Critical Coarsening without Surface Tension: the Voter Universality Class

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We show that the two-dimensional voter model, usually considered to only be a marginal coarsening system, represents a broad class of models for which phase-ordering takes place without surface tension. We argue that voter-like growth is generically observed at order-disorder nonequilibrium transitions solely driven by interfacial noise between dynamically symmetric absorbing states.

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Coarsening phenomena occur in a large variety of situations in and out of physics, ranging from the demixion of alloys [1] to population dynamics [2]. At a fundamental level, phase-ordering challenges our capacity to deal with nonequilibrium systems and our understanding of the mechanisms determining different universality classes. In many cases, phase competition is driven by surface tension, leading to 'curvature-driven' growth. Coarsening patterns are then characterized by a single length scale $L(t) \sim t^{1/z}$, where the exponent z only depends on general symmetry and conservation properties of the system [1]. For instance, $z = 2$ for the common case of a non-conserved scalar order parameter (NCOP), a large class including the Ising model. In this context, the two-dimensional voter model (VM) [3], a caricatural process in which sites on a square lattice adopt the opinion of a randomly-chosen neighbor, stands out as an exception. Its coarsening process, which gives rise to patterns with clusters of all sizes between 1 and $\sqrt{t}$ (Fig. 1) [4,5], is characterized by a slow, logarithmic decay of the density of interfaces $\rho \sim 1/\ln t$ (as opposed to the algebraic decay $\rho \sim 1/L(t) \sim t^{-1/z}$ of curvature-driven growth). The marginality of the VM is usually attributed to the exceptional character of its analytic properties [4–6].

In this Letter, we show that, in fact, large classes of models exhibit the same type of domain growth as the simple VM, without being endowed with any of its peculiar symmetry and integrability properties. We argue that voter-like coarsening is best defined by the absence of surface tension and that it is generically observed at the transitions between disordered and fully-ordered phases in the absence of bulk fluctuations, when these nonequilibrium transitions are driven by interfacial noise only. Finally, we discuss the universality of the scaling properties associated with voter-like critical points.

We first review the properties of the usual two-state VM, emphasizing those of importance for defining generalized models. A 'voter' (or spin) residing on site $\mathbf{x}$ of a hypercubic lattice can have two different opinions $s_{\mathbf{x}} = \pm 1$. In any space dimension d, an elementary move consists in randomly choosing one site and assigning to it the opinion of one of its randomly chosen nearest neighbors (n.n.). This ensures that the two homogeneous configurations (where all spins are either $+$ or $-$) are absorbing states, and that the model is Z_2-symmetric under global inversion ($s_{\mathbf{x}} \rightarrow -s_{\mathbf{x}}$). Recasting the dynamic rule in terms of n.n. pair updating, a pair of opposite spins $+-$ is randomly selected, and evolves to a $++$ or $--$ pair with equal ($1/2$) probabilities. The rates of creation of $+$ and $-$ spins being equal, any initial value of the global magnetization m is conserved in the limit of large system sizes.

Another prominent feature of the VM is a 'duality' [3] with a system of coalescing random walks: going backward in time, the successive ancestors of a given spin follow the trail of a simple random walk (RW); comparing the values of several spins shows that their associated RWs necessarily merge upon encounter. This correspondence allows to solve many aspects of the kinetics of the VM, because it implies that the correlation functions between an arbitrary number of spins form a closed hierarchy of diffusion equations [5,6]. In particular, the calculation of the density of interfaces $\rho_m(t)$ (i.e. the fraction of $+-$ n.n. pairs) starting from random initial conditions (r.i.c.) of magnetization m, is ultimately given by the probability that a RW initially at unit distance from the origin, has not yet reached it at time t. Therefore, owing to the recurrence properties of RWs, the VM shows coarsening for $d \leq 2$ (i.e. $\rho_m(t) \rightarrow 0$ when $t \rightarrow \infty$). For the 'marginal' case $d = 2$ —on which we mainly focus henceforth—, one finds the slow logarithmic decay [5,6]:

$$\rho_m(t) = (1 - m^2) \left[\frac{2\pi D}{\ln t} + \mathcal{O}\left(\frac{1}{\ln^2 t} \right) \right] , \qquad (1)$$

D being the diffusion constant of the underlying RW ($D = 1/4$ for the standard case of n.n., square lattice walks, when each spin is updated on average once per unit of time). This peculiar behavior, which contrasts with the algebraic decay encountered in NCOP growth, thus seems tantamount to the fact that $d = 2$ is the crit-

ical dimension for the recurrence of RWs.

Beyond these analytic properties, the nature of the coarsening displayed by the VM in two dimensions is highlighted by studying the evolution of a 'bubble' of one phase embedded into another. For curvature-driven NCOP growth, the volume of a bubble decreases linearly with time under the effect of surface tension. In the VM, however, large bubbles do not shrink but slowly disintegrate as their boundary roughens diffusively to reach a typical width of the order of their initial radius r_0 (Fig. 1, top). As long as $\sqrt{t} \ll r_0$, conservation of the magnetization is still effective, and implies that radially-averaged magnetization profiles $m(r,t)$ have a stationary middle point. This indicates that curvature plays no role, a fact further confirmed by the observation that the $m(r,t)$ curves are the same as the transverse profiles of an infinite, straight interface, whose derivatives can be shown to be given by a simple Gaussian of variance $2Dt$.
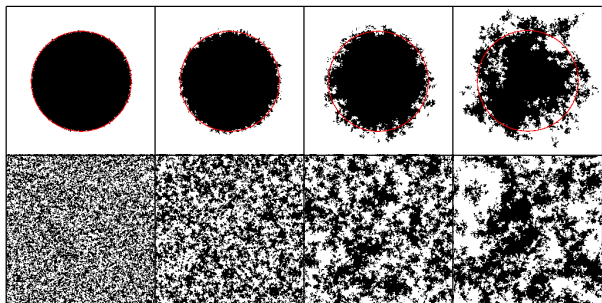

FIG. 1. Domain growth in the usual VM (system size 256^2). Top: snapshots at $t = 4, 16, 64, 256$ during the evolution of a bubble of initial radius $r_0 = 180$ (thin circle). Bottom: same from symmetric r.i.c.

We interpret the above behavior as characteristic of the absence of surface tension, a 'physical' property which, together with the fact that coarsening does occur (i.e., $\lim_{t\to\infty} \rho = 0$), we conjecture to be constitutive of $d = 2$ voter-like domain growth (as defined by the logarithmic decay of $\rho_m(t)$). To investigate which of the properties outlined above (existence of two absorbing states, Z_2-symmetry, duality with coalescing RWs, m-conservation) is necessary to produce voter-like growth, we now embed the simple VM studied above into large families of rules.

Consider first that a $+-$ pair is now only updated with probability p ($\Pr(+- \to +-) = 1 - p$), and that $\Pr(+- \to ++) = pq$ and $\Pr(+- \to --) = p(1-q)$, so that for $q \neq \frac{1}{2}$ the m-conservation is broken. Suppose next that p and q depend on the local configuration around the $+-$ pair, for instance via n^+ and n^-, the numbers of $+$ (resp. $-$) n.n. of the $-$ (resp. $+$) site in the central pair. (Note that as soon as the transition probabilities do vary with the local configuration, the duality with coalescing RWs breaks down.) Conservation of m ($\Pr(+- \to ++) = \Pr(+- \to --)$) is then simply expressed by the conditions $q_{n^+,n^-} = \frac{1}{2}$, to be

obeyed $\forall n^+, n^- \in \{1, 2, 3, 4\}$, whereas $p_{n^-,n^+} = p_{n^+,n^-}$ and $q_{n^-,n^+} = 1 - q_{n^+,n^-}$ stand for Z_2-symmetry.

With the exception of the usual VM ($p = 1, q = 1/2$), the m-conserving and Z_2-symmetric rules, defined by $q_{n^+,n^-} = 1/2$ and $p_{n^-,n^+} = p_{n^+,n^-}$, do *not* benefit from the duality property used above to derive, e.g., Eq. (1). Nevertheless, we have found (by coarsening, bubble and line experiments [9]) that they behave like the simple VM, but with a diffusion constant $D \neq \frac{1}{4}$. Thus, integrability is not a necessary condition for voter-like growth.

The subset of m-conserving rules without Z_2-symmetry is of particular interest: no surface tension is expected, but one may wonder about the effects of the asymmetry on growth properties. Below, we use the case $p_{n^+,n^-} = 1/n^+$, which is nothing but the interface growth model introduced by Kaya, Kabakçıoğlu, and Erzan (KKE) at its 'delocalization' transition [7]. The asymmetry of the rule manifests itself at the microscopic level (by looking at local 3-spins configurations [9]), but also at the macroscopic level, by the evolution of the magnetization profiles in 'line' or 'bubble' experiments. As for the usual VM, the profiles obtained in both cases are identical, their width scale as $2D_{\mathrm{KKE}}\sqrt{t}$ with $D_{\mathrm{KKE}} = 0.36(2)$ [8], but they are asymmetric (Fig. 2a). Voter-like behavior is further confirmed by the study of phase ordering following r.i.c. of magnetization m. The density $\rho_m(t)$ displays the signature scaling spelled out by Eq. (1) (Fig. 2b), and the ensuing value $D = 0.34(1)$ is in fair agreement with the value D_{KKE} determined above, which suggests that the KKE rule asymptotically behaves like the usual VM with an *effective* diffusion constant.
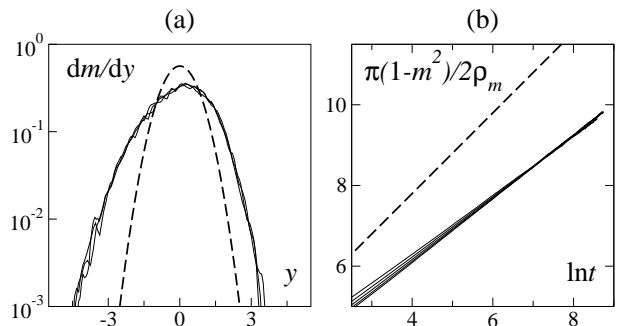

FIG. 2. Domain growth in the KKE model and the n.n. ($D = \frac{1}{4}$) VM. (a): derivative of rescaled magnetization profiles at times $t = 16, 64, 256$ (data from a bubble experiment; initial conditions: $m(r,0) = \mathrm{sign}(r - r_0)$ with $r_0 = 4096$). Dashed line: exact result for the VM $dm/dy = (2/\sqrt{\pi})e^{-y^2}$ with $y = (r - r_0)/\sqrt{t}$; (b): scaling of $\rho_m(t)$ (as suggested by Eq. (1)) in coarsening experiments for $m = 0, \pm0.2, \pm0.4$ in systems of size 16384^2; dashed line: exact result for the VM.

All the other m-conserving, asymmetric rules that we have studied reveal a similar voter-like behavior. We conjecture that they almost all do so, as long as the p_{n^+,n^-} are all strictly positive (to avoid possible 'blocking' in certain configurations). In other words, m-conservation

appears as a sufficient condition for voter-type growth, even in the absence of Z_2-symmetry or integrability. We now show that m-conservation is not a necessary condition since some non-conserving Z_2-symmetric rules with two absorbing states exhibit voter-like features.

If pair-update rules are convenient to control m-conservation, single-site update models suffice to study rules which need not possess this property. We thus consider a family of 'kinetic Ising models', in which a spin is flipped with a probability $r_{s,h}$ depending on its value $s = \pm 1$ and on its local field $h \in \{-4, -2, 0, 2, 4\}$, i.e. the sum of the values of its four n.n.. The Z_2-symmetric rules of this family, obeying $r_{-s,-h} = r_{s,h}$, can be parameterized by five transition probabilities $r_h \equiv r_{1,h}$. Such rules are usually not integrable. Again, the VM, given by $r_h = \frac{1}{2} - \frac{h}{8}$, is an exception. Its marginal situation in this context has already been noted [10,11] in the particular case where the transition probabilities only depend on h, a condition expressed here by $r_{-s,h} = 1 - r_{s,h}$. One is then left with only two free parameters, $p_{\rm b} \equiv r_4$, $p_{\rm i} \equiv r_2$, which can be respectively interpreted [11] as a measure of the strength of 'bulk' and 'interfacial' noise. In the $(p_{\rm i}, p_{\rm b})$ plane, the VM ($p_{\rm i} = \frac{1}{4}$, $p_{\rm b} = 0$) turns out to be the endpoint of a line of Ising-like (second-order) phase transitions [10,11], at the border between the ordered and disordered phases in the absence of bulk noise ($p_{\rm b} = 0$).

We claim that this situation is generic: the 4-parameter space of the general kinetic Ising model above without bulk noise (space defined by $r_4 = 0$) is divided into a disordered and an ordered region separated by a codimension-1 *critical* manifold of voter-like rules [12]. In Fig. 3, we show a section of this manifold in a plane which contains the VM. Its location was determined by studying the evolution of ρ following r.i.c. with zero magnetization at various parameter values. Crossing the critical manifold, the characteristic logarithmic decay $\rho \propto 1/\ln t$ separates the region where $\rho \propto 1/\sqrt{t}$ (curvature-driven growth typical of the ordered phase) from the disordered region where ρ saturates to finite values (Fig. 3b). A peculiar feature of this voter-like critical manifold is that, using the language of the renormalization group (RG), $m = 0$ is a 'weakly attracting' fixed point of the dynamics: starting from any $m_0 \neq 0, \pm 1$ ($m_0 = m(0)$), one observes that $m(t) \propto 1/\ln t$ (Fig. 3c). Yet, the decay of the interface density behaves like in the usual VM (Fig. 3d), even for $m_0 \neq 0$, because the slow evolution of $m(t)$ introduces only $\mathcal{O}(1/\ln^2 t)$ corrections to Eq. (1) and therefore does not alter the value of the effective diffusion constant. This also explains why our bubble and line experiments —which strictly speaking probe only a zero-measure set of random initial conditions— performed for rules on the voter-like manifold, do give the same results as for the usual VM [9], in spite of the absence of m-conservation.

Similar investigations have been undertaken in various planes of the 4-parameter space, always yielding the same results. This suggests to conjecture that all critical Z_2-symmetric rules without bulk noise form a codimension-1 'voter-like' manifold separating order from disorder, characterized by the logarithmic decay of both ρ and m.

Does the above picture extend to non-conserving, asymmetric rules without bulk noise? In this case, even though two absorbing states exist, their dynamical roles are not symmetric, and such order-disorder transitions are known [13,14] to belong generically to the universality class of directed percolation (DP). We have indeed observed [9] that for such rules both m and ρ scale at criticality with the *same* exponent $\beta^{\rm DP}/\nu_\parallel^{\rm DP} = 0.450(1)$ [14].
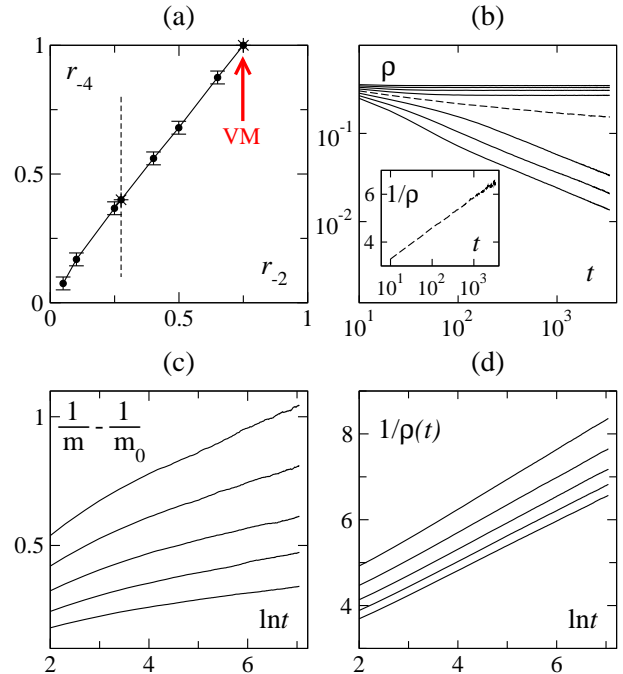

FIG. 3. Voter-like rules not conserving m: (a): phase diagram in the plane (r_{-2}, r_{-4}) for $r_0 = \frac{1}{2}, r_2 = r_{-4}/4$. (b): $\rho(t)$ from $m_0 = 0$ r.i.c. increasing r_{-4} (from top to bottom) with $r_{-2} = 0.275$ (dashed line in (a)). Inset: interface density voter-like decay at the critical point (system size 2048^2). (c,d): $m(t)$ (c) and $\rho(t)$ (d) from r.i.c. with various m_0 at the critical point described in (b) (system size 16384^2).

Having specified the required conditions to observe voter-like coarsening, we now turn to the investigation of the scaling laws associated to voter-like points and to an assessment of their universality. To our knowledge, there has been only one attempt to determining the scaling exponents governing the approach to the usual VM. Within the reduced $(p_{\rm i}, p_{\rm b})$ parameter plane defined above, the authors of [10] noted first that increasing $p_{\rm i}$ along the zero-bulk-noise line $p_{\rm b} = 0$, up to the VM point ($p_{\rm i} = \frac{1}{4}$), the (static) magnetization $m_{\rm s}$ jumps from 0 to ± 1. Thus, the order parameter exponent $\beta = 0$, and, for such a first-order $d = 2$ nonequilibrium critical point, the susceptibility and the correlation length exponents should satisfy $\gamma = 2\nu$. A finite-size scaling (FSS) analysis of the fluctuations of m_s as $\varepsilon \equiv p_{\rm i} - \frac{1}{4} \to 0^-$ confirmed the lat-

ter relation, and gave $\gamma \approx 1.25$. However, repeating these simulations with much better statistics, we evidenced [9] a systematic decrease of the local exponent γ as $\varepsilon \to 0^-$. On general grounds, corrections to scaling in this problem are expected to be logarithmic [15] and, thus, the above γ value must be taken cautiously. In fact, a more reliable approach of the VM is from the ordered side ($\varepsilon \to 0^+$), along the zero-bulk noise line. The curvature-driven-growth regimes $\rho \sim \xi(\varepsilon)/L(t) \sim \varepsilon^{-\nu}/t^{1/2}$, which eventually settle then in large enough systems, are less prone to the logarithmic corrections mentioned above. The scaling of the correlation length $\xi(\varepsilon)$ yields $\nu = 0.45(7)$, compatible with the simple 'diffusive' value $\nu = \frac{1}{2}$ (Fig. 4).
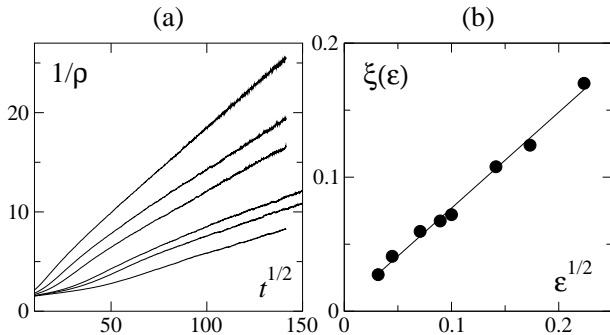

(a) (b)

FIG. 4. Phase-ordering from $m = 0$ r.i.c. approaching the VM from the ordered side (system size 8192^2). (a) ρ vs $\sqrt{t}$ for various ε between 10^{-3} and 5.10^{-2} (from bottom to top); (b) extracted correlation length ξ vs $\sqrt{\varepsilon}$.

Compounding the above results, a conservative interpretation of the data is that $\beta = 0$, $\gamma = 1$, and $\nu = \frac{1}{2}$, with appropriate logarithmic corrections to scaling. This is further confirmed by similar, preliminary, results obtained when approaching any of the voter-like models studied above [9]. We finally note that these exponents would obey all the standard scaling relations valid at a critical point, such as $\beta = [d - \lambda(H)]/\lambda(T_i)$, $\gamma = [2\lambda(H) - d]/\lambda(T_i)$, or $\nu = 1/\lambda(T_i)$, with (in $d = 2$) $\lambda(T_i) = \lambda(H) = 2$. We tentatively associate the former RG-eigenvalue with the relevance of interfacial noise, and the latter with the presence of a 'dynamically self-induced' magnetic field breaking the Z_2-symmetry (as happens in KKE-like models). The behavior of the classes of voter-like models we have defined is consistent with such a set of exponents, which would thus characterize (in $d = 2$) the genuine voter critical point. For instance, converting the static critical behavior $m \sim \varepsilon^\beta$ to a dynamical one via $\xi \sim \varepsilon^{-\nu} \sim L(t) \sim t^{1/z}$ gives $m \sim t^{-\beta/\nu z}$. Now, if $\beta = 0$ strictly, this implies $m(t) = \text{Cst}$, while if $\beta = 0^+$ is interpreted, as is customary, including logarithmic corrections, then $m(t) \sim 1/\ln t$. It is also nice to note that, despite the presence of the *a priori* relevant eigenvalue $\lambda(H) = 2$, m-conserving rules without Z_2-symmetry may still display voter-like growth, because the critical exponent δ (which describes how the order parameter behaves under a magnetic field: $m \sim H^{1/\delta}$),

would be formally given by $1/\delta = -1 + d/\lambda(H) = 0$.

Naturally, the RG picture sketched above needs to be substantiated by the study of an appropriate field theory, an endeavor left for future work. In particular, one would like to have a better understanding of the RG flow around the voter critical point as the space dimension varies, and of the special role played by $d = 2$, which appears as the *upper critical dimension* where voter-like phase-ordering can be induced by the sole presence of interfacial noise [16,9]. At any rate, we believe the results presented here reveal that the voter model, which was, up to now, perceived as a marginal system, embodies in fact a broad class of models for which coarsening occurs in the absence of surface tension. Our findings also suggest that the critical behavior associated to this class of systems is perhaps best characterized as an order-disorder transition driven by the interfacial noise between two absorbing states possessing equivalent 'dynamical roles', this symmetry being enforced either by Z_2-symmetry of the local rules, or by the global conservation of magnetization.

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