A Langevin approach to stock market fluctuations and crashes

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Abstract. We propose a non linear Langevin equation as a model for stock market fluctuations and crashes. This equation is based on an identification of the different processes influencing the demand and supply, and their mathematical transcription. We emphasize the importance of feedback effects of price variations onto themselves. Risk aversion, in particular, leads to an "up-down" symmetry breaking term which is responsible for crashes, where "panic" is self reinforcing. It is also responsible for the sudden collapse of speculative bubbles. Interestingly, these crashes appear as rare, "activated" events, and have an exponentially small probability of occurence. The model leads to a specific "shape" of the falldown of the price during a crash, which we compare with the October 1987 data. The normal regime, where the stock price exhibits behavior similar to that of a random walk, however reveals non trivial correlations on different time scales, in particular on the time scale over which operators perceive a change of trend.

PACS. 02.50.Ey Stochastic processes – 89.90.+n Other areas of general interest to physicists

1 Introduction

Stock market fluctuations exhibit several statistical peculiarities which are still awaiting for a satisfactory interpretation. More strikingly, many of these statistical properties are common to a wide variety of markets and instruments. The most prominent features are [1–3,5–7]:

1. On short time scales, the variations of stock prices are strongly non-Gaussian.
2. Market "volatility" (*i.e.* the conditional variance of the fluctuations) is itself time dependent, with a slowly decreasing, power-law like, autocorrelation function.
3. On very long time scales, the log of the price tends to grow linearly with time with rare, large drops corresponding to market crashes.

The first two properties are observed on a certain range of time scales, ranging from an hour to several weeks, but do not hold for very large time scales (several years) where macroeconomic factors enter into consideration, nor for very short time scales (minutes or so, the typical duration of a transaction) where the detailed structure of the market has to be taken into account.

These "anomalies" have drawn considerable attention recently, because of their intrinsic importance, but also because of possible analogies with physical phenomena such as earthquakes or avalanches. The point is that crashes correspond to a collective effect, where a large proportion of the actors in a market decide simultaneously to sell their stocks; it is thus tempting to think of a crash as some kind of critical point where (as in statistical physics models undergoing a phase transition) the response to a small external perturbation becomes infinite, because all the subparts of the system respond cooperatively. Correspondingly, it has been suggested that "crash precursors" might exist, and in particular "log-periodic" oscillations before the crash [8]. However, no microscopic model has been proposed which substantiate such a claim. Actually, there are as yet no convincing model which "explains" the statistical features described above, although many proposals have been put forward [9,11].

A crucial ingredient in model building is the specification of the level (in our case, the time scale) at which one aims to describes the properties of the system. There are currently two major approaches to market dynamics in the economics and finance literature. One approach is a "temporary equilibrium" approach which assumes that supply and demand equilibrate quickly enough to be considered at instantaneous equilibrium at all times [9]. The other one is that of market microstructure theory [10] which examines the implications of market structure, behavorial assumptions about market participants and specific trading rules on price behavior at the transaction level.

However, our aim here is to describe market dynamics on time scales where, according to empirical observations, some interesting regularities which are *common* between markets with *different* microstructures appear [2,5]. At the same time, these time scales are not long enough to allow the market to reach equilibrium: empirical studies

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show that at intraday time scales there is an imbalance between supply and demand [4]. The level of description adopted here is therefore intermediate between the macroeconomic level which is that of the market equilibrium models [9] and the individual agent level which is that of the market microstructure theory [10].

The aim of this paper is to propose an alternative description of the dynamics of speculative markets with a simple Langevin equation. This equation is built from general arguments, encapsulating what we believe to be the essential ingredients; in particular, the feedback of the price fluctuations on the behaviour of the market participants. We try to motivate as much as possible each term in the equation, and the value of the corresponding parameters is estimated by comparing with empirical data. Our basic idea is that although a reasonably simple description of each individual market participant is impossible in quantitative terms, the collective behavior of the market and its impact on the price in particular can be represented in statistical terms by a stochastic dynamical equation with a few number of terms. Our approach is in the spirit of many phenomenological, "Landau-like" approaches to physical phenomena [12].

2 A phenomenological Langevin equation

We denote the price of the stock at time t as $x(t)$. At any given instant of time, there is an instantaneous demand $\phi_+(t)$ and an instantaneous supply $\phi_-(t)$ for the asset. The first dynamical equation describes the effect of an offset between supply and demand, which tends to push the price up (if $\phi_+ > \phi_-$) or down in the other case. In general, one can write a relation between the instantaneous return $u(t)$ and the excess demand $\Delta\phi$ of the type

$$\frac{dx}{dt} = u(t) = \mathcal{F}(\Delta\phi) \qquad \Delta\phi := \phi_+ - \phi_- \qquad (1)$$

where $\mathcal{F}$ is an increasing function, such that $\mathcal{F}(0) = 0$. In the following, we will frequently assume that $\mathcal{F}$ is linear (or else that $\Delta\phi$ is small enough to be satisfied with the first term in the Taylor expansion of $\mathcal{F}$), and write

$$u(t) = \frac{\Delta\phi}{\lambda} \qquad (2)$$

where λ is a measure of *market depth i.e.* the excess demand required to move the price by one unit. When λ is high, the market can "absorb" supply/demand offsets by very small price changes. Now, we try to construct a dynamical equation for the supply and demand separately. Consider for example the number of buyers ϕ_+. Between t and $t + dt$, a certain fraction of those get their deal and disappear (at least temporarily). This deal is usually ensured by *market makers*, which act as intermediaries between buyers and sellers. The role of market markers is to absorb the demand (and supply) even if these do not match perfectly. Of course, the market makers will absorb buy orders more quickly if they know that the number of

sellers is high, and *vice versa*. (In a non-centralized market, orders are satisfied when a buyer meets a seller, which is all the more probable if there are larger numbers of buy orders and sell orders on the market.) The simplest way to model this effect is by taking a number of orders satisfied per unit time proportional to the number of buy (and sell) orders present on the market at time t. The effect of absorption of orders by the market (MM) can thus be modelled as:

$$\left.\frac{d\phi_\pm}{dt}\right|_{MM} = -\Gamma_\pm(\phi_\mp)\phi_\pm \qquad (3)$$

where Γ are rates (inverse time scales). We assume that market makers act symmetrically, *i.e.*, that $\Gamma_+ = \Gamma_-$. To lowest order in ϕ, we write:

$$\Gamma(\phi) = \gamma + \gamma'\phi + ... \qquad (4)$$

On liquid markets, the time scale $1/\gamma$ before which a deal is reached is short; typically a few minutes or less (see also below for another interpretation of $1/\gamma$).

There are several other effects which must be modeled to account for the time evolution of supply and demand. One is the spontaneous (SP) appearance of new buyers (or sellers), under the influence of new information, individual need for cash, or particular investment strategies. This gives rise to a variation in supply and demand, which we model as the sum of a trend and a random term:

$$\left.\frac{d\phi_\pm}{dt}\right|_{SP} = m_\pm(t) + \eta_\pm(t) \qquad (5)$$

where $\eta_\pm$ have zero mean, and a short correlation time τ_c. $m_\pm$ is the average increase of demand (or supply), which actually also depends on time through the time dependent anticipated return $R(t)$ and the anticipated risk $\Sigma(t)$. It is quite clear that both these quantities are constantly reestimated by the market participants, with a strong influence of the recent past. For example, "trend followers" extrapolate a local trend into the future. On the other hand, "fundamental analysts" estimate what they believe to be the intrinsic value of the stock; if the observed price is above this "true" value, the anticipated trend is reduced, and *vice versa*. An important empirical fact is that since the information on the "true" value is obviously incomplete, the past price fluctuations themselves are often considered as containing some information, at least on what other market participants anticipate. In mathematical terms, these effects can be represented as:

$$R(t) = R_0 + \alpha\overline{u} - k(x - x_0);$$
$$\overline{u} = \int_{-\infty}^{t} dt' K_R(t - t')u(t') \qquad (6)$$

where K_R is a normalized kernel (*i.e.* of integral one) defining how the past local trend $\overline{u}$ is measured by the agents. The coefficient α measures the impact of the observed recent trend on the anticipated return. One actually expects α to depend on $\overline{u}$, and to be larger for negative $\overline{u}$ than for positive $\overline{u}$: loosing money is worse than not

gaining as much as one could have had. We shall therefore write: $\alpha = a - a'\overline{u}$ in the following. Finally k is a mean-reversion force, towards an average (over the fundamental analysts) "true price" x_0[1].

Similarly, agents use past variations of the price to infer a "volatility" parameter for the asset, indicative of its riskiness. Typically such a parameter is calculated as a short-term historical standard deviation of returns[2]

$$\Sigma(t) = \Sigma_0 + \beta\overline{u^2}; \qquad \overline{u^2} = \int_{-\infty}^{t} dt' K_{\Sigma}(t - t')[u(t')]^2.$$
(7)

The higher the volatility of the asset (as perceived by the market), the lower will be the demand of a risk averse investor for this asset: market excess demand will thus be a decreasing function of volatility. The simplest functional form for the dependence of excess demand on volatility is thus a decreasing affine function. Correspondingly, expanding $m_{\pm}(R, \Sigma)$ to lowest order, one has:

$$m_{\pm} = m_{0\pm} + \alpha_{\pm}\overline{u} + \beta_{\pm}\overline{u^2} - \kappa_{\pm}(x - x_0)$$
(8)

where the signs of the different coefficients are set by the observation that m_+ is an increasing function of anticipated return R and a decreasing function of risk Σ, and *vice versa* for m_-. Equation (8), with $\alpha_{\pm}$ expanded to first order in $\overline{u}$, contains all the terms to order u^2 which arise if one assumes that the agents try to reach a trade-off between risk and return: the demand for an asset decreases if is recent evolution shows high volatility and increases if it shows an upward trend. This is the case for example if the investors follow a mean-variance optimisation scheme [13], with adaptive estimates of risk $\Sigma(t)$ and return $R(t)$.

Yet another contribution to the change of demand and supply, which has a similar structure, comes from the existence of option markets, where traders hedge their option positions by buying or selling the underlying stock. The Black-Scholes rule for hedging relates the number of stock to be held to the price of the underlying by a non linear formula [14]. A change of price thus leads to an increase in the demand or supply which can also be represented by $\alpha_{\pm}$ and $\beta_{\pm}$ type of terms, reflecting an average of the so-called "Δ"'s' and the "γ"s' of the different options [14]. In particular, the Black-Scholes hedging strategy is a positive feedback strategy of the trend following type; actually, the 1987 crash is often attributed (at least in part) to the automatic use of the Black-Scholes hedging strategy, which automatically generates sell orders when the value of the stock goes down (portfolio insurance)[14].

We are now in position to write an equation for the supply/demand offset $\Delta\phi$ by summing all these different contributions:

$$\frac{d\Delta\phi}{dt} = -\gamma\Delta\phi + m_0 + a\overline{u} - a'\overline{u}^2$$
$$- b\overline{u^2} - k(x - x_0) + \eta(t)$$
(9)

with $a, a', b, k > 0$. Note in particular that $a', b > 0$ reflects the fact that agents are risk averse, and that an increase of the local volatility always leads to negative contribution to $\Delta\phi$. This feature will be crucial in the following. For definiteness, we will consider η to be Gaussian white noise and normalize it as:

$$\langle \eta(t)\eta(t')\rangle = 2\lambda^2 D\delta(t - t')$$
(10)

where D measure the susceptibility of the market to the random external shocks, typically the arrival of information. In principle, D should also depend on the recent history, reflecting the fact that an increase in volatility induces a stronger reactivity of the market to external news. In the same spirit as above, one could thus write:

$$D = D_0 + D_1\overline{u^2}$$
(11)

For simplicity, we will neglect the influence of D_1 in the following sections, but comment on its effect in the concluding section.

Finally, let us note that equation (9) can be extended to allow for agents with different reaction times. For example, the term $a\int_{-\infty}^{t} dt' K_R(t - t')u(t')$ can be generalized as:

$$\sum_i a_i \int_{-\infty}^{t} dt' K_R^i(t - t')u(t')$$
(12)

where the K_R^i have different ranges which can reflect the fact that there exists several population of traders with different time horizons. Note that $a_i < 0$ corresponds to contrarian traders.

3 Analysis of the linear theory Liquid vs. illiquid markets

Let us consider the linear "risk neutral" case where $a' = b = 0$. For simplicity, we will assume in the following that $K_R(t) = \Gamma \exp{-(\Gamma t)}$, and first consider the local limit where Γ is much larger that γ (short memory time). In this case, the equation for x becomes that of an harmonic oscillator [3]:

$$\frac{d^2x}{dt^2} + \left(\gamma - \frac{a}{\lambda}\right)\frac{dx}{dt} + \frac{k}{\lambda}(x - \tilde{x}_0) = \frac{1}{\lambda}\eta(t)$$
(13)

where m_0 has been absorbed into a redefinition of $\tilde{x}_0 := x_0 + m_0/k$. For liquid markets, where λ and γ are large

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[1] Note that x_0 is actually itself time dependent, although its evolution in general takes place over rather long time scales (months/years).

[2] In principle, a long term average return squared should be substracted from equation (7). This correction is however negligible since fluctuations are in general much larger than average trends.

[3] In an oral seminar given in Jussieu in June 1997, Doyne Farmer also presented a second-order equation for the price. We are not aware of the existence of a written version, and do not know to what extend his analysis is similar to ours.

enough, the "friction" term $\tilde{\gamma} := \gamma - a/\lambda$ is positive. In this case the market is stable, and the price oscillates around an equilibrium value $\tilde{x}_0$, which is higher than the average fundamental price if the spontaneous demand is larger than the spontaneous supply (*i.e.* m_0 is positive), as expected when the overall economy grows. One can also compute the time correlation function of the price fluctuations. The important parameter is:

$$\epsilon := \frac{k}{\lambda \tilde{\gamma}^2} . \tag{14}$$

For liquid markets, $\epsilon \ll 1$. The correlation is found to be the sum of two exponentials, with correlation times τ_1, τ_2:

$$\tau_1 \simeq \frac{1}{\tilde{\gamma}} \qquad \tau_2 \simeq \frac{\tau_1}{\epsilon} \tag{15}$$

and amplitudes $\mathcal{A}_{1,2}$ such that $\mathcal{A}_2 \simeq \epsilon^2 \mathcal{A}_1$. Thus, on a time scale τ_1, the correlation function falls to a very small value $\sim \epsilon^2$. This allows one to identify τ_1 with the correlation time observed on liquid markets, which is of the order of several minutes [2], thereby fixing the order of magnitude of $\tau_1^{-1} = \tilde{\gamma} \simeq \gamma$. Thus, on time scales such that $\tau_1 \ll t \ll \tau_2$, the stock price behaves as a simple biased random walk with volatility $\sigma^2 = 2D\tau_1^2$, before feeling the confining effect of the "fundamental" price. Since the fundamental price is surely not known to better than – say – 10%, and that the typical variation of the price of a stock is also around 10% per year, it is reasonable to assume that the time scale τ_2 beyond which "fundamental" effects (as represented by the harmonic term) play a role is of the order of a year [4]. This leads to $\epsilon \simeq 10^{-4}$. Hence, for liquid markets, the role of the confining term can probably be neglected, at least on short time scales.

The situation is rather different for illiquid markets, or when trend following effects are large, since $\tilde{\gamma}$ can be negative. In this case, the market is unstable, with an exponential rise or decay of the stock value, corresponding to a speculative bubble. However, in this case, dx/dt grows with time and it soon becomes untenable to neglect the higher order terms, in particular the risk aversion terms proportional to a', b. We will comment on this case below. Let us however start by analyzing the role of b for liquid markets for which, as explained above, it is reasonable to set $k = 0$.

4 Risk aversion induced crashes as activated events

Still focusing on the limit where the memory time Γ^{-1} is very small [5], one finds the following non linear Langevin

[4] Beyond the year time scale, however, the evolution of x_0 itself cannot be neglected.

[5] We assume in the following that $K_\Sigma = K_R$; note that in this limit the coefficient a' can be absorbed in a redefinition of b.
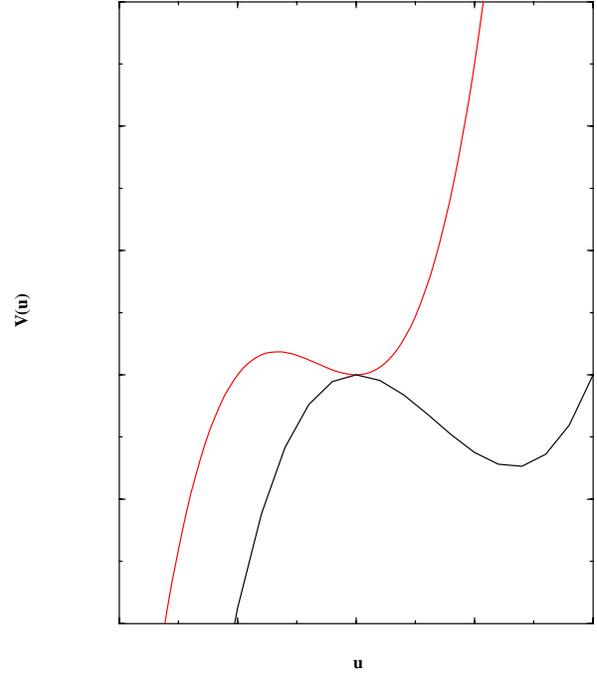
Effective Potential

Fig. 1. Shape of the effective potential $V(u)$. For liquid markets, $\tilde{\gamma} > 0$ and $V(u)$ is minimum for $u \propto m_0$. For illiquid markets, or when the "trend following" effect is large, the minimum moves to to positive value unrelated to m_0. The interesting point is the presence of a potential barrier, separating a normal random walk like regime from a crash regime.

equation for the instantaneous return $u(t)$:

$$\frac{du}{dt} = \frac{m_0}{\lambda} - \tilde{\gamma} u - \frac{b}{\lambda} u^2 + \frac{1}{\lambda} \eta(t) \equiv -\frac{\partial V}{\partial u} + \frac{1}{\lambda} \eta(t). \tag{16}$$

This equation represents the evolution of the position u of a viscous fictitious particle in a potential $V(u)$ represented in Figure 1.

In order to keep the mathematical form simple, we set the average trend m_0/λ to zero (no net average offset between spontaneous demand and spontaneous supply); this does not qualitatively change the following picture, unless m_0 is negative and large (in which case $V(u)$ looses its minimum). The potential $V(u)$ can then be written as:

$$V(u) = \frac{\tilde{\gamma}}{2} u^2 + \frac{b}{3\lambda} u^3 \tag{17}$$

which has a local minimum for $u = 0$, and a local maximum for $u^* = -\lambda\tilde{\gamma}/b$, beyond which the potential plumets to $-\infty$. The "barrier height" V^* separating the stable region around $u = 0$ from the unstable region is given by:

$$V^* = V(u^*) - V(0) = \frac{\tilde{\gamma} u^{*2}}{6} . \tag{18}$$

The nature of the motion of u in such a potential is the following: starting at $u = 0$, the particle has a random harmonic-like motion in the vicinity of $u = 0$ until an "activated" event (*i.e.* driven by the noise term) brings the particle near u^*. Once this barrier is crossed, the fictitious particle reaches $-\infty$ in finite time. In financial terms, the regime where u oscillates around $u = 0$ and where b can be neglected, is the "normal" random walk regime discussed in the previous paragraph. (Note that the random walk is biased when $m_0 \neq 0$.) This normal regime can however be interrupted by "crashes", where the time derivative of the price becomes very large and negative, due to the risk aversion term b which enhances the drop in the price. The point is that these two regimes can be clearly separated since the average time t^* needed for such crashes to occur can be exponentially long, since it is given by the classical Arrhenius-Kramers formula [15,16]:

$$t^* = 2\pi\tau_1 \exp\left(\frac{V^*}{D}\right) = \frac{2\pi}{\gamma} \exp\left(\frac{u^{*2}\tau_1}{3\sigma^2}\right) \qquad (19)$$

where D is the variance of the noise η and $\tau_1 = 1/\tilde{\gamma}$. Taking $t^* = 10$ years, $\sigma = 1\%$ per day, and $\tau_1 = 10$ minutes, one finds that the characteristic value u^* beyond which the market "panics" and where a crash situation appears is of the order of -1% in ten minutes, which not unreasonable. The ratio appearing in the exponential can also be written as the square of $u^*\tau_1/\sigma\sqrt{\tau_1}$; it thus compares the value of what is considered to be an anomalous drop on the correlation time $(u^*\tau_1)$ to the "normal" change over this time scale $(\sigma\sqrt{\tau_1})$.

Note that in this line of thought, a crash occurs because of an improbable succession of unfavorable events, and not due to a single large event in particular. Furthermore, there are no "precursors" (characteristic patterns observed before the crash): before u has reached u^*, it is impossible to decide whether it will do so or whether it will quietly come back in the "normal" region $u \simeq 0$. Note finally that an increase in the liquidity factor γ reduces the probability of crashes. This is related to the stabilizing role of market makers, which appears very clearly.

An interesting prediction concerns the behaviour of the price once one enters the crash regime *i.e.* once u becomes larger (in absolute value) than u^*. Neglecting the noise term, one finds that the stock price is given by:

$$x(t) = x^* + \frac{\lambda}{b} \ln\left[\exp(\tilde{\gamma}(t_f - t)) - 1\right] \qquad (20)$$

which diverges logarithmically towards $-\infty$ when t reaches a final time t_f. Of course, in practice, this divergence is not real since when the price becomes too low, other mechanisms, which we have not taken into account in the model, come into play. One thus expects that some external mechanism interrupts the crash, which in the Langevin language, correspond to a "reinjection" of the particle around $u = 0$. Formula (20) is compared in Figure 2 to the observed price of the S&P index during the 1987 October crash, where we have fixed t_f to be the time when the price reaches its minimum. This leads to

Crash of the S&P in 1987
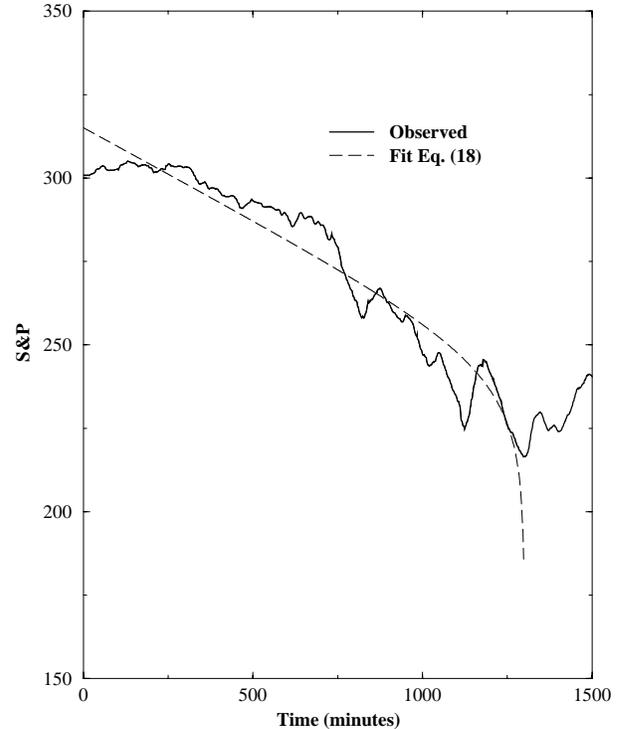
From Oct 15, 14.20 to Oct 20, 16.30

Fig. 2. Evolution of the New-York S&P index during the 1987 October crash. The dotted line is a fit with the noiseless formula (20), where t_f is taken to be the time when the index reached its minimum. At this point, our model certainly breaks down, since other effects, not taken into account in the present approach, come into play. This fit thus does not really probe the logarithmic divergence, but is useful to fix the values of $\tilde{\gamma}$ and λ/b.

$\tilde{\gamma} = 4.5 \times 10^{-3}$ (in minutes^{-1}) and $\lambda/b = 12.9$ (S&P points), from which we estimate $u^* = 3.5$ S&P points per hour (more than 1% per hour). The last figure is not unreasonable; however, the order of magnitude found for $\tilde{\gamma}$ is much smaller than expected, on the basis that τ_1 is ten minutes or so. We shall come back to this point below, in Section 6.

5 Illiquid markets: speculative bubbles and collapse

Suppose now that the trend following tendency is strong so that $\tilde{\gamma} < 0$. The potential $V(u)$ has a minimum for $u = u^*$ which is now *positive*, and a maximum for $u = 0$. The price increment u oscillates around a positive value unrelated to m_0, which means that there is a non zero trend not based on true growth but entirely induced by the fact that a price increase motivates more people to buy – this is called a speculative bubble. After a time t, the price has risen on average by an amount u^*t.

If this increase is too large, it becomes illegitimate to neglect the role of the "fundamental" price x_0. The full potential in which u evolves is actually given by $V(u) + k(x - x_0)u/\lambda$. It is easy to see that this potential has a local minimum (which leads to the above sustained growth) only when:

$$\tilde{\gamma}^2 > 4\frac{kb}{\lambda^2}(x - x_0) \qquad (21)$$

but that this minimum disappears for larger values of $x - x_0$. Assuming that $x(t = 0) = x_0$, we thus find a time t_b where the bubble has to collapse, since the (metastable) equilibrium around u^* is no longer present. The lifetime of the bubble is given by:

$$t_b \simeq \frac{a - \lambda\gamma}{k} . \qquad (22)$$

As could be expected, the stronger the "spring" term k pulling the price back towards the fundamental value x_0, the shorter will be the duration of speculative bubbles. This spring term may be interpreted as proportional to the fraction of "fundamentalists" in the market.

6 Memory effects

Up to now, we have assumed that the impact of a change of price on the behaviour of the market participants was instantaneous, *i.e.* that the kernels K_R and K_Σ used to estimate the average return and risk have a typical memory time shorter than any other time scale in the problem, in particular γ^{-1}. Since we have argued that γ^{-1} is of the order of a few minutes on liquid markets, this assumption is not very realistic: it is more reasonable to assume that agents judge the evolution of risk and return on longer time scales, at least several days. We are thus actually in the opposite limit where $\Gamma \ll \gamma$. Fortunately the case of an exponential memory kernel still leads to a tractable model, provided $a' = 0$. It is easy to show that the dynamical equation now reads:

$$\frac{d^2u}{dt^2} = -(\gamma + \Gamma)\frac{du}{dt} - \Gamma\left[\frac{m}{\lambda} - \tilde{\gamma}u - \frac{b}{\lambda}u^2\right] + \frac{1}{\lambda}\left[\Gamma\eta(t) + \frac{d\eta}{dt}\right] \qquad (23)$$

which indeed leads back to equation (16) in the limit $\Gamma \to \infty$. Equation (23) governs the evolution of a massive particle in the very same potential as the one above (Fig. 1). One can show that in this case, for $b = m_0 = 0$, and in the limit where $\Gamma \ll \gamma$, the correlation function of the increment u is given by:

$$\langle u(t)u(t')\rangle = \sigma^2\left[(\tilde{\gamma}^2 + \tilde{\gamma}\tilde{\Gamma})\exp(-\gamma|t - t'|)\right.$$
$$\left. + (\tilde{\Gamma}^2 + \tilde{\gamma}\tilde{\Gamma})\exp-(\Gamma|t - t'|)\right] \qquad (24)$$

(with $\tilde{\Gamma} = 2\Gamma a/\lambda\gamma$) which thus decays rapidly (on a time $\tilde{\gamma}^{-1}$) to a value which can be negative if $a < 0$, before

slowly going to zero on time scales $\sim \Gamma^{-1}$. Interestingly, the empirical correlation function of short time increments indeed shows a *negative* minimum on time scales of the order of 15 minutes [2,5]. The relative amplitude of this minimum (as compared to $\langle u(t)^2\rangle$) is of the order of a few %. This could thus be interpreted as the effect of fast "contrarian" traders superposed to the regulatory action of the market makers (contributing to a negative a).

For $b > 0$, one still has a sharp distinction between a "normal" regime, where the stock price performs a random walk with volatility σ (except that, as just discussed, the increment correlation function has a small tail decaying on time scales Γ^{-1}), and a "crash" regime, when the "particle" manages to reach the top of the potential barrier. The theory of activated processes can be extended to massive particles. In the limit $\Gamma \ll \tilde{\gamma}$, the average time between crashes t^* is given by a formula very close to the one above [15,16]:

$$t^* = \frac{2\pi}{\sqrt{\tilde{\gamma}\Gamma}}\exp\left(\frac{u^{*2}\tau_1}{3\sigma^2}\right) \qquad (25)$$

i.e., only the prefactor of the exponential is changed. Note that, as could be expected intuitively, the fact that there is a delay in the reaction of traders tends to stabilize the market, since the crash time is multiplied by a factor $\sqrt{\tilde{\gamma}/\Gamma}$.

Finally, the dynamics of the price when the crash has started is also affected by the presence of a memory. The truly asymptotic behaviour of equation (23) (for zero noise) is given by:

$$u(t) \simeq -\frac{6\lambda}{\Gamma b}(t_f - t)^{-2} \qquad (26)$$

which leads to a $(t_f - t)^{-1}$ divergence of the price itself. However, as noticed above, this divergence is certainly interrupted by effects which our model cannot describe. In order to compare with empirical data, in particular that of the crash of 1987, one can notice that the time scale over which the crash took place (days) is much larger than $\tilde{\gamma}^{-1}$. It is thus reasonable to neglect the second derivative term as compared to the first. In the limit where $\Gamma \ll \gamma$, we are thus led to:

$$\frac{du}{dt} = -\Gamma u - \frac{b\Gamma}{\lambda\gamma}u^2 \qquad (27)$$

the solution of which is of the same form as the one without memory, except for the coefficients:

$$x(t) = x^* + \frac{\lambda\gamma}{b\Gamma}\ln[\exp(\Gamma(t_f - t)) - 1]. \qquad (28)$$

The same fit as in Figure 2 is thus adequate. However, interestingly, one finds that it is now Γ, rather than $\tilde{\gamma}$, which appears in the exponential. In other words, the time scale during which the crash develops is much longer; from the fit we find (see Fig. 2): $\Gamma^{-1} \simeq 220$ minutes (half a day). The estimate of u^*, as $\lambda\gamma/b$, is unaffected.

7 Concluding remarks

We hope to have convinced the reader that the above Langevin equation, which is based on an identification of the different processes influencing supply and demand and their mathematical transcription, captures many of the features seen on markets. We have in particular emphasized the role of feedback, in particular through risk aversion, which leads to an "up-down" symmetry breaking non linear term $u(t)^2$. This term is responsible for the appearance of crashes, where "panic" is self reinforcing; it is also responsible for the sudden collapse of speculative bubbles. Interestingly, however, these crashes are rare events, which have an exponentially small probability of occurence (see Eq. (19)). We predict that the "shape" of the falldown of the price during a crash should be given by equation (28), which is compatible with empirical data (Fig. 2). An interesting feature of our simple model is that it implies a relation between parameters describing the statistical properties of returns during the "normal" regime and the frequency of crashes (Eq. (19)).

The "normal" regime, where the stock price behaves as a random walk, reveals non trivial correlations on the time scale over which operators perceive a change of trend. In particular, a small negative dip related to the existence of contrarian traders can appear. In this respect, it is important to stress that within these models the presence of correlations lead, in principle, to simple winning strategies (on average). It is however easy to convince oneself that if the level of correlations is small (for example, as seen above, of order of a few percent after tens of minutes), the transaction costs are such that arbitrage cannot be implemented in practice [2]. Therefore, we believe that non trivial correlations can be observed on financial data, and do actually arise naturally when feedback effects are included.

Before closing, we would like to discuss briefly several other points. The first one concerns the fact that we have considered x to be the price, rather than the log of the price. Of course, on short time scales, this does not matter, and actually a description in terms of the price itself is often preferable on short time scales [2]. On longer time scales, however, the log of the price should be preferred since it describes the evolution of prices in *relative* rather than absolute terms. However, on these long time scales, one should also take into account the evolution of the model parameters (such as the fundamental price x_0, or the average trend m_0), which are related to economic factors and thus not amenable to such a simple statistical treatment, based on consideration of a purely speculative market. Second, we have identified a "normal" regime, where u oscillates around zero, and a crash regime for $|u| > u^*$. In the model presented above, the "normal" fluctuations are Gaussian[6] if η is Gaussian and if the re-

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[6] Note that in our model, "normal" fluctuations and crashes describe two very different regimes of the same dynamical equation. In this sense, we agree with the idea that market crashes are indeed "outliers" from a statistical point of view [17], since the shape of the tail of the distribution in the normal regime

lation between price changes and supply/demand unbalance is linear. In order to account for the large kurtosis observed on markets during "normal" periods (*i.e.* excluding crashes), one necessarily has to take into account either the non linearity of the price change and/or the intermittent (non-Gaussian) nature of the random shocks, in particular the role of the feedback term D_1 introduced in equation (11), which can easily be shown to lead to "fat tails". Although the quantitative formulae given above are affected by such effects, the qualitative picture will remain.

Finally, the above model, where crashes appear as activated events, suggests a tentative interpretation for "log-periodic" oscillations seen before crashes [8]. Imagine that each time u reaches – by accident – an anomalously negative value (but above u^*), the market becomes more "nervous". This means that its susceptibility to external disturbances like news will increase. In our model, this can be described by an increase of the parameter $D \propto \sigma^2$, through the term D_1 in equation (11). If D increases by a certain value δD at every accident and since D appears in an exponential, this implies that the average time Δt before the next "accident" is decreased by a certain factor which, to linear order in δD, is constant:

$$\Delta t_{n+1} = \Delta t_n S^{-\delta D/D} \qquad S = \exp\left(\frac{u^{*2}\tau_1}{3\sigma^2}\right). \qquad (29)$$

This leads to a roughly log-periodic behaviour, which indeed predicts that the time difference between two events is a geometric series. However, our scenario is *not* related to a critical point: the crash appears when u exceeds u^*, and not when $\Delta t \to 0$, *i.e.*, when crash events accumulate. In this respect, it should be noted that according to the critical log periodic theory, there should have been another crash near the end of November 1997, and then again roughly 10 days later, which did not occur [18].

References

1. C.W.J. Granger, Z.X. Ding, *Stylized facts on the temporal distributional properties of daily data from speculative markets*, Working Paper, University of California, San Diego (1994); D.M. Guillaume, M.M. Dacorogna, R.R. Davé, U.A. Müller, R.B. Olsen, O.V. Pictet, Finance and Stochastics **1**, 95-130 (1997); R. Mantegna, H.E. Stanley, Nature **376**, 46-49 (1995).
2. J.P. Bouchaud, M. Potters, *Théorie des Risques Financiers* (Aléa-Saclay, Eyrolles, 1997).
3. R. Cont, M. Potters, J.P. Bouchaud, in *Scale invariance and beyond*, edited by B. Dubrulle, F. Graner, D. Sornette (EDP-Springer, 1997); R. Cont, Science & Finance Working Paper 97-01 (cond-mat/9705075).

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cannot be usefully extrapolated to the unstable regime. Imposing a constant current in the unstable region of equation (16) actually leads to $P(u) \sim |u|^{-2}$ for large negative u's.

4. see *e.g.*: Gourieroux, Lefol, Meyer, *Analysis of Order Queues*, CREST Working Paper 9653 (1996); Gourieroux, Jasiak, Lefol, *Intraday Market Activity*, CREST Working Paper 9633 (1996).

5. R. Cont, *Statistical Finance: empirical study and theoretical modeling of price fluctuations in financial markets*, Ph.D. dissertation, Université de Paris XI (1998).

6. P. Cizeau, Y. Liu, M. Meyer, C.K. Peng, H.E. Stanley, preprints cond-mat/9708143 and 970621.

7. M. Potters, R. Cont, J.P. Bouchaud, Europhys. Lett. **41**, 239 (1998).

8. D. Sornette, A. Johansen, J.P. Bouchaud, J. Phys. **6**, 167 (1996); J.A. Feigenbaum, P.G.O. Freund, Int. J. Mod. Phys. **10**, 3737 (1996); N. Vandewalle, M. Ausloos, P. Boveroux, A. Minguet, Eur. Phys. J. B **4**, 139 (1998).

9. W.A. Brock, D.A. Hsieh, B. LeBaron, *Nonlinear dynamics, chaos and instability: statistical theory and economic evidence* (Cambridge MA: MIT Press, 1991); J.M. Grandmont, in *Handbook of mathematical economics*, edited by K.J. Arrow, M.D. Intriligator (Elsevier, 1981); W.A. Brock, C.F. Hommes, Econometrica **65**, 1059 (1997).

10. For a review of market microstructure theory see: M. O'Hara, *Market microstructure theory* (Blackwell Publishers, 1995).

11. For recent attempts in the physics community, see: M. Lévy, H. Lévy, S. Solomon, J. Phys. **5**, 1087 (1995); P. Bak, M. Paczuski, M. Shubik, Physica A **246**, 430 (1997); G. Caldarelli, M. Marsili, Y.C. Zhang, Europhys. Lett. **40** 479 (1997); R. Cont, J.P. Bouchaud, preprint cond-mat/9712318.

12. For a "manifesto" defending such an approach in the physical context see *e.g.*: T. Hwa, M. Kardar, Phys. Rev. A **45**, 7002 (1992).

13. H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments* (J.Wiley and Sons, New York, 1959); E.J. Elton, M.J. Gruber, *Modern Portfolio Theory and Investment Analysis* (J.Wiley and Sons, New York, 1995).

14. for a review, see: J. Hull, *Options, futures and other derivative securities* (Prentice-Hall, 1997). For a physicist approach, see [2].

15. S. Chandrasekhar, in *Noise and Stochastic Processes*, edited by N. Wax (Dover, 1954).

16. P. Hanggi, P. Talkner, M. Borkovec, Rev. Mod. Phys. **52**, 251 (1990)

17. A. Johansen, D. Sornette, *Stock Market crashes are outliers*, Eur. Phys. J. B **1**, 141 (1998).

18. L. Laloux, M. Potters, R. Cont, J.P. Aguilar, J.P. Bouchaud, submitted to Europhys. Lett., preprint (http:xxx.lanl.gov/cond-mat/9804111).