Bistable-monostable transition in the Ising model on two connected complex networks

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In this paper, we investigate the behavior of the Ising model on two sparsely connected complex networks. The networks have the topology of a random graph or Barabási and Albert scale-free networks. We extend our previous analysis and show that a bistable-monostable phase transition occurs in such systems. During this transition, the magnetization undergoes a discontinuous jump. We calculate the critical temperature analytically for regular random graphs and study a more general case using an iterative map corresponding to mean-field dynamics. The calculations are confirmed by numeric simulations based on Monte Carlo approach.

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I. INTRODUCTION

Along with the emergence of complex networks research which started with the breakthrough paper by Barabási and Albert [1] came studies of the Ising model in such systems [2–6]. Many aspects of the model have been studied, from simple antiferromagnetic interactions [7,8] and spin-glasses [9,10] to the directed structure of the network [11]. Recent research also involves investigations of spin behavior on Cayley tree structures [12] and on modular, hierarchical, and fractal networks [13]. In the present paper, we investigate the Ising model where Ising spins are placed at nodes of two interconnected complex networks. Such a topology is a special limit of modular systems that are frequently studied, e.g., as models of community structures [14–16]. In fact, modular networks are abundant around us. Social groups that consist of smaller tight groups, companies clustered as larger corporations, groups of strongly interdependent species influencing each other, or even the twin hemispheres of the brain seen as a neural network are examples of such systems. Our subject and approach is mostly relevant to social systems, where social influence is often similar to the majority dynamics of the Ising model. Our aim is not to attempt to build a model of such systems since they are far too diverse and complex to encompass within a single model; rather we aim to explore this simplified case as a possible reference for studying more complex models.

In our previous work [17] we investigated the Ising model on a pair of connected networks. Dynamical phenomena for such a topology have been studied for other majority models in [18]. Our recent research concerning the Ising model indicates that there is a bistability for low temperatures in such a system (after we disregard the two states which are symmetric through reversal of all spins). As a result, a transition between bistable and monostable phases occurs at a critical temperature. Here we show that a discontinuous jump can occur during the bistable-monostable transition. Our analytical calculations in the present paper are backed up by numerical simulations.

II. MODEL

In our study, we consider two interconnected complex networks, where at each node we place an Ising spin and where the interactions between the spins are ferromagnetic only.

The analytic part is based on mean-field (MF) approximation and initially takes no assumptions toward network structure other than it being random and without correlations. Further on, detailed MF investigations are limited to two cases: regular random graphs and Barabási-Albert (B-A) networks. Regular random graphs are networks that possess random connections, but with all nodes having exactly the same degree k. As a representative of random networks without a constant degree, we have used Barabási-Albert network. The B-A model is a model of a growing network [1] and possesses a scale-free topology with a degree distribution of $P(k) \sim k^{-3}$.

Our two networks are interconnected by E_{AB} links (Fig. 1). Each of these links connects a node in network A with a node in network B. The nodes to be connected are chosen preferentially, i.e., the probability to pick a given node i equals $\Pi_{Ai}=k_{AAi}/\Sigma_j k_{AAj}$. If we perform linking in this way, the internetwork degree k_{ABi} of a node is statistically proportional its to intranetwork degree k_{AAi}. It is worthwhile noting that despite preferential choices of nodes on both ends of a

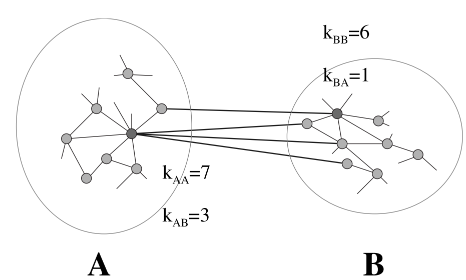

FIG. 1. Two connected B-A networks. A few nodes from each network are shown. The intranetwork degrees k_{AA} and k_{BB} and internetwork degrees k_{AB} and k_{BB} for two sample nodes are presented.

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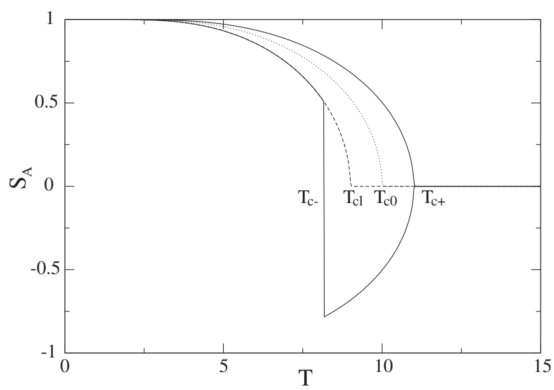

FIG. 2. Plots of mean-field $S_A(T)$ for $N_A = N_B = 5000$ and $k = \text{const} = 10$ (regular random graphs). The dotted line is for $p=0$ (unconnected networks, ferromagnetic-paramagnetic transition at T_{c0}) and the rest are for $p=0.1$. The solid lines correspond to an actual model with a discontinuous transition at T_{c-}. The dashed line corresponds to an enforced second-order phase transition at T_{cl}. See Sec. II for a detailed explanation.

link, the resulting connections are not correlated. The distribution of degrees on one end of the link does not depend on the degree of the actual node on the other. We consider only the cases where internetwork connections are sparse, i.e., the number of links between different networks is smaller than links inside networks.

Let us consider a *single* complex network, in the form of the B-A model where the node degree distribution possesses a scale-free form

$$P(k) = 2m^2 k^{-3} \tag{1}$$

for $k \geq m$, where N is the number of nodes and the parameter m is dependent on the mean node degree, $\langle k \rangle = 2m$ [1]. If we consider Ising interactions between spins placed in nodes of such a network, then we observe [2] that a spontaneous magnetization in such systems exists for temperatures $T < T_{c0}$. Assuming the degree of the largest hub is $m\sqrt{N}$ (mean value), this temperature can be calculated [3] as

$$T_{c0} = \frac{mJ}{2} \log(N). \tag{2}$$

The temperature will be measured in units of Boltzmann constant (k_B). The characteristic temperature T_{c0} can be considered the temperature during the ferromagnetic-paramagnetic phase transition. Since the number of nodes N is finite, no true phase transition is possible here and the sign of the order parameter can be reversed by a *very* large fluctuation. A remarkable feature of the solution [Eq. (2)] is its nontrivial dependence on the systems size N, which is a direct effect of scale-free node degree distribution [Eq. (1)] and consequent existence of highly connected nodes (hubs) in the network.

The problem of the Ising model on *coupled* random networks was considered in [17]. Let us use Fig. 2 as an illustration and explain the general behavior of such a system. The figure shows how the average magnetization of one regular random graph (out of the interconnected two) depends on the temperature. We use a regular random graph for the sake of picture clarity. In the case of B-A networks, the shape of the plots would be different making it less readable. However, the interesting points would remain the same.

The plots in Fig. 2 show four different situations and the lines are obtained through numerical MF map iterations (see Sec. IV). The first case (dotted line) is that of unconnected graphs, and since it shows magnetization of one subnetwork, it is in fact a single network. For low temperatures, the spins are ordered, while for high temperatures they are disordered. There is a typical ferromagnetic-paramagnetic phase transition occurring at T_{c0}.

The second case (solid lines) illustrates connected networks. In low temperatures, the system is ferromagnetic and bistable. The stable states correspond to ordering with both magnetizations parallel or antiparallel. In this temperature range, there are two solid lines in Fig. 2. At certain temperature T_{c-}, the antiparallel state ceases to be stable and the system undergoes a transition from the bistable to the monostable phase. If the actual system state is antiparallel, then there is a discontinuous jump of the magnetization, clearly visible at the figure. Above temperature T_{c-}, but below T_{c+}, the system is ferromagnetic and monostable (both solid lines overlap for $T > T_{c-}$). When the temperature increases further, the system undergoes another transition—this time a typical second-order ferromagnetic-paramagnetic transition at temperature T_{c+}. This temperature is always higher than T_{c0} because subnetworks with the same magnetization signs reinforce each other's internal orders, resulting the need for in higher temperature to dissolve them.

The last plot (dashed line) in Fig. 2 is the case of a specially modified model, where the spin of one network is enforced to be opposite to that of the second [more precisely weighted spins of subnetworks are opposite: $S_B = -S_A$, see Eq. (7) for definition of weighted spin]. This emulates antiparallelly ordered networks in the bistable phase, but forces the system to undergo a second-order phase transition at T_{cl} instead of discontinuous transition at T_{c-}. To understand why this case was included in the figure, we have to explain the MF approach used to describe the system of coupled networks. The MF analytic approach can be found in Sec. III. If one assumes that the magnetizations are very small, one can use a linear approximation of the hyperbolic tangents present in the equations. Such an approach leads to equations that are relatively easy to solve [17] and produces two critical temperatures—one corresponding to the bistable-monostable transition and the second to the ferromagnetic-paramagnetic transition. However, the assumption of small magnetizations is equivalent to assuming a second-order phase transition. In this paper, we show that such an approach is inadequate for the lower-temperature transition. The nonlinearized equations show that this transition is discontinuous. We refer to this approach as *linear approximation* and note the resulting critical temperature of the bistable-monostable transition as T_{cl}. The dashed line in Fig. 2 corresponds to the forced relation $S_B = -S_A$, and consequently is the realization of a system that is described accurately by linear approximation.

III. ANALYTIC APPROACH

We use a MF approach to solve the problem of the Ising model. The self-consistent equation for the average spin is as follows:

$$\langle s_i\rangle = \tanh\left(\beta\sum_j J_{ij}\langle s_j\rangle + \beta h_i\right). \tag{3}$$

The above equation can be rewritten as follows:

$$\langle s_i\rangle = \tanh\left[\beta J\sum_j\left(\frac{k_ik_j}{E}\langle s_j\rangle\right) + \beta h_i\right], \tag{4}$$

where $\beta=1/T$, the temperature T is measured in units of inverse Boltzmann constant $(1/k_B)$, averaging is over the canonical ensemble, h_i is the external field acting on node i, and E is the total number of links in the network. The factor J_{ij} was replaced by Jk_ik_j/E. This substitution works for all network topologies that are random, without correlations between connections.

If we consider two networks that interact, we can treat the influence of the second network as an external field h_i in MF approximation. Since the number of internetwork links is proportional to number of intranetwork links, we can write the full set of equations for the two networks as follows:

$$\langle s_{Ai}\rangle = \tanh\left[\beta J_{AA}\sum_j\left(\frac{k_{Ai}k_{Aj}}{E_A}\langle s_{Aj}\rangle\right) + \beta J_{BA}\sum_l\frac{k_{ABi}k_{BAl}}{E_{BA}}\langle s_{Bl}\rangle\right], \tag{5}$$

$$\langle s_{Bi}\rangle = \tanh\left[\beta J_{BB}\sum_j\left(\frac{k_{Bi}k_{Bj}}{E_B}\langle s_{Bj}\rangle\right) + \beta J_{AB}\sum_l\frac{k_{BAi}k_{ABl}}{E_{AB}}\langle s_{Al}\rangle\right]. \tag{6}$$

We introduce the following weighted average spin:

$$S = (1/E)\sum_i k_is_i. \tag{7}$$

It is an order parameter for the Ising model on a random network with nonhomogeneous degree distribution. Additionally, we use the substitution $k_{ABi}=p_Ak_{Ai}$, $k_{BAi}=p_Bk_{Bi}$ using the fact that internetwork degrees are proportional to intranetwork degrees (see Sec. II). As a result, we obtain following equations for the weighted average spins:

$$S_A = \sum_i\frac{k_{Ai}}{E_A}\tanh(\beta J_{AA}k_{Ai}S_A + \beta J_{BA}p_Ak_{Ai}S_B), \tag{8}$$

$$S_B = \sum_i\frac{k_{Bi}}{E_B}\tanh(\beta J_{BB}k_{Bi}S_B + \beta J_{AB}p_Bk_{Bi}S_A). \tag{9}$$

The parameters p_A and p_B, which we introduced, can be interpreted as relative internetwork link densities. When $p_A=p_B=0$ the networks are not connected. When p_A or p_B reach 1, the density of links between networks approaches the density of links within networks, so the subnetworks are no longer distinguishable. It is worth noting that p_A and p_B are not independent; indeed a following relation exists between

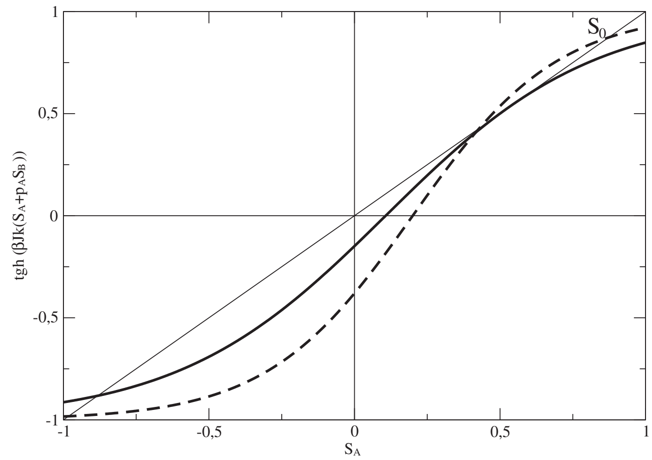

FIG. 3. Hyperbolic tangent plot. The dashed line is Eq. (8) for $T<T_{c-}$ when Eq. (8) has three solutions and two of them are stable. The solid line is for $T=T_{c-}$, when one of the two stable solutions of Eq. (8) merges with the unstable solution resulting in a solution is not stable. The thin line is a diagonal $y=x$. The upper-right intersection of the dashed plot with the diagonal is point S_0 [see the discussion below Eq. (9)].

them and the number of internetwork links: $E_{AB}=E_Ap_A=E_Bp_B$.

We shall show that in the considered system, the bistable-monostable transition can occur with a discontinuous magnetization change.

Until now, our analysis was general, describing any random network without correlations. Now let us limit our investigation to a pair of random networks of the same size, the same link density, and k=const (regular random graph). The last condition is a serious simplification, but we will later show that the behavior of the original nonsimplified system is similar. As an additional simplification, let us assume that $J_{AA}=J_{BB}=J_{AB}=J_{BA}=J$.

It follows that the first of our two equations [Eq. (8)] becomes

$$S_A = \frac{k_A}{E_A}\tanh(\beta Jk_AS_A + \beta Jp_Ak_AS_B). \tag{10}$$

The right side of the Eq. (10) is a hyperbolic tangent, shifted by the value $h=Jk_Ap_AS_B$ along the x axis. When the temperature T is low, value of β is high and the equation has three solutions (dashed line in Fig. 3). The situation is similar to the situation without the shift, except the value of solution S_0 is slightly different. Let us now consider what happens when the temperature increases and approaches the critical value of T_{c-}. The hyperbolic tangent gradually becomes flatter. This decreases the value of S_0. It is important to note that the second network, which is actually responsible for shift h, undergoes similar changes. Thus, the value of the shift decreases as the temperature increases. It turns out, however, that the shift does not go to 0. At certain temperature T_{c-}, the hyperbolic tangent becomes tangential to the $y=x$ line (solid line in Fig. 3). This is the unstable point, where any fluctuation of S_A or S_B can cause the system to leave the antiparallel state and order parallelly. Up to this point, the antiparallel state was symmetric with regard to magnetization values (because the networks have the same parameters) and therefore $S_B = -S_A$. Above this point, such an assumption is no longer valid, but can be still used to find the critical point.

At T_{c-}, the tangent is tangential to the $y=x$ line. We can write the following conditions for β_{c-} and S_{Ac-}:

$$\frac{\tanh(\beta_{c-}Jk_AS_A + \beta_{c-}Jk_Ap_AS_B)}{S_A} = 1, \qquad (11)$$

$$\frac{\partial \tanh(\beta_{c-}Jk_AS_A + \beta_{c-}Jk_Ap_AS_B)}{\partial S_A} = 1. \qquad (12)$$

After considering the differential, we applied the condition $S_B = -S_A$ and calculated the values of β_{c-} and S_{Ac-}. We obtained the following equations:

$$S_{Ac-} = \frac{\ln(\sqrt{\beta_{c-}Jk_A} + \sqrt{\beta_{c-}Jk_A - 1})}{\beta_{c-}Jk_A(1-p_A)}, \qquad (13)$$

$$\beta_{c-} = \frac{\ln[(1+S_{Ac-})/(1-S_{Ac-})]}{2Jk_A(1-p_A)S_{Ac-}}. \qquad (14)$$

This set of equations determines the critical point (T_{c-}, S_{Ac-}) for the transition between the bistable and monostable phases. At this point, the antiparallel state ceases to be stable.

If we multiply Eq. (13) by β_{c-} and Eq. (14) by S_{Ac-}, we can compare the right sides of both equations and obtain the following relation:

$$S_{Ac-} = \frac{\beta_{c-}Jk_A + \sqrt{\beta_{c-}Jk_A(\beta_{c-}Jk_A - 1)} - 1}{\beta_{c-}Jk_A + \sqrt{\beta_{c-}Jk_A(\beta_{c-}Jk_A - 1)}}. \qquad (15)$$

Comparing this with Eq. (13) results in a single implicit equation for β_{c-} and p_A, which can be simplified as follows:

$$p_A = 1 - \frac{1+\sqrt{1 - 1/(\beta_{c-}Jk_A)}}{\beta_{c-}Jk_A + \sqrt{\beta_{c-}Jk_A(\beta_{c-}Jk_A - 1)} - 1} \cdot \ln(\sqrt{\beta_{c-}Jk_A} + \sqrt{\beta_{c-}Jk_A - 1}). \qquad (16)$$

Drawing $p_A(\beta_{c-})$ and changing axes yield a dependence of T_{c-} on parameter p_A (see Fig. 4).

We can also approximate the behavior of the solution for small p_A. Our conditions [Eq. (11) and (12)] can be written in the following manner:

$$\tanh[\beta_{c-}Jk_A(1-p_A)S_{Ac-}] = S_{Ac-}, \qquad (17)$$

$$\cosh^2[\beta_{c-}Jk_A(1-p_A)S_{Ac-}] = \beta_{c-}Jk_A. \qquad (18)$$

If we multiply the equations sidewise, we obtain a single equation for a multiple $X = \beta_{c-}Jk_AS_{Ac-}$,

$$\sinh[2(1-p_A)X] = X. \qquad (19)$$

We know that for very small p_A the value of S_{Ac-} is also very small. Consequently we can use Taylor expansion of hyperbolic sinus around zero to obtain the following approximation:

$$2(1-p_A)X + [2(1-p_A)X]^3/6 \approx 2X. \qquad (20)$$

As a result, we get an approximate value of X,

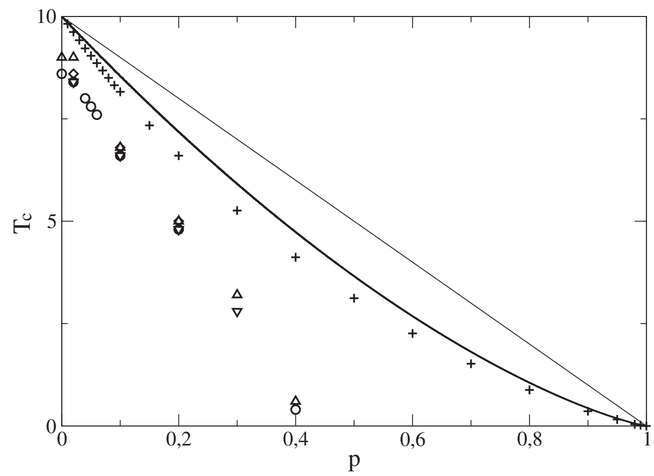

FIG. 4. Dependence of critical temperature T_{c-} on parameter p for two regular random graphs with $k=10$. The thin straight line is $T_{cl}(p)$, which is calculated by using linear approximation (see Sec. II). The solid curved line is an analytical prediction of T_{c-} [Eq. (16)]. The plus symbols are map iterations (see Sec. IV), while circles, triangles, and diamonds are results of numerical Monte Carlo simulations (see Sec. V). Circles are for $\tau=100$, triangles pointing upward are for $\tau=30$, and triangles pointing downward are for $\tau=200$. Diamonds are for $\tau=100$, but for network sizes $N=N_A=N_B=50\,000$.

$$X \approx \sqrt{\frac{3}{2}p_A}. \qquad (21)$$

Putting the result into the Eq. (18), we obtain the following equation:

$$\cosh^2[(1-p_A)\sqrt{(3/2)p_A}] = \beta_{c-}Jk_A. \qquad (22)$$

Approximating $\cosh^2 x = 1 + x^2$, we finally obtain $T_{c-} \approx k_A[1-(3/2)p_A]$. So far, we have concentrated on coupled regular random graphs, where the node degree k is constant. We also have assumed both subnetworks are of the same size. Without such simplifications, the equations are very hard to solve analytically. We have studied more complex cases using map iterations and Monte Carlo simulations.

IV. MAP ITERATIONS

In the case of connected networks possessing nontrivial degree distributions that are different from regular random graphs, the problem of the exact temperature of the transition could not be solved analytically and we had to use numerical methods.

We consider a two-dimensional map,

$$S_A^{(t+1)}(T) = \sum_{k_A} P(k_A)\frac{k_A}{E_A}\tanh[\beta J_{AA}k_AS_A^{(t)}(T) + \beta J_{BA}p_Ak_AS_B^{(t)}(T)], \qquad (23)$$

$$S_B^{(t+1)}(T) = \sum_{k_B} P(k_B)\frac{k_B}{E_B}\tanh[\beta J_{BB}k_B S_B^{(t)}(T) + \beta J_{AB}p_B k_B S_A^{(t)}(T)], \tag{24}$$

where the $S_A^{(t)}(T)$ and $S_B^{(t)}(T)$ are time-dependent variables and the rest are constant parameters, including given degree distributions $P(k_A)$ and $P(k_B)$. Due to our definition of weighted spin [Eq. (7)], S_A and S_B are in the range of $[-1,1]$. We assume $J_{AA}=J_{BB}=J_{AB}=J_{BA}=J$, so we can omit these constants in the equations and have $\beta=1/T$.

We investigate the dependence of stable state spins $[S_A(T),S_B(T)]$ on the temperature, assuming the antiparallel initial conditions $S_A^{(t=0)}(T)=1$ and $S_B^{(t=0)}(T)=-1$. Since the system is symmetric, below T_{c-} we have $S^{(t)}(T)=S_A^{(t)}(T)=-S_B^{(t)}(T)$. At T_{c-}, the system cannot remain in the antiparallel state and jumps to the parallel ordering. In our map iterations, the system always ordered with the negative spin values. This consistency is obvious, considering the deterministic nature of the map iterations. It is worth mentioning that since the system can leave the unstable state in two ways (network A changing to negative or B to positive) and both ways are mathematically equivalent, the final result cannot be predicted by equations. The final result depends on how the computer calculates and represents numbers.

By observing $S(T)$, we can find the critical temperature T_{c-}, where the antiparallel state disappears and a jump between positive and negative spin values occurs (see Fig. 2). We define our stable spin values as follows: $S_A(T)=S_A^{(tm)}(T)$ and $S_B(T)=S_B^{(tm)}(T)$, where the time $tm=1000$ is the number of iterations of the map that have been performed before we assumed it reached a stationary solution.

We investigated various T ranges, usually around critical temperature T_{c-}, with a temperature step $\Delta T=0.2$. Our networks were sized $N_A=N_B=5000$ and possessed either a power-law degree distribution $P(k_A)=P(k_B)$, taken from a B-A network growth simulation, or a constant degree $k=\text{const}$ for testing the analytical equations. Since the networks were the same, so $p_A=p_B=p$.

Figure 4 presents values of T_{c-} for coupled networks with constant $k=10$. As can be seen, the map iterations do not exactly agree with the analytical equations. This is probably due to the limited accuracy of the numerical calculations; indeed, near the critical point such limited accuracy can play a crucial role. Numerical noise can tip the system over the edge into the parallel state, reducing the temperature at which the jump takes place and therefore decreasing observed critical temperature value.

Figure 5 presents results for coupled B-A networks. It is evident that the linear approximation (see Sec. II) leads to incorrect results. For small p, the first-order phase-transition critical temperature is linearly dependent on parameter p, but with a different factor than is predicted by linear approximation. For a higher internetwork connection number, the dependence is not linear.

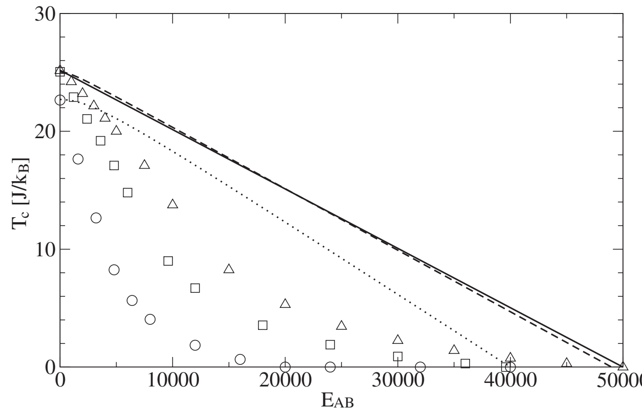

FIG. 5. Dependence of critical temperature T_{c-} on the number of internetwork connections E_{AB} for two B-A networks. Lines are analytic predictions using linear approximation (T_{cl}, see Sec. II), while symbols are critical temperatures obtained from map iterations (T_{c-}, see Sec. IV). The solid line and triangles correspond to $N_A=N_B=5000$, the dashed line and squares correspond to $N_A=6000$ and $N_B=4000$, and the dotted line and circles correspond to $N_A=8000$ and $N_B=2000$.

V. MONTE CARLO SIMULATIONS

We performed Monte Carlo simulations of the Ising model on two interconnected B-A networks and regular random graphs, where $N_A=N_B=5000$ and $\langle k_A\rangle=\langle k_B\rangle=10$.

The simulation for each temperature T is independent. It starts from an antiparallel ordered system ($S_A=-S_B=1$). The system then performs $\tau=100$ time steps of Monte Carlo dynamics. After that, we measure the averages of $|(S_A+S_B)/2|$ and $|S_A|$ or S_A and $|S_A|$ during another $\tau=100$ time steps. One time step equals N_A+N_B random single node updates, which is on average one update per node. We chose time τ so that the network has enough time to relax to the equilibrium state, but not enough time for the temperature noise to switch the system between both states in the bistable phase. We selected $\tau=100$ since there were no large differences in the results of our numerical experiments for larger τ values.

The simulation for given parameters was repeated 100 times for each case. Consequently the results have statistical meaning. The values S_A, $|S_A|$, and $|(S_A+S_B)|$ are averages over both time and multiple simulations (i.e., $|S_A| \equiv \langle\langle|S_A|\rangle_{\text{time}}\rangle_{\text{simulations}}$).

The method of finding the critical temperature T_{c-} relies on measuring two values, which behave in significantly different ways in bistable and monostable phases, thus allowing us to find the point where the transition between these phases occurs. Let us look at Fig. 6 as we explain our method in detail. In this case we measured and plotted $|(S_A+S_B)/2|$ and $|S_A|$. Since the figure shows systems for four different interconnection densities, let us focus on just one—the dotted lines that correspond to $p=0.2$. There are two such lines because one corresponds to $|(S_A+S_B)/2|$ and the second to $|S_A|$.

Since we start from antiparallel ordering, the system persists in the antiparallel state in the bistable phase. In such a situation, $|(S_A+S_B)/2|$ is close to zero, as both networks are the same size and have equal weighted spin S values but of opposite signs. It should be noted that this value is not exactly zero because of fluctuations. Since we measure the absolute value, those fluctuations do not cancel each other but add up, resulting in a small nonzero total value. On the other

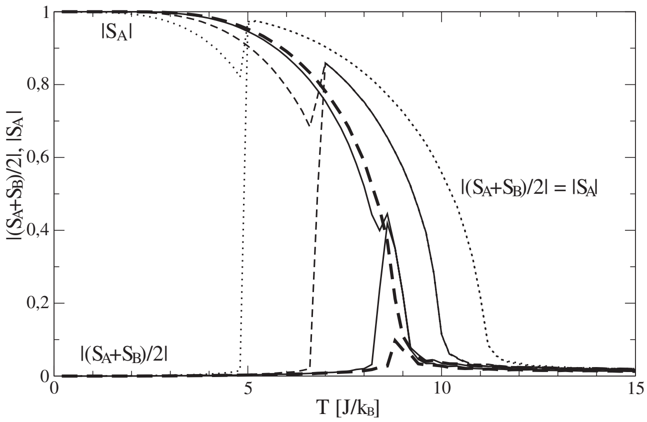

FIG. 6. Dependence of $|(S_A+S_B)/2|$ and $|S_A|$ on temperature T for Monte Carlo simulations of two regular random graphs with $N_A=N_B=5000$ and $\langle k_A\rangle=\langle k_B\rangle=10$. The thick dashed lines are for $p=0$, the solid lines are for $p=0.02$, the thin dashed lines are for $p=0.1$, and the dotted lines are for $p=0.2$. The simulations start from antiparallel ordering ($S_A=-S_B=1$). See Sec. V for a discussion. The labels in the figure refer to the dotted lines.

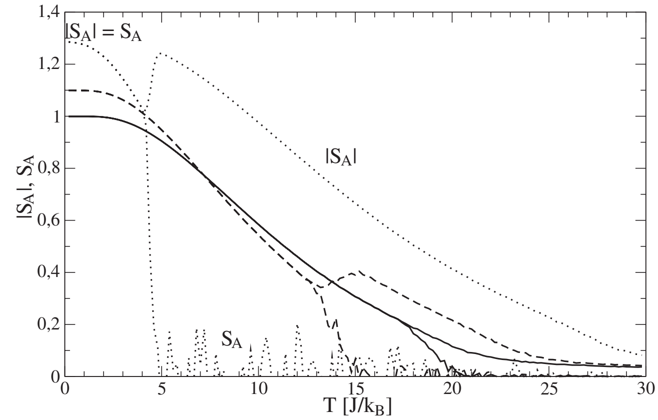

FIG. 7. Dependence of the weighted spin of single subnetwork S_A and its absolute value $|S_A|$ on temperature T for two B-A networks with $N_A=N_B=5000$ and $\langle k_A\rangle=\langle k_B\rangle=10$. The solid lines are for $p=0$ (unconnected networks), the dashed lines are for $p=0.1$ and the dotted lines are for $p=0.3$. See Sec. V for a discussion. The labels in the figure refer to the dotted lines.

hand, $|S_A|$ is quite high because the network is ordered.

Above T_{c-}, in the monostable phase, the antiparallel state is no longer stable. Consequently the system always switches from an initial antiparallel state to a parallel state. Because of the symmetry there is no way to predict whether the system will order parallelly with positive or negative magnetizations. As we measured the absolute values, it does not matter to us at all. What matters is that both networks always have the same magnetizations ($S_A=S_B$) so $|(S_A+S_B)/2|$ has the same value as $|S_A|$ and plots for both overlap.

During the transition at T_{c-}, the value of $|(S_A+S_B)/2|$ changes dramatically, from a near-zero value to a value that is close to 1. This is because in an antiparallel state below T_{c-}, the magnetizations of both subnetworks cancel out, while in a state above they add up. The value of $|S_A|$ increases because the magnetization value for parallel ordered system is higher than that of the antiparallel ordered system in the same temperature.

At higher temperatures T, another transition occurs. This time there is a transition between monostable ferromagnetic and paramagnetic phases. It is a second-order phase transition, as can be clearly seen from both $|(S_A+S_B)|$ and $|S_A|$ approaching continuously to zero. We do not investigate this transition in our paper, as such studies have been conducted before [17].

For other cases in Fig. 6—dashed and solid lines—the values of $|(S_A+S_B)/2|$ and $|S_A|$ behave in the same way. The thick dashed line is a special case of unconnected networks. In such a case, temperatures T_{c-} and T_{c+} coincide and there is no monostable phase. The rise of $|(S_A+S_B)/2|$ is, in this case, caused purely by the large fluctuations. In close vicinity of ferromagnetic-paramagnetic phase transition, the size of fluctuations increases to encompass the whole system, causing subnetworks to flip randomly to opposite magnetization. This causes a significant increase in $|(S_A+S_B)/2|$.

We have investigated the dependence of T_{c-} on the number of internetwork links $E_{AB}\sim p$ using the method described above. First, we have taken the case of $k=\text{const}$ to test how the simulations fit with the analytic results and map iterations. The results (Fig. 4) indicate that critical temperature T_{c-} is different than predicted analytically but the deviation is not large. The fact that the temperatures drop to zero at around $p=0.5$ not around $p=1$ shows that MF method does not describe the dynamics of the system accurately. For such strong internetwork interactions, the MF approximation is poor. Aggregation of individual spins into the averages does not work well when the interactions between individual spins belonging to different groups become comparable to the internal interactions within the groups.

Our main results concern the B-A networks. We have, however, encountered a difficulty in relation to our methodology. For small p values, the plot of $|(S_A+S_B)/2|$ does not distinguish clearly between bistable and monostable phases and the jump in $|S_A|$ is also small. Because of this, we have modified our methodology somewhat. Instead of observing $|(S_A+S_B)/2|$ and $|S_A|$, we observed S_A and $|S_A|$ (see Fig. 7). In the bistable phase, the system preserved the initial antiparallel ordering. Since the initial spin was fixed, the S_A values averaged over several realizations all added up, giving the same averaged value for S_A and $|S_A|$. This means that for bistable phase $S_A=|S_A|$. In the monostable phase, half of the time the observed network flipped to opposite spin. Thus half of the time S_A was negative and half of the time it was positive. This caused S_A to go to zero and fluctuate around it (the measured value is signed, so the fluctuations do not add up). The point where the plots of S_A and $|S_A|$ split is the point where the system starts going to the parallel state, i.e., point where monostable phase begins. The critical temperature was defined as temperature T at the point where plots of S_A and $|S_A|$ split—this point coincides with the minimum value of $|S_A|$ (the same measure as before) in cases where the minimum is visible (for large p). As with the previous figure, this occurs for all lines corresponding to connected networks (dotted and dashed). The solid line is a special case of unconnected networks. In this case, the split is caused not by

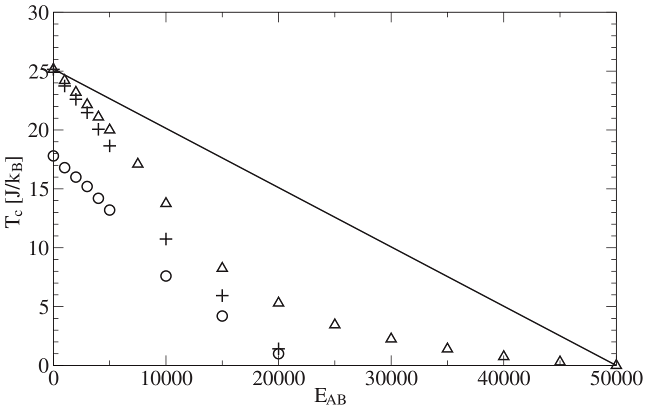

FIG. 8. Dependence of T_{c-} on the internetwork link number $E_{AB} \sim p$. The line represents T_{cl}, which is calculated using linear approximation (see Sec. II). The triangles are map iterations that display the first-order phase transition (see Sec. IV), while circles correspond to the data obtained from Monte Carlo simulations (see Sec. V). The plus symbols are the same as the circles, but are re-scaled to have the same value at $E_{AB}=0$ as map iterations. All considered simulations, iterations, and analytic results are for two B-A networks of sizes $N_A=N_B=5000$ and average degrees $\langle k_A \rangle = \langle k_B \rangle = 10$. Note: For such parameters, $E_{AB}=50\,000$ corresponds to $p=1$.

flipping due to entering into the monostable phase, but the flipping due to large fluctuations that appear close to the critical temperature of the ferromagnetic-paramagnetic transition T_{c+}.

The $|S_A|$ values that jumped far from zero prove that the linear approximation is indeed wrong and that the order parameter changes discontinuously. If the linear approximation were correct, the weighted spin would need to touch zero at the transition point. This is clearly not the case. The dependence of T_{c-} on E_{AB}, obtained from data that are partially shown in Fig. 7, is shown in Fig. 8. The critical temperature T_{c-} is much lower than predicted by either analytics or map iterations. This is because the MF method predicts critical temperature values for the Ising model in the B-A network that are higher than they really are. As a first approximation, we assume (based on experience with results regarding the critical temperature in the B-A network) that the difference is only a constant factor. This allows us to work around the problem that arises in a single network and see how connecting two networks influences the critical temperature. We rescale our results by a constant factor so that the value for unconnected networks is the same. After that, we obtain relatively good agreement with map iterations. Above $E_{AB} \approx 15\,000$, the results from map iterations and simulations start to differ greatly. This is the result of the limited system size; the delicate antiparallel ordering is quickly destroyed by fluctuations accompanied by strong internetwork interaction, and the system reverts to lower-energy parallel ordering. However, for lower interconnection densities, the simulations agree with map iterations.

VI. CONCLUSIONS

We have shown that in a system of two sparsely connected networks one of the two transitions that occur is a transition between a bistable (parallel/antiparallel) state and a monostable (parallel only) state. It is possible for a system to change the order parameter discontinuously during this transition. The critical temperatures corresponding to bistable-monostable (T_{c-}) and ferromagnetic-paramagnetic (T_{c+}) transitions depend on the interaction strength between the networks. The critical temperature T_{c-} decreases as the interaction strength E_{AB} grows in a nonlinear fashion. The critical temperature values T_{c-} are lower than those obtained by using linear approximation during the analytic calculations (T_{cl}). The analytic results are backed by numeric simulations.

Our exact results regarding bi/monostability will probably not apply to real social systems since real social dynamics are never as simple as Ising model. However, we expect that bi/monostability itself might be observable or provide a better understanding of more complex behavior.

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