Voter model on Sierpinski fractals

Krzysztof Suchecki and Janusz A. Hołyst *

Faculty of Physics and Center of Excellence for Complex Systems Research, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland

PACS numbers: 89.75.-k, 05.45.Df, 05.50.+q

Abstract

We investigate the ordering of voter model on fractal lattices: Sierpinski Carpets and Sierpinski Gasket. We obtain a power law ordering in all cases, but the dynamics is found to differ significantly for finite and infinite ramification order of investigated fractals.

1 Introduction

The Ising model is a well known dynamical model that was investigated in complex networks and fractal structures [1–6]. However, aside from that model, there are many other possible dynamics, sharing little in common with behavior of the Ising model. The voter model is an example of such a model, that exhibits different qualities at a very basic level. Unlike the Ising model, the voter model has no surface tension and defines a broad universality class [7]. While the Ising model dynamics has been studied on fractal lattices [3–6] little is known about the behavior of voter model in such geometries.

We have investigated the behavior of the voter model on Sierpinski carpets and on Sierpinski gasket. It is known [8] that for non-fractal systems the evolution of the voter model depends on the dimensionality of the lattice. For a large time t the ordering process obeys the following equations

$$\rho(t) \sim \begin{cases} t^{-\alpha}, & D < 2 \\ (\ln t)^{-1}, & D = 2 \\ 1, & D > 2 \end{cases} \tag{1}$$

* Corresponding author. Tel.: +48 22 660 7133; fax: +48 22 628 2171
Email address: jholyst@if.pw.edu.pl (Janusz A. Hołyst).

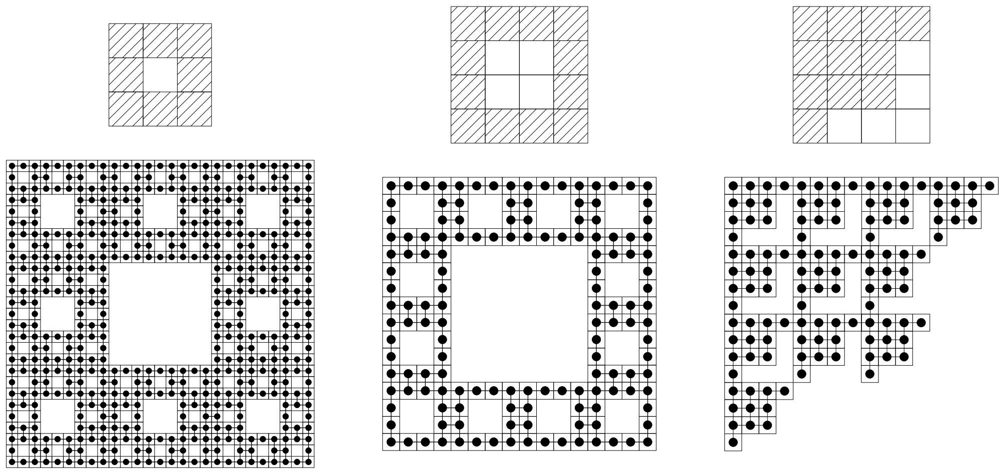


Fig. 1. The construction of Sierpinski Carpet (SC) fractal network. Three different basic patterns are at the top, and corresponding fractal networks of level 3 and 2 are at the bottom. The hatched area of patterns are full positions. The first two carpets from the left have an infinite ramification, while the third one possesses the ramification $\mathcal{R} = 4$.

where ρ is a fraction of links that form interfaces, i.e. they connect opposite spins, D is dimensionality, t is time and $\alpha = 1 - D/2$ is the predicted exponent. The derivation was made for a hypercubic lattice with an arbitrary number of neighbors, what directly translates into dimensionality. Later on, the correlation functions were calculated for the voter model dynamics in such lattices. The lattices were initially assumed to have an integer dimensionality, but the resulting equations have a critical point at $D = 2$. The final result in [8] was given in the form of such inequalities to emphasize the criticality.

We will focus on the problem of voter model dynamics on fractal lattices. Since the analytical derivations [8] were not made for fractals, they are not expected to describe the dynamics correctly, but they form a good reference point for investigations and comparison.

2 Models

The voter model is a very simple model of opinion formation. Nodes in the network are agents, each one having an opinion. There are only two possible opinions, and typically they are considered as +1 and -1, just as Ising spins. The dynamic rule is simple — the node opinion changes to an opinion of one randomly chosen neighbor.
The implementation is following: we choose one node at random, and then

one of its neighbors randomly. The first node assumes the state of the second. One time step of the dynamics corresponds to the number of individual node updates equal to the number of nodes in the network, so on average each node is updated once every time step.

We investigate the voter model behavior on two fractal networks: Sierpinski Carpet (SC) and Sierpinski Gasket (SG). The SC is constructed according to a chosen basic pattern. The pattern is a square, divided into $n \times n$ squares that can be full or empty (Fig.1). First, single nodes are taken, and arranged into the pattern, putting nodes into full positions and skipping empty positions. In the next step, the resulting structures are arranged into the same pattern. All neighboring nodes in the resulting pattern are connected creating the fractal network. The fractal dimension of SC depends on the basic pattern. Classical SC has 3×3 pattern with all the squares full except the central one. Such SC has a fractal dimension $d = \frac{\ln 8}{\ln 3} \approx 1.893$. We have investigated two types of SC. One has an *infinite* ramication and dimensions ranging from $\frac{\ln 28}{\ln 8} \approx 1.6025$ to $\frac{\ln 8}{\ln 3}$. The basic pattern is an empty interior surrounded by a full positions frame, similar to the classical SC (see Fig.1). Second type has the *finite* ramification $\mathcal{R} = 4$ and dimensions ranging from $\frac{\ln 6}{\ln 3} \approx 1.6309$ to $\frac{\ln 38}{\ln 7} \approx 1.8693$. The basic pattern is full, except for the right and down edge, where only single positions are full (see Fig.1). Since it is impossible to numerically investigate true, infinite fractals, we will call the number of steps in what the network was made a *fractal level*.

The ramification is the minimum number of links that one needs to remove to separate a macroscopic part of infinite fractal. The finite ramification means that the structure has some "weak points" where only a finite number of links connect together two parts of an infinite network. The infinite ramification means that infinite parts of infinite network are connected by infinite number of links. For example, a regular square lattice has an infinite ramification, while a tree has the ramification equal to 1.

The SG network is created in the following way (Fig.2). Three nodes are taken and connected into a triangle. In the middle of each edge a node is created and the three new nodes are connected between themselves. This way the whole triangle is divided into four smaller ones. In the next step all three non-central triangles are treated in the same way, adding nodes in middle of the triangle edges and linking them toghether. SG has the fractal dimension $\frac{\ln 3}{\ln 2} \approx 1.5850$ and it possesses a *finite* ramification $\mathcal{R} = 4$.

While in the case of SC, it is easy to create a general class of SC fractals with different fractal dimensions, we are not aware of any generalization of SG model that allows easy tuning of fractal dimensions.

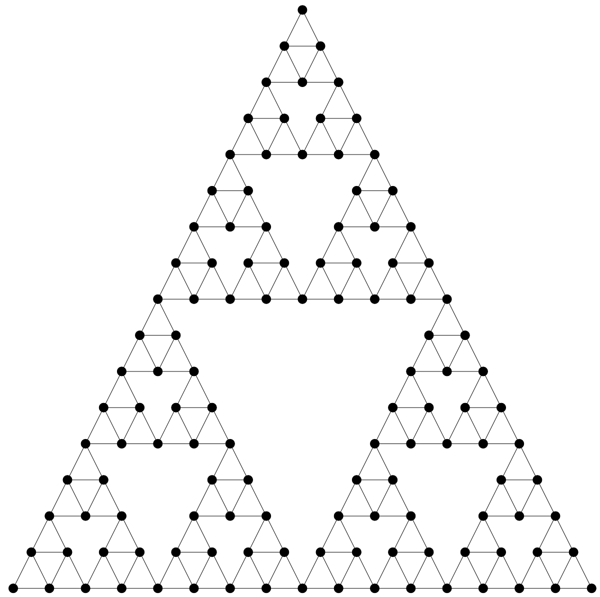


Fig. 2. The construction of Sierpinski Gasket (SG) fractal network of level 5.

3 Results

We have investigated ordering of the voter model in SC and SG fractals. To measure the disorder, we have used the fraction of interfaces $\rho = I/E$, where I is a number of interfaces – links connecting nodes with different spins, $E = N \langle k \rangle /2$ is the total number of links in the network, k is the node degree – number of connections the node possesses.

The system orders (Fig.3) with the interface fraction ρ decreasing as a power of time t. However, due to the finite system size, there are fluctuations around the power-law. Since the power-law decay becomes slower with time, the fluctuations become more significant, and they push the system into a completely ordered absorbing state after some time.

To extract the power-law trend, we have averaged the results of many simulations, but to avoid the exponential decay due to complete ordering of the individual simulations, in a given time step we have averaged only over the simulations that were not completely ordered at that time. This way we have circumvented the fluctuations ordering the system and have obtained an approximation of an infinite network (Fig.4).

We have observed the evolution of the interface fraction ρ in time for networks

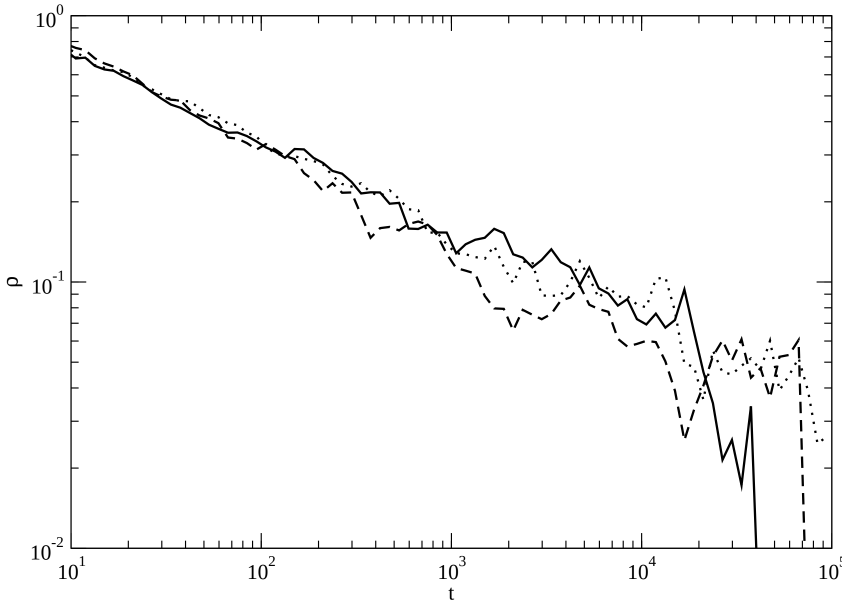


Fig. 3. The ordering process in the SG models for three different simulations. The data are for fractals of level 8. ρ is the fraction of links that are interfaces.

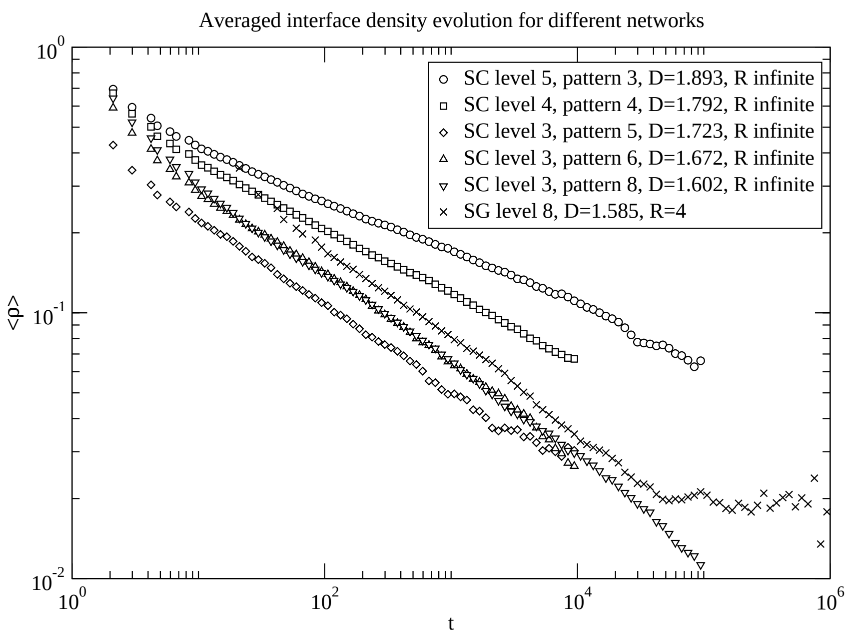
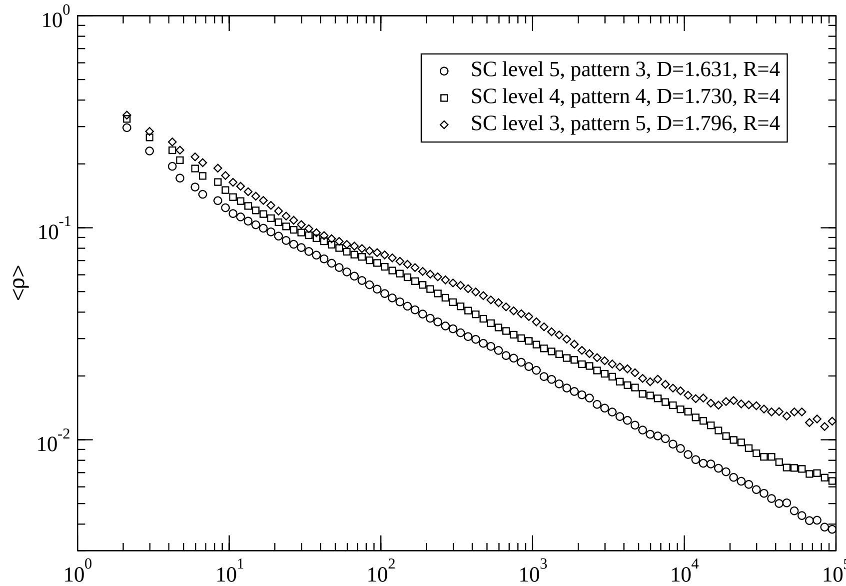


Fig. 4. The ordering process after averaging over simulations that did not order completely. The data are for SC networks with infinite ramification and for SG networks. All the data are averaged over 100 network simulations and log-binned. The exponents α_{exp} are obtained from the slopes. The SG simulation data has been taken with 10 time steps intervals, thus the data starts later than for SC. The plateau in SG simulations is a combined effect of finite system size and our method of averaging only over active simulations. $\langle\rho\rangle$ is the mean fraction of links that are interfaces. "Pattern" is the linear size of the pattern used in carpet creation.



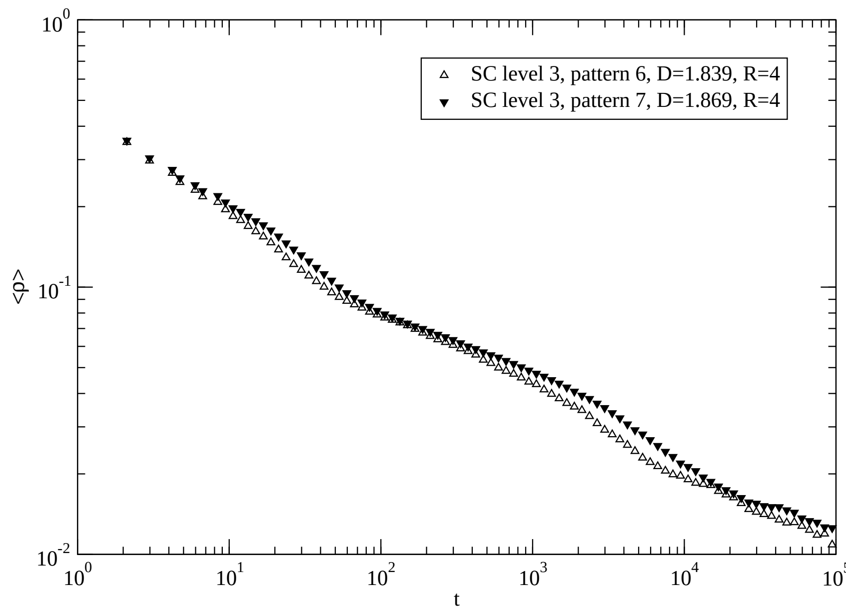


Fig. 5. The ordering process after averaging over simulations that did not order completely. The data are for SC networks with finite ramification and averaged over 100 network simulations and log-binned. The exponents α_{exp} are obtained from the slopes.

with various fractal dimensions between 1 and 2 and both with finite and infinite ramification.

The dynamics of SC networks with finite and infinite ramification clearly shows different behavior (Fig.4,Fig.5). For finite ramification networks oscillations are present around the general power-law trend. The oscillations are exponential decay periods, each with different time scale. This can be explained as a

type	ramification	level	pattern	dimension	α_{theory}	α_{exp}
SC	infinite	5	3	1.8928	0.0536	0.1908 ± 0.0007
SC	infinite	4	4	1.7925	0.1038	0.2484 ± 0.0007
SC	infinite	3	5	1.7227	0.1387	0.3136 ± 0.0029
SC	infinite	3	6	1.6720	0.1640	0.3362 ± 0.0028
SC	infinite	3	8	1.6025	0.1988	0.3339 ± 0.0017
SC	4	5	3	1.6309	0.1845	0.3763 ± 0.0008
SC	4	4	4	1.7297	0.1351	0.3393 ± 0.0015
SC	4	3	5	1.7959	0.1021	0.3246 ± 0.0029
SC	4	3	6	1.8394	0.0803	0.3087 ± 0.0038
SC	4	3	7	1.8693	0.0653	0.2947 ± 0.0046
SG	4	9	-	1.5850	0.2075	0.3456 ± 0.0034
regular	infinite	-	-	1.0000	0.5000	0.4973 ± 0.0006

Table 1
Theoretical and experimental exponents α (see Eq.1) for ordering processes in various networks. The results are averaged over 100 individual simulations. The levels of fractals were maximized while keeping a number of nodes that allowed the actual simulations to be completed in a reasonable amount of time. The regular network was a simple 1-dimensional chain with $\langle k \rangle = 4$ (first and second nearest neighbors connected) and periodic boundary conditions. The pattern column shows the linear size of the pattern used in carpet creation.

complete ordering of weakly connected modules of certain sizes. The modules are weakly connected due to finite ramification, and the complete ordering occurs because of random fluctuations. When nearly all modules of a given size are ordered, the ordering at the next hierarchical level starts, with a longer time scale. The ordering due to random fluctuations produces an exponential decay of average number of interfaces, and repeating the process in following scales produces the power-law overall behavior. The effect is less visible for smaller pattern sizes, since modules in such case are smaller and connected relatively stronger. Additionally oscillations have shorter period, making them even less visible.

We have measured the exponent α of the power-law for all investigated lattices, and compare it to the theoretical value [8] (Eq.1). It is worth to remind that the theoretical values are only reference points, and were not originally calculated for fractal structures.

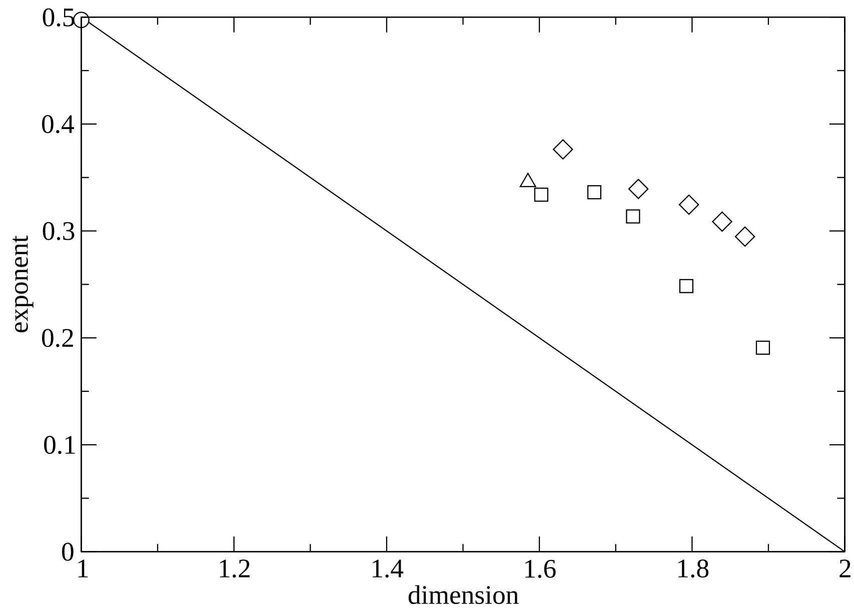


Fig. 6. Theoretical and experimental exponents for the ordering processes in function of fractal dimension. The line is the analytic formula (Eq.1), the squares are exponents for SC networks with infinite ramification, the triangle is exponent for SG network, the diamonds are exponents for SC networks with finite ramification, the circle in the left upper corner is the exponent for the one-dimensional network. Error bars are smaller than symbol sizes. Exact values and additional data can be found in Table 1.

We also measured the dependence of the ordering exponents on the fractal size. Simulations for SC fractals of different sizes has been performed in finite and infinite ramification cases.

The ordering exponent is clearly dependant on the fractal size, at least for the fractals with infinite ramification (Fig.7). The lack of such a strong dependence for finite ramification fractals shows that the dynamics in these two cases are different. Additionally, one can see that for larger fractals, the ordering exponents tend to close to the analytical formulas (Eq.1), especially for infinite ramification networks. We do not know what are the infinite size limits of the ordering exponents for these fractals, and whether they would converge to analytical predictions or not.

4 Conclusions

The results we have obtained show that the ordering of voter model in fractals is described by a power law, similarly to regular networks below critical dimension $D = 2$. The dynamics is different though, and analytic calculations

type	ramification	level	pattern	size	dimension	α_{exp}
SC	infinite	2	3	64	1.8928	0.3094 ± 0.0077
SC	infinite	3	3	512	1.8928	0.2755 ± 0.0017
SC	infinite	4	3	4096	1.8928	0.2263 ± 0.0014
SC	infinite	5	3	32768	1.8928	0.1908 ± 0.0007
SC	4	2	3	36	1.6309	0.2563 ± 0.0075
SC	4	3	3	216	1.6309	0.4055 ± 0.0028
SC	4	4	3	1296	1.6309	0.3909 ± 0.0033
SC	4	5	3	7776	1.6309	0.3763 ± 0.0008
SC	4	6	3	46656	1.6309	0.3717 ± 0.0016

Table 2
Measured exponents α for ordering processes in fractals of various size. The results are averaged over 100 individual simulations, except for level 6 fractal data that was averaged only over 10 individual simulations. The pattern column shows the linear size of the pattern used in carpet creation.

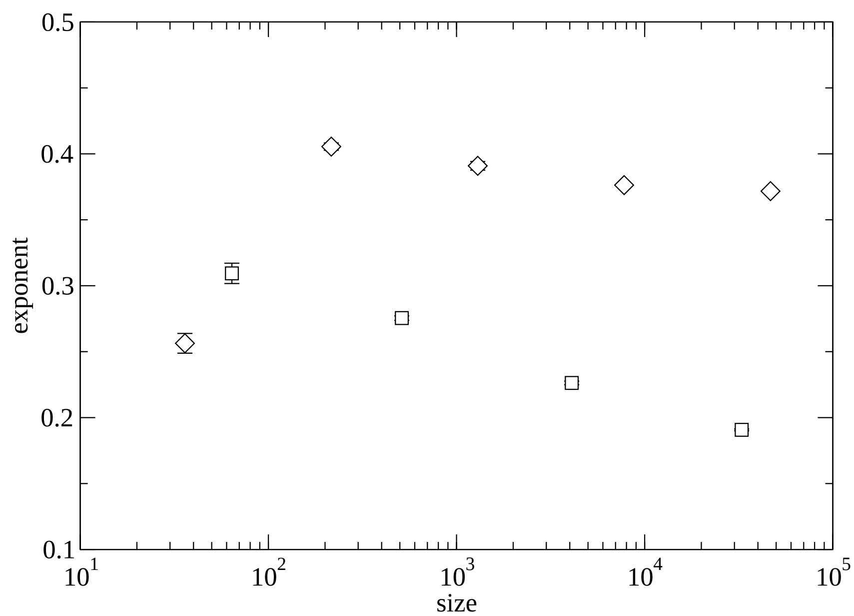


Fig. 7. The exponents for the ordering processes in SC fractals of various sizes. The squares are for infinite ramification and $D \approx 1.8928$, the diamonds are for finite ramification and $D \approx 1.6309$. Exact values and additional data can be found in Table 2.

[8] (Eq.1) for regular networks do not give correct exponents if simply applied to fractal lattices (Fig.6).
The ordering process strongly depends whether the fractal has finite or infinite ramification. For fractals with infinite ramification, the ordering is similar to regular lattice and is determined by recurrence properties of the random walk [7]. In finite ramification fractals the dynamics are different and the ordering is driven by complete ordering of weakly connected modules in following scales. Additionally the results further confirm the known fact, that for fractal structures the dimension alone does not determine dynamics in such systems [3,6].

This work was partially supported by a EU Grant *Measuring and Modelling Complex Networks Across Domains* (MMCOMNET) and by State Committee for Scientific Research in Poland (Grant No. 1P03B04727).

References

[1] A. Aleksiejuk, J.A. Hołyst, D. Stauffer, Physica A 310 (2002), 260-266.

[2] G. Bianconi, Phys. Lett. A 303 (2002), 166-168.

[3] Y. Gefen, B.B. Mandelbrot, A. Aharony, Phys. Rev. Lett. 45 (1980), 855.

[4] J.M. Carmona, U.M.B. Marconi, J.J. Ruiz-Lorenzo, A. Tarancón, Phys. Rev. B 58 (1998), 14387.

[5] T. Stošic, B.D. Stošic, S. Milošević, H.E. Stanley, Physica A 233 (1996), 31-38.

[6] G. Kohring, Phys. Rev. B 33 (1986), 610.

[7] I. Dornic, H. Chaté, J. Chave, H. Hinrichsen, Phys. Rev. Lett 87 (2001), 045701.

[8] L. Frachenbourg, P.L. Krapivsky, Phys. Rev. E 53 (1996), 3009.