Behavior of ϕ^4 kinks in the presence of external forces

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The influence of external forces on the properties of kinks in the ϕ^4 equation is investigated. Depending on characteristic parameters, the model describes the interaction of a kink with a material impurity or a phase boundary. In the presence of external forces, there are differences between stability conditions for a kink treated as a point particle and for a kink investigated as an extended object. These differences come from interactions of kink wings with zeros of external forces. An inhomogeneous external force changes the spectrum of small oscillations around a kink, and additional bound states can appear. When this force has the form of a static two-well potential, chaotic kink motion is possible, provided the amplitude of an additional ac force is higher than a critical value.

I. INTRODUCTION

Solitons and solitary-wave solutions play an important role in different branches of physics.[1,2] Recently there has been great interest in the influence on solitons of various additional forces.[3–5] The physical origin of such forces may be, for example, the presence of impurities (nonmagnetic ions in spin chains), the inhomogeneity of external fields, or the damping or coupling to other degrees of freedom. Usually it is assumed that these forces are appropriately small because this is a condition of the applicability of various perturbation methods.[4]

In the simplest of such methods, the so-called adiabatic approach,[6] a soliton is treated as a structureless pointlike particle and the perturbation influences only the position and velocity of such an object. More exact results can be obtained using the collective-coordinates approach (CCA), i.e., employing an orthonormal set of eigenfunctions which arises in the stability analysis of an *unperturbed* soliton.[2] Still another possibility is to combine the perturbation analysis with the inverse scattering transform,[4] however, this powerful tool can be used only when the unperturbed system is completely integrable.

There are, however, some examples that show that the magnitude of external perturbations plays an important role in the system behavior. In fact,[7] for a completely integrable magnetic chain with two local anisotropies,[8] the influence of Gilbert damping can be balanced by an additional external magnetic field so the kink can move with constant velocity provided that the external field is lower than some critical value (Walker field).[9] Moreover, recent CCA-based calculations[10] indicate the existence in this system of another critical velocity connected to the effect of the emission of spin waves by a moving kink (or a domain wall).

In the present work we assume that the total external force consists of two parts, $F(x)$ and $G(x,t)$ (and a damping term). The force $F(x)$ possesses such a form that we are able to find an *exact* (static) kink solution $\varphi_k(x)$ in the presence of this force although we do not assume that this force is small. Then we analyze the effect of any *small* force $G(x,t)$, applying the CCA with respect to the solution $\varphi_k(x)$.

For simplicity we consider the ϕ^4 model,[11] which is used, for example, in Ginzburg-Landau systems.[12] Similar calculations can also be performed for other kink-bearing systems like sine-Gordon,[1] double-sine-Gordon,[14] or anisotropic spin models.[7]

The plan of this work is as follows. In Sec. II, we analyze the influence of a static homogeneous force on a kink motion. Section III deals with the problem of a special inhomogeneous static force $F(x)$. We find a static kink solution and we perform a stability analysis of this solution. The problem of the small force $G(x,t)$ added to the force $F(x)$ is considered in Sec. IV. In Sec. V, we apply the results of earlier sections to study the chaotic movement of a kink.

II. HOMOGENEOUS STATIC FORCE

Let us consider the following Hamiltonian describing the one-dimensional ϕ^4 model with some additional space-time perturbation:

$$H=\int\left[\tfrac{1}{2}(\varphi_t)^2+\tfrac{1}{2}(\varphi_x)^2+\tfrac{1}{8}(\varphi^2-1)^2-\varphi F(x,t)\right]dx \ . \quad (1)$$

The scalar field $\varphi(x,t)$ (for convenience we use dimensionless time and space variables) is coupled here to external force $F(x,t)$ which can represent an impurity, a boundary between two different phases, or an additional external field. The standard equation of motion for the system (1) can now be written as

$$\varphi_{xx}-\varphi_{tt}-\gamma\varphi_t+\tfrac{1}{2}\varphi-\tfrac{1}{2}\varphi^3=-F(x,t) \ , \quad (2)$$

where γ (>0) is a damping constant. In the absence of the force $F(x,t)$ and damping γ, there exist well-known kink (antikink) solutions of Eq. (2) that are given by

$$\varphi_k(x,t)=\pm\tanh[\tfrac{1}{2}(x-vt-x_0)/(1-v^2)^{1/2}] \ , \quad (3)$$

where $|v|<1$ and the higher (lower) sign refers to a kink (an antikink). We are interested in the existence and properties of soliton solutions similar to Eq. (3) for a wide class of external forces $F(x,t)$ and in the presence of the nonzero damping γ.

Let us start from the simplest case, i.e., $F(x,t)=\text{const}$. After some algebra we obtain the result that, if

$$F^2<\tfrac{1}{27}\,, \tag{4}$$

then there exists the following solution:

$$\varphi_k(x,t)=C\tanh[D(x-vt-x_0)/(1-v^2)^{1/2}]+E\ , \tag{5}$$

where $C=\pm(\varphi_3-\varphi_1)/2$, $D=(\varphi_3-\varphi_1)/4$, $E=(\varphi_3+\varphi_1)/2$, the velocity v fulfills the equation

$$\gamma v/(1-v^2)^{1/2}=\pm\varphi_2/2\ , \tag{6}$$

and $\varphi_1<\varphi_2<\varphi_3$ are the roots of the equation $\varphi-\varphi^3=-2F$. For $C>0$ (<0) the solution (5) represents a kink (an antikink). Because $\text{sgn}(\varphi_2)=-\text{sgn}(F)$, we see that the sign of the velocity v of the kink (of the antikink) is opposite to (is the same as) the sign of the force F. Simple energetic arguments for this fact can be found in Chap. (4.3) of the first of Refs. 2. Thus, in the presence of the force F and damping γ, the kink (or antikink) tends to move with a velocity which is determined by Eq. (6) that describes the balance between the effects of the driving force and the damping (compare this result to that from Ref. 5).

III. INHOMOGENEOUS STATIC FORCE

Now, let us consider the case of an inhomogeneous but static force $F(x,t)=F(x)$ and we shall look for a static kink solution. Because of the sensitivity of direction of kink motion to the sign of force F [for $F(x,t)\equiv\text{const}$], in the presence of an inhomogeneous force the position of the center of a static kink (or antikink) will be in the neighborhood of the first-order zeros x_0 of $F(x)$. For forces fulfilling the relation $F(x_0+x)=-F(x_0-x)$, the point x_0, *coincides* with the center of a kink (antikink) trapped by such an inhomogeneity. For such a case there is also a simple condition of the stability of the equilibrium position of the kink (antikink) at the point x_0,

$$\left[\frac{dF}{dx}\right]_{x_0} \overset{(<)}{>} 0\ , \tag{7}$$

provided that there are no other zeros of $F(x)$ besides x_0. However, there is also an analogue of condition (4) in this case. It can be formulated in the following way. If the stability condition (7) is not fulfilled, then there can exist an unstable kink (or antikink) (centered at the point x_0) provided that

$$[F(\pm\infty)]^2<\tfrac{1}{27}\ . \tag{8}$$

Now let us consider the special form of the (static) force $F(x)$ that is parametrized by constants A and B:

$$F^{AB}(x)=\tfrac{1}{2}A(A^2-1)\tanh(Bx) + \tfrac{1}{2}A(4B^2-A^2)\sinh(Bx)\cosh^{-3}(Bx)\ , \tag{9}$$

(Fig. 1), where the first term belongs to a nonlocal part of this force and the second term represents a local part. There are (at least) two interesting special cases of the force (9): (a) If $A^2=1$, then the force $F^{AB}(x)$ represents an impurity localized at the point $x=0$. (b) If $A^2=4B^2$, then force $F^{AB}(x)$ represents a typical boundary between two different phases at $x=0$. The advantage of the parametrization (9) is that, putting (9) into (2), we obtain immediately a static kink (or antikink) solution of the later equation in the simple form

$$\varphi_k(x,t)=A\tanh(Bx)\ . \tag{10}$$

To investigate the stability of this solution, we consider small-amplitude oscillations around $\varphi_k(x,t)$, i.e., we put

$$\varphi(x,t)=\varphi_k(x)+f(x)e^{\lambda t}\ , \tag{11}$$

where $|f(x)|\ll A$. Linearizing in a standard way[2] Eq. (2) around $\varphi_k(x)$, we get for the function $f(x)$ the following equation:

$$-f_{xx}(x)+[\tfrac{3}{2}A^2-\tfrac{1}{2}-\tfrac{3}{2}A^2\cosh^{-2}(Bx)]f(x)=\Gamma f(x)\ , \tag{12}$$

where $\Gamma=-\lambda^2-\lambda\gamma$. This equation is equivalent to the Schrödinger equation for a particle in the so-called Pöschl-Teller potential.[15] The spectrum for this potential consists of a discrete and continuum part. The discrete

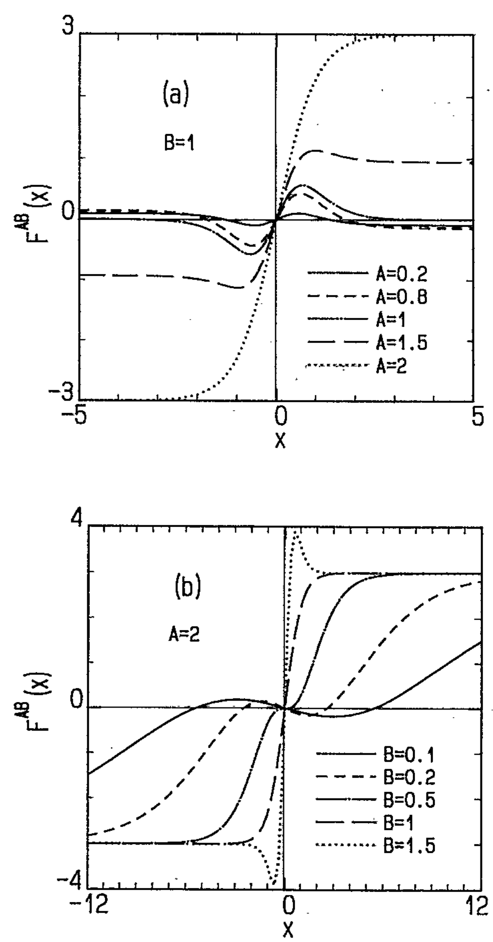

FIG. 1. Behavior of the function $F^{AB}(x)$ for various parameters A and B.

levels are described by the formula

$$\Gamma_n = -\tfrac{1}{2} + B^2(\Lambda + 2\Lambda n - n^2) , \tag{13}$$

where the parameter Λ (>0) is defined by the equation $3A^2 = 2B^2\Lambda(\Lambda+1)$, $n=0,1,2\ldots$, and $n<\Lambda$. The corresponding eigenfunctions f_n (they mean soliton-phonon bound states) can be found in Refs. 15 and 16. We note here only that the ground-state function (soliton translational mode) is given by

$$f_0(x) = \cosh^{-\Lambda}(Bx) . \tag{14}$$

From (13) we see that the lowest eigenvalue is $\Gamma_0 = B^2\Lambda - \frac{1}{2}$. Thus, the condition for stability of solution (10) can be written as

$$2B^2\Lambda > 1 . \tag{15}$$

It is interesting that, if one uses for our force (9) the relation (7), then one gets the stability condition

$$4B^2 > 1 . \tag{16}$$

We see that the stability condition (16) (that is, a condition based on the hidden assumption that a kink can be treated as a point particle) is a sufficient condition for stability of a kink treated as an extended object [Eq. (15)] provided that $\Lambda \geq 2$. However, it is easy algebraically to show that the last inequality combined with Eq. (16) means that our force function $F^{AB}(x)$ possesses only one zero $x_0 = 0$. In fact, function (9) possesses only one zero provided that

$$4B^2 \overset{(\geq)}{\leq} 1 , \quad \tfrac{2}{3}B^2\Lambda(\Lambda+1) \overset{(\geq)}{\leq} 1 \tag{17}$$

[excluding the point $4B^2=1$, $\frac{2}{3}B^2\Lambda(\Lambda+1)=1$, where the function $F^{AB}(x)$ disappears]. If we hold condition (16) and diminish the parameter Λ, then, for $\Lambda(\Lambda+1) < 3/(2B^2)$, there appear two additional zeros $x_{1(2)}$ of the function $F^{AB}(x)$ (see Fig. 1). These zeros can be treated approximately as two unstable equilibrium positions of a kink reduced to a point particle. In the limit $\Lambda(\Lambda+1) \to 3/(2B^2) + 0_+$, we have $|x_{1(2)}| \to \infty$, so that an interaction of these zeros with kink wings (i.e., regions where $|Bx| \gg 1$) is very small and the kink is still stable at the position $x_0 = 0$. However, when we hold Eq. (16) and diminish Λ below the value $1/2B^2$, then these two additional zeros are closer to the kink center $x_0 = 0$ and interactions of kink wings with these zeros are sufficiently strong to make the whole kink unstable. One can say that this instability occurs due to the extended (nonlocal) character of the kink. Relations between regions of parameters B, Λ, where (a) the kink stability condition (15) holds, (b) the "point-particle" (PP) stability condition (16) is valid, and (c) there is only one zero of the force function $F^{AB}(x)$, i.e., condition (17) holds, are depicted at Figs. 2 and 3. In region I, the PP is stable, there is only one (stable) zero $x_0 = 0$ and the kink is stable. In region II, the PP is stable and there are three zeros (one stable and two unstable) but the kink is still stable. In region III, the PP is stable, there are three zeros but the kink is already unstable because two unstable zeros are close enough to the point x_0 to make the whole kink unstable, due to their interactions with kink wings. In region IV,

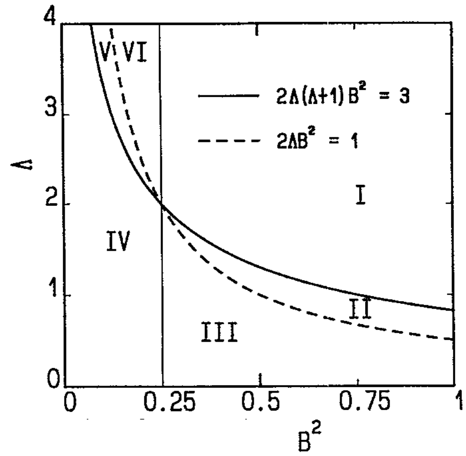

FIG. 2. Different stability regions as functions of parameters B and Λ (see discussion in text).

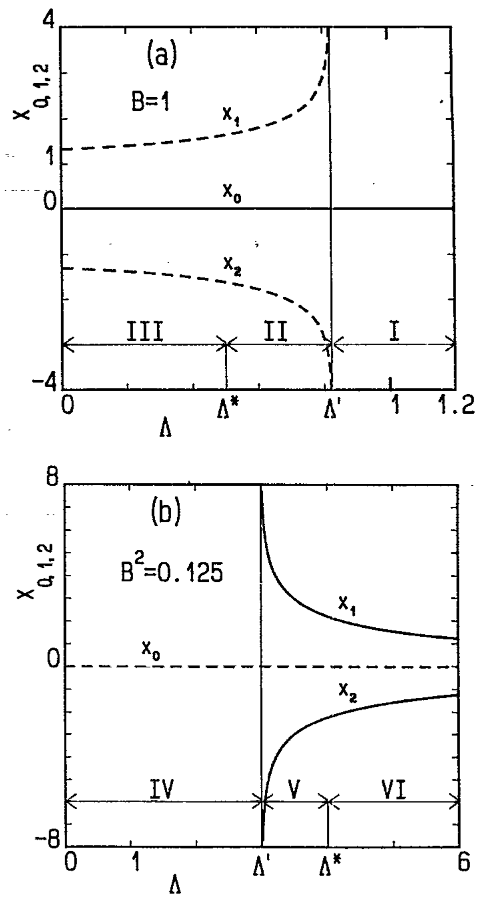

FIG. 3. Positions of stable (solid lines) and unstable (dashed lines) zeros of the function $F^{AB}(x)$. Characteristic values of the parameter Λ are given by equations: $\Lambda' = [(1+6/B^2)^{1/2} - 1]/2$ and $\Lambda^* = 1/(2B^2)$ (compare with Fig. 2).

the PP is unstable, there is only one (unstable) zero $x_0=0$ and the kink is unstable. In region V, the PP is unstable, there are three zeros (one unstable and two stable) but the kink is still unstable. In region VI, the PP is unstable, but the kink is already stable because two stable zeros are close enough to the point x_0 to insure the kink stability due to their interactions with kink wings.

Moreover, it follows from (13) that, depending on the values of parameters B and Λ (or B and A), there can be more than one negative "energy level" Γ_n, which means physically an instability of the kink (10) against the perturbations of internal structure. Besides the bound states $f_n(x)$ [and, connected with them, discrete levels Γ_n given by (13)], there are also continuum states $f_k(x)$ representing extended phonon states with eigenvalues

$$\Gamma_k > \tfrac{3}{2}A^2 - \tfrac{1}{2} . \tag{18}$$

IV. INHOMOGENEOUS TIME-DEPENDENT FORCE

Now let us consider a *small* space-time-dependent perturbation $G(x,t)$ that exists beside the constant force $F^{AB}(x)$ given by (9), i.e., we consider the equation

$$\varphi_{xx} - \varphi_{tt} - \gamma\varphi_t + \tfrac{1}{2}\varphi - \tfrac{1}{2}\varphi^3 = -F^{AB}(x) - G(x,t) \tag{19}$$

with $|G(x,t)| \ll |F^{AB}(x)|$. We can expand the function $G(x,t)$ in a basis spanned by the system of eigenfunctions $f_n(x)$ and $f_k(x)$

$$G(x,t) = \sum_n f_n(x) g_n(t) + \int_{-\infty}^{+\infty} f_k(x) g_k(t) dk \tag{20}$$

and we write a solution in the following form:

$$\varphi(x,t) = A\tanh(Bx) + \sum_n f_n(x) h_n(t) + \int_{-\infty}^{+\infty} f_k(x) h_k(t) dk . \tag{21}$$

In the standard way[2,13] we can now write equations of motion for the functions $h_k(t)$ in the form of a system of uncoupled, driven, damped oscillators (an overdot means a time derivative):

$$\ddot{h}_{n(k)}(t) + \gamma\dot{h}_{n(k)}(t) + \Gamma_{n(k)} h_{n(k)}(t) = g_{n(k)}(t) . \tag{22}$$

These equations describe the dynamics of a soliton under the action of the force $F^{AB}(x)$ and the perturbation $G(x,t)$. If the force $g_{n(k)}(t)$ is time periodic, e.g.,

$$g_{n(k)}(t) = g^0_{n(k)} \cos(\omega_{n(k)} t) ,$$

then there is a possibility of resonances if any frequency $\omega_{n(k)}$ obeys the condition

$$\omega_{n(k)} = (\Gamma_{n(k)} - \gamma^2/4)^{1/2} . \tag{23}$$

Let us now consider an example, i.e., the case when $\Lambda=2$. Instead of Eq. (19), we now write

$$\varphi_{xx} - \varphi_{tt} - \gamma\varphi_t + \tfrac{1}{2}\varphi - \tfrac{1}{2}\varphi^3 = B(1-4B^2)\tanh(Bx) - g_0(t)\cosh^{-2}(Bx) - g_1(t)\sinh(Bx)\cosh^{-2}(Bx) \tag{24}$$

[we assume that there is no force $g_k(t)$ conjugated to the continuum spectrum]. Instead of Eq. (22), we get two equations

$$\ddot{h}_0(t) + \gamma\dot{h}_0(t) + (2B^2 - \tfrac{1}{2})h_0(t) = g_0(t) , \tag{25a}$$

$$\ddot{h}_1(t) + \gamma\dot{h}_1(t) + (5B^2 - \tfrac{1}{2})h_1(t) = g_1(t) , \tag{25b}$$

and an appropriate equation describing the behavior of the continuum spectrum. If $g_0(t), g_1(t) \equiv 0$ and the stability condition for the translational mode holds, i.e., $4B^2 > 1$, then the kink can perform damped oscillations around the position $x=0$ combined with damped oscillations of the kink shape. In the opposite case, the kink can move away. Moreover, if $10B^2 < 1$, then the shape mode becomes unstable and the kink is "plastic" against shape deformations (compare this to the result in a double-sine-Gordon model[14]). One can also show that the condition for the stability of the continuum spectrum is $12B^2 > 1$. We stress that the instability of the shape mode and the instability against the emission or absorption of phonons occur in the situation where the kink is accelerating due to the instability of the translational mode. If the external force $g_0(t)$ is not a constant, then if the translational mode of the kink is stable, it is possible to sustain the permanent oscillations of the kink center; on the other hand, the periodic force $g_1(t)$ can cause oscillations of the shape (mostly of the width of the kink) whose amplitude depends on the amplitude and frequency of the force $g_1(t)$.

V. CHAOTIC MOTION

We shall now show the kink in the presence of some special inhomogeneous force $F(x)$ and the external force $G(x,t)$ can exhibit a chaotic motion. Let us assume that

$$F(x,t) = \begin{cases} B(4B^2-1)\tanh[B(x+x_0)] & \text{for } x<0 , \\ 0 & \text{for } x=0 , \\ B(4B^2-1)\tanh[B(x-x_0)] & \text{for } x>0 , \end{cases} \tag{26}$$

where $4B^2 > 1$ and we choose the force $G(x,t)$ in the form

$$G(x,t) = \begin{cases} P_0\cos(\omega t)\cosh^{-2}[B(x+x_0)] & \text{for } x<0 , \\ P_0\cos(\omega t)\cosh^{-2}[B(x-x_0)] & \text{for } x>0 . \end{cases} \tag{27}$$

If the amplitude P_0 equals zero, then the two stable equilibrium positions of the kink center are approximately x_0 and $-x_0$, i.e., there are stable solutions $\varphi_k(x) \cong 2B\tanh[B(x \pm x_0)]$ (the point $x=0$ is an unstable position of the center of such a kink). This fact means that the force (26) acts on the kink $\varphi_k(k)$ like some effective two-well potential on the point particle. If $P_0 \neq 0$ but $|P_0| \ll 1$, then it follows from Eq. (22) that, depending on the initial condition, the center of the kink oscillates harmonically around the positions $\pm x_0$. However, for larger values of the amplitude P_0, nonlinear effects can occur that are connected to (a) jumping of the kink center between two positions $x = x_0$ and $x = -x_0$, and (b) coupling between different modes f_n and f_k. The first of these effects can be compared to the behavior of a point particle whose motion is given by the Duffing equation[17]

$$z_{tt} + \gamma z_t - z + z^3 = p_0 \cos(\omega t) \ . \quad (28)$$

It is well known[17] that there is chaotic behavior of the solution of (28) for some set of parameters γ, p_0, and ω; thus, we can also expect that the motion of the kink in the presence of the forces (26) and (27) will be chaotic. This chaotic behavior should especially occur if the kink jumps between two wells of the effective potential coming from the force (26). An approximate condition for such jumps can be obtained as follows. Suppose that the kink is in the neighborhood of point x_0. Taking into account (21), we can write an approximated form of such a kink as

$$\varphi_k(x,t) = 2B \tanh[B(x - x_0)] + h_0(t) \cosh^{-2}[B(x - x_0)] \ . \quad (29)$$

However, if we calculate the center of mass of solution (29), it differs from x_0 for the value $|x_c| \simeq |h_0/(2B^2)|$. So, the condition of the jumping of the kink between the wells centered at $x = \pm x_0$ is $|h_0^0/(2B)^2| > x_0$, where h^0 is the amplitude of the translational mode $h_0(t) = h_0^0 \cos(\omega t + \delta)$. This amplitude follows from (25a) and can be written as

$$|h_0^0| = |P_0| / [(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2} \ , \quad (30)$$

where $\omega_0^2 = 2B^2 - \frac{1}{2}$.

VI. SUMMARY AND CONCLUSIONS

We have investigated the behavior of a kink of the ϕ^4 equation in the presence of an additional time- and space-dependent force $F(x) + G(x,t)$ and a damping. For the case of a homogeneous, static force, we found moving, stationary solutions provided the magnitude of the force is smaller than some critical value. In the presence of an inhomogeneous, static force there exist static kinks whose centers are connected (or coincide) with the positions of zeros of this force. We investigated in greater detail a family of inhomogeneous, static forces $F^{AB}(x)$, special cases of which correspond to the presence of a localized impurity or a boundary between different phases (or materials). We found a static kink solution for this family of forces and we performed a stability analysis of this solution. It occurs that the inhomogeneous force changes significantly the spectrum of bound states and extended states (phonons) that can exist as small-amplitude oscillations around a kink solution. Although for a kink solution with no external force there are only two bound-state solutions (and continuum states), in the presence of the external force the kink acts on phonon states like a Pöschl-Teller potential well on a quantum particle. It follows that, depending on the values of the parameters of the force $F^{AB}(x)$, there are several bound states (besides continuum states). Contrary to the case of an unperturbed ϕ^4 kink, in the presence of the external force the eigenvalue connected to the lowest bound state can be different from zero. This can be easily understood because, due to the presence of the inhomogeneity, there is no translational invariance of the system so there is no place for a Goldstone mode in this case. The inhomogeneity can cause this eigenvalue to be positive, which means that the kink is stable against translation (trapping of the kink by the impurity or the material boundary), or it can change this eigenvalue to be negative, which means the kink is unstable against translation movement. Moreover, it can occur that not only the lowest bound state is unstable but also higher bound states (shape modes) or even a part of the continuum spectrum can lose its stability.

The condition of kink stability, i.e., the condition that the lowest eigenvalue of the stability operator is positive, is not equivalent to the stability condition that one obtains from simple energetic considerations treating the kink as a point particle. The reason for this fact is that, if the force function $F^{AB}(x)$ possesses several zeros, it influences not only the kink center but also the kink wings, i.e., the spatial extended character of the kink is important in such a case.

We also considered the problem of the time-dependent force. We found that, if such a force possesses an appropriate space shape (that is, it coincides with one of the eigenfunctions of the kink stability operator), then it is possible to get a resonance of the kink provided the frequency of such a force coincides with the resonance frequency of this mode. It means, for example, that one can pump energy to the translational kink mode (one can move a kink) using some time-dependent force that is fitted to the shape of the translational mode. We used this fact to suggest the possibility of chaotic motion of the kink if the static inhomogeneous force acts on the kink as a two-well potential on the point particle and there is the additional time periodic force that can move a kink from one potential well to another.

In comparison to other studies of the soliton-impurity interactions,[4,18] we find that it is important to consider the behavior of higher soliton-phonon bound states in the presence of the impurity. These higher bound states are connected to the kink shape and, if one precludes investigations to the stability of the translation mode only, then a kink is treated as a point particle; i.e., one neglects its extended character.

Finally, we remark that, because bound states related to the internal soliton structure are important from the point of kink-kink and kink-antikink scattering,[19] the numerical simulations of such events should give especially interesting results in the presence of impurities.

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[1]G. Eilenberger, *Solitons: Mathematical Methods for Physicists,* Springer Series in Solid-State Sciences Vol. 19 (Springer, Berlin, 1981); R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982); A. Hasegawa, *Optical Solitons in Fibers* (Springer, Berlin, 1989).

[2]A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D **1**, 1 (1980); in *Nonlinearity in Condensed Matter,* edited by A. R. Bishop, D. K. Campbell, P. Kumar, and S. E. Trullinger, Springer Series in Solid-State Sciences Vol. 69 (Springer-Verlag, Berlin, 1987); J. F. Currie, J. A. Krumhansl, A. R. Bishop, and S. E. Trullinger, Phys. Rev. B **22**, 477 (1980).

[3]*Nonlinear Coherent Structures,* edited by M. Barthes and J. Léon, Lecture Notes in Physics Vol. 353 (Springer-Verlag, Berlin, 1990).

[4]Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).

[5]J. A. González and J.A. Hołyst, Phys. Rev. B **35**, 3643 (1987).

[6]D.W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1652 (1978).

[7]J. A. Hołyst, A. Sukiennicki, and J. J. Żebrowski, Phys. Rev. B **33**, 3492 (1986).

[8]E. K. Sklyanin (unpublished); A. E. Borovik and V. N. Robuk, Theor. Math. Phys. **46**, 371 (1981).

[9]L. R. Walker (unpublished); see J. F. Dillon, in *A Treatise on Magnetism,* edited by G. T. Rado and H. Suhl (Academic, New York, 1963), Vol. III.

[10]D. Bouzidi and H. Suhl, Phys. Rev. Lett. **65**, 2587 (1990).

[11]J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B **11**, 3535 (1975).

[12]S. Aubry, J. Chem. Phys. **62**, 3217 (1975).

[13]P. C. Dash, Phys. Lett. **109A**, 307 (1985).

[14]P. Sodano, M. El-Batanouny, and C. R. Willis, Phys. Rev. B **34**, 4936 (1986); C. R. Willis, M. El-Batanouny, M. Boesch, and P. Sodano, *ibid.* **40**, 686 (1989); K. M. Leung, *ibid.* **26**, 226 (1982); **27**, 2877 (1983).

[15]S. Flügge, *Practical Quantum Mechanics* (Springer, Berlin, 1971).

[16]J. A. Hołyst and H. Benner, Phys. Rev. B **43**, 11 190 (1991).

[17]J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, Berlin, 1986).

[18]T. Bountis, S. Pnevmatikos, S. Protogerakis, and G. Sohos, in *Nonlinear Coherent Structures* (Ref. 3).

[19]D. K. Campbell and M. Peyard, Physica D **18**, 47 (1986); D. K. Campbell, M. Peyard, and P. Sodano, *ibid.* **19**, 165 (1986).