N}$ possible graphs which can be formed from the $n$ (labelled) vertices $P_1, P_2, \ldots, P_n$ by selecting $N$ edges from the $\binom{n}{2}$ possible edges $\widehat{P_iP_j}$ $(1 \leq i < j \leq n)$. Thus the effective number of vertices of $\Gamma_{n,N}$ may be less than $n$, as some points $P_i$ may be not connected in $\Gamma_{n,N}$ with any other point $P_j$; we shall call such points $P_i$ *isolated* points. We consider the isolated points also as belonging to $\Gamma_{n,N}$. $\Gamma_{n,N}$ is called completely connected if it effectively contains all points $P_1, P_2, \ldots, P_n$ (i. e. if it has no isolated points) and is connected in the ordinary sense. In the present paper we consider asymptotic statistical properties of random graphs for $n \to +\infty$. We shall deal with the following questions:

1. What is the probability of $\Gamma_{n,N}$ being completely connected?
2. What is the probability that the greatest connected component (sub-graph) of $\Gamma_{n,N}$ should have effectively $n-k$ points? $(k=0, 1, \ldots)$.
3. What is the probability that $\Gamma_{n,N}$ should consist of exactly $k+1$ connected components? $(k=0, 1, \ldots)$.
4. If the edges of a graph with $n$ vertices are chosen successively so that after each step every edge which has not yet been chosen has the same probability to be chosen as the next, and if we continue this process until the graph becomes completely connected, what is the probability that the number of necessary steps $\nu$ will be equal to a given number $l$?

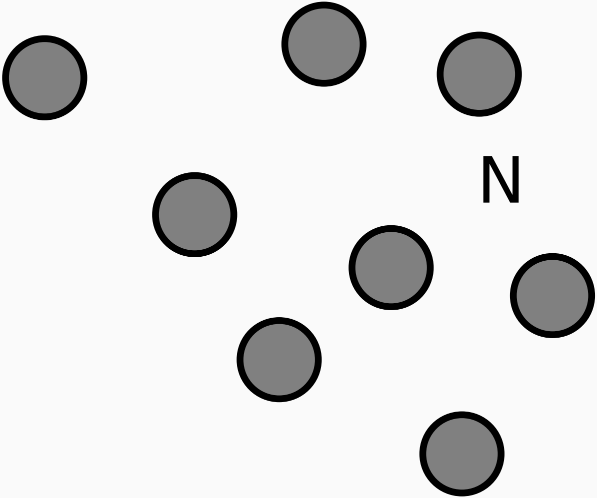
As (partial) answers to the above questions we prove the following four theorems. In Theorems 1, 2, and 3 we use the notation

$$N_c = \left[\frac{1}{2} n \log n + cn\right] \tag{1}$$

Two variants of the model:

- $G_{N,E}$ (Erdős-Rényi),
- $G_{N,p}$ (E. Gilbert)

# Erdős-Rényi model in the $G_{N,E}$ form



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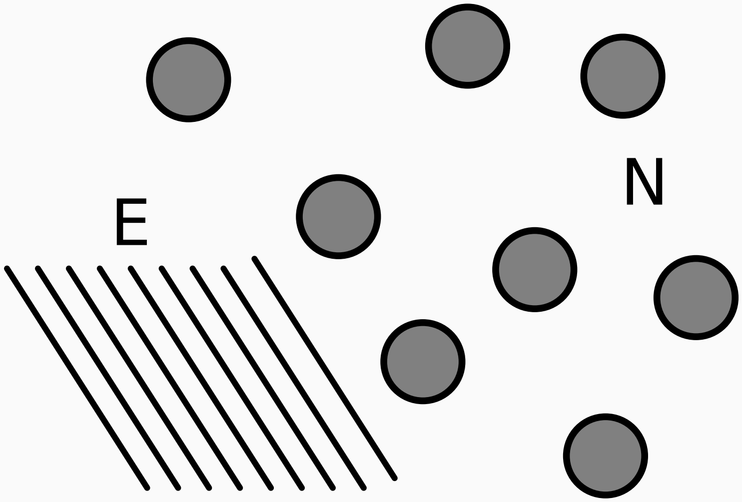

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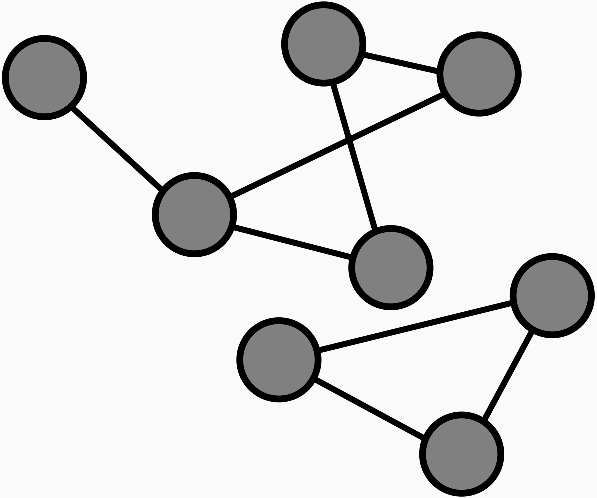

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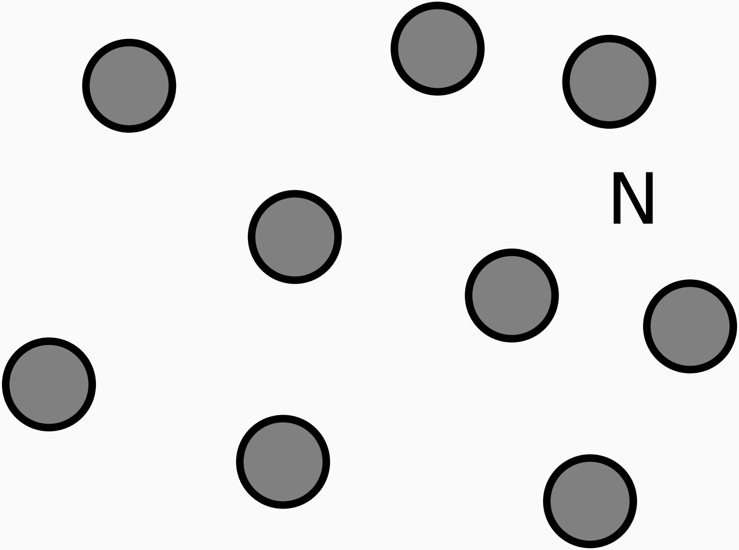

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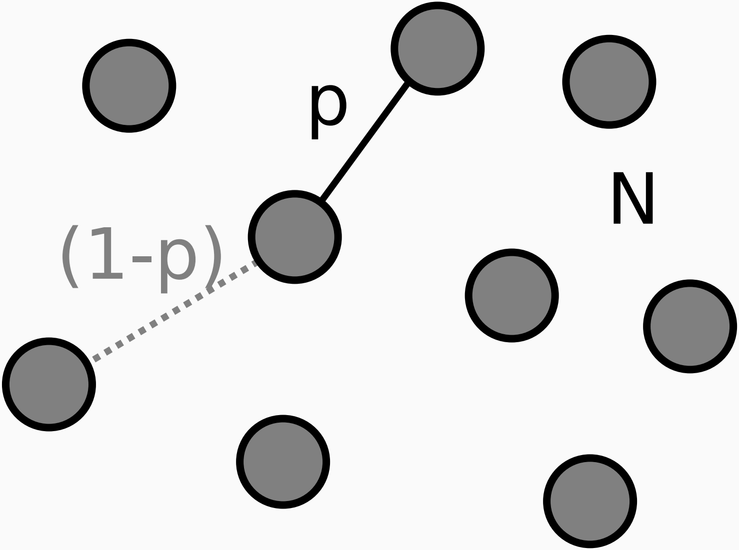

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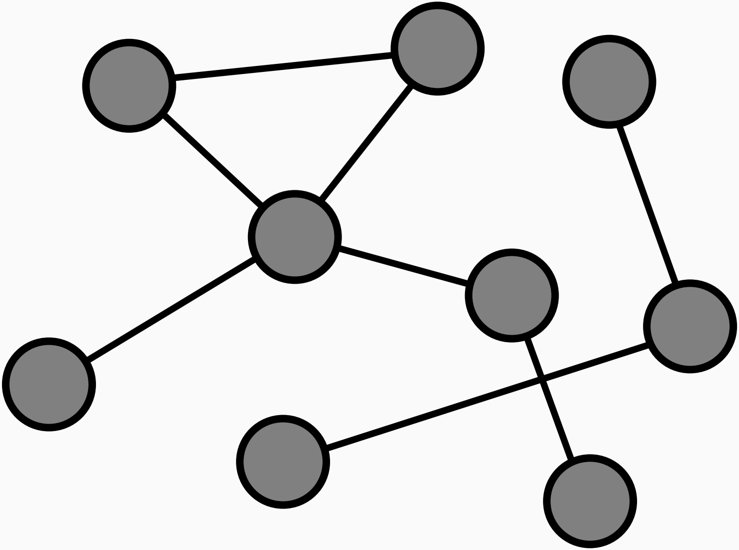
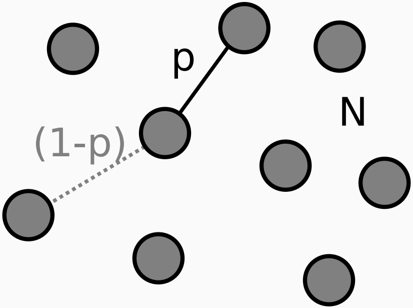


## Let us focus on the $G_{N,p}$ model

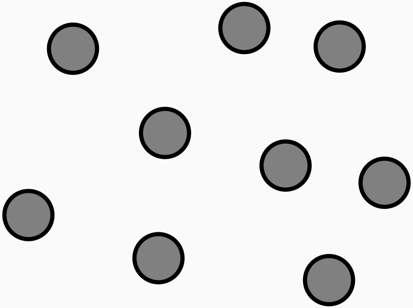
### Assumptions

We analyze the ensemble described by two parameters:

- $N$ number of vertices,
- $p$ probability of two vertices connection.



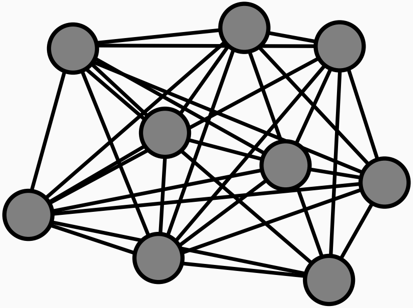
# Limiting case: $p = 0$



**Conclusion:**

$p = 0$ is a trivial case.

# Limiting case: $p = 1$



**Conclusion 1:**

This is a very unsuccessful visualization...

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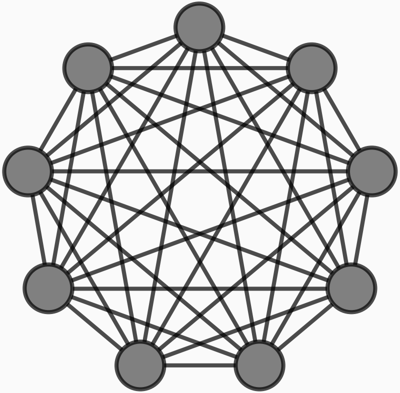


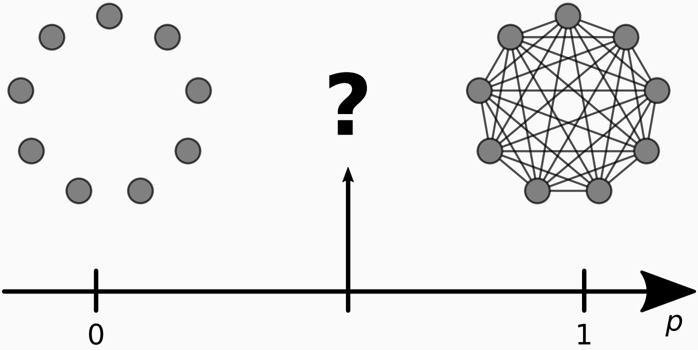
Conclusion 1:

~~This is a very unsuccessful visualization...~~

Conclusion 2:

$p = 1$ is a complete graphs.

# ER graphs – dependence on the parameter $p$



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which with the handshake lemma leads to

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**So we know the average degree**

What about the degree distribution of the vertices?

# Solution of the $G_{N,p}$ model

Random variable describing vertex degree $K$

$$K = \sum_{i=1}^{N-1} X_k,$$

where $X_k$ are iid variables with $\mathbb{P}(X_k = 1) = p$ $\mathbb{P}(X_k = 0) = 1 - p$.

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Poissonian approximation

$$\mathcal{P}(k) \approx \frac{e^{-\langle k \rangle} \langle k \rangle^k}{k!},$$

because $\langle k \rangle \approx Np$.

# Non-physicality of ER graphs

Let's calculate variance for the poissonian case

$$\mathbb{E}(K) = \sum_{k=0}^{\infty} \frac{k e^{-\langle k \rangle} \langle k \rangle^k}{k!} = \cdots = \langle k \rangle,$$

$$\mathbb{E}(K^2) = \sum_{k=0}^{\infty} \frac{k^2 e^{-\langle k \rangle} \langle k \rangle^k}{k!} = \cdots = \langle k \rangle + \langle k \rangle^2.$$

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**Details?**

During the project.

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Such small variance is not observed in the real life networks...

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**Clustering coefficient: reminder**

We are looking for the value of the clustering coefficient for ER graphs

$$C_i = \frac{2E_i}{k_i(k_i - 1)}.$$

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**Answer:**

$$\langle C \rangle = \frac{p\langle k \rangle(\langle k \rangle - 1)}{\langle k \rangle(\langle k \rangle - 1)} = p.$$

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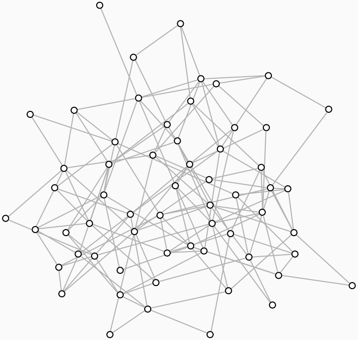
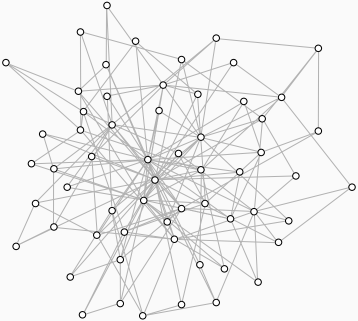
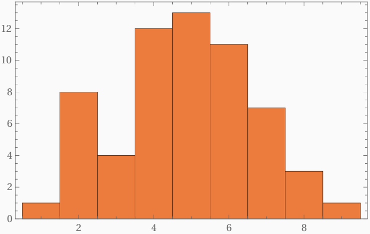
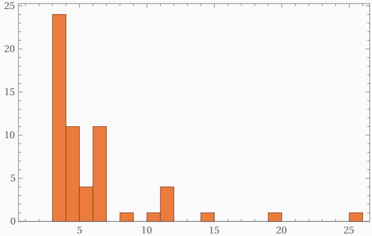
# Non-physicality of ER graphs







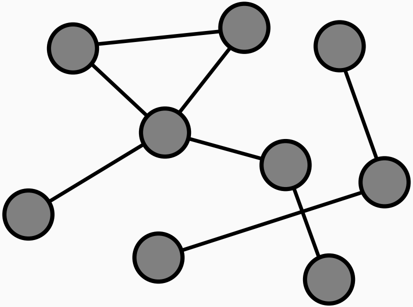




In the following classes, a more realistic BA model. ↗

# Configuration model

**Idea:**

Let's build a graph from the bricks we have.



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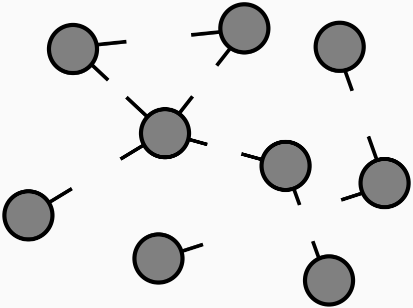

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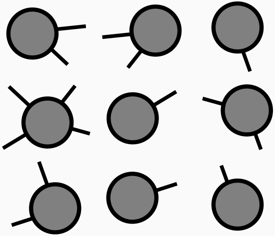

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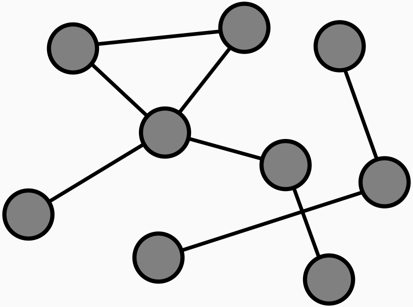

What if you replace the actual bricks with selected *a priori*?

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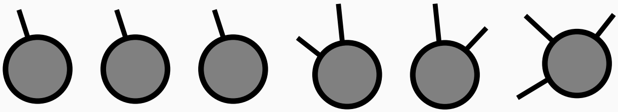

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Is it always possible?

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Question:

Find a (small) counterexample that the proposed procedure may not always be completed.

# Stochastic block model

Generalization of the ER graphs

$$\begin{bmatrix} [p_{11}] & [p_{12}] & \dots & [p_{1N}] \\ [p_{21}] & [p_{22}] & \dots & [p_{2N}] \\ \dots & \dots & \ddots & \dots \\ [p_{N1}] & [p_{N2}] & \dots & [p_{NN}] \end{bmatrix}$$

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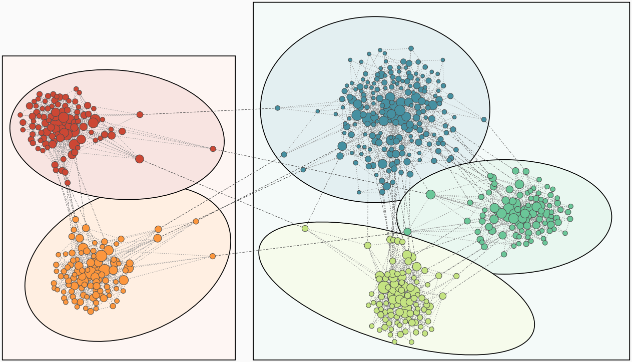
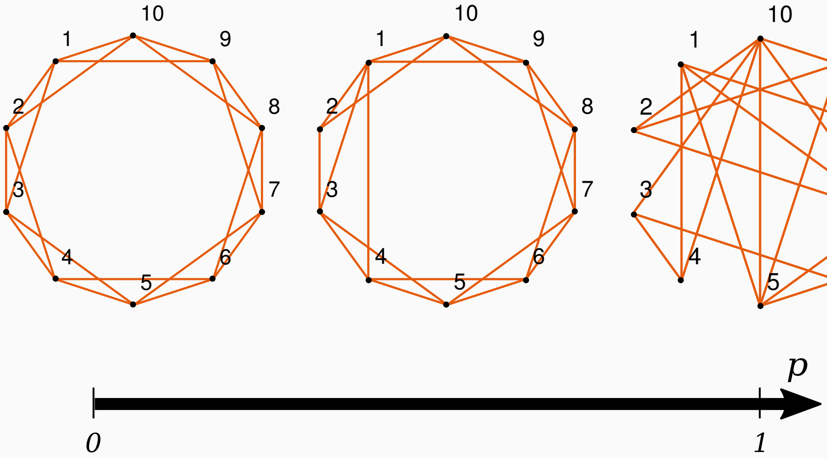


# Watts-Strogatz model



# Networks with given hamiltonian

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i.e. set $M_N = \mathbb{M}^{N\times N}(\{0,1\})$. We want to define a probability distribution on it.

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It *only* remains to solve this equation

$$\frac{\partial \mathcal{L}}{\partial \mathcal{P}(G)} = 0.$$

# Summary

# Homework:

**Read about:**

- Dulbecco rule,
- Matthew effect,
- rich get richer rule,
- Yule processes.

**Suggested source:**

- M. Perc, *Journal of The Royal Society Interface* **11** (2014)

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MSc program in Data Science has been developed
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Thank you for your attention!