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Solitary waves in one-dimensional damped systems

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An analysis of general properties of traveling waves that can exist in one-dimensional damped systems is presented. It is shown that, depending on the symmetry of the potential and the presence or absence of dissipation, different classes of localized solutions in the form of kinks, pulses, and semisolitons are possible.

It is now generally accepted that solitons or solitary waves[1,2] play a crucial role for many phenomena appearing in one- or quasi-one-dimensional models of solids.[3,4] The main aim of this Brief Report is to discuss the general properties of various solitary waves that can exist in the form of kinks or pulses in certain nonlinear, driven damped systems. Although similar models have been widely explored by several authors[5-9] as well as by ourselves[10,11] we think that there is a necessity to generalize some hitherto obtained results and to show how these results can be achieved in a more direct way, as well as to point out that there are several interesting types of solitary waves that have not been considered up to now in corresponding systems.

We analyze the classical scalar one-dimensional model described by the following equation of motion:

$$\frac{\partial^2\phi(x,t)}{\partial x^2}-c^{-2}\frac{\partial^2\phi(x,t)}{\partial t^2}-R\frac{\partial\phi(x,t)}{\partial t}+G(\phi)=0\ , \quad (1)$$

where $G(\phi)=-[dU(\phi)/d\phi]$. We assume that the potential $U(\phi)$ is an analytical function of ϕ and that it possesses three extrema, which means two minima in points ϕ_1 and ϕ_3 and a maximum in a point ϕ_2, $\phi_1<\phi_2<\phi_3$ (Fig. 1). We also assume that $(d^2U/d\phi^2)_{\phi=\phi_2}<0$ and $U(\phi_1)=0$. The parameter R in Eq. (1) plays the role of a linear damping constant ($R\geq 0$), while the constant c is a characteristic velocity of the considered model. Similar equations are widely used to describe, for example, one-dimensional ferroelectrics[6] or charge-density-wave systems.[3]

We are interested in the solutions of Eq. (1) in the form of traveling waves, e.g., $\phi(x,t)=\phi(x-vt)$, where a velocity v is a constant not depending on time. It is convenient to introduce the variable $z=(x-vt)[1-(v/c)^2]^{-1/2}$. The partial differential equation (1) can now be rewritten in a form of ordinary differential equation,

$$\frac{d^2\phi(z)}{dz^2}+\gamma\frac{d\phi}{dz}=-\frac{dV(\phi)}{d\phi}\ , \quad (2)$$

where $\gamma=Rv/[1-(v/c)^2]^{1/2}$, $V(\phi)=-U(\phi)$. We see that Eq. (2) may be treated as a Newtonian equation of motion for a damped *unit-mass point particle* moving in the *inverted* potential $V(\phi)$ (Fig. 1). The variable z plays in Eq. (2) the role of "time" for the *particle*. We shall further widely explore the useful equivalence between a trajectory of the *point particle* from Eq. (2) and the shape of a traveling-wave solution from Eq. (1).

If a traveling wave possesses a shape of a localized solitary wave then $\lim_{z\to-\infty}\phi(z)=\phi_i$, $\lim_{z\to\infty}\phi(z)=\phi_j$, where $i,j=1,2,3$. This means that the corresponding particle from Eq. (2) moves for an infinitely long time between certain extrema of the potential $V(\phi)$. In a natural way one can distinguish several special cases for Eq. (1).

(1) In the simplest case of zero damping ($R=0$) and a degeneracy of maxima of the potential $V(\phi)$, e.g., $V(\phi_1)=V(\phi_3)$, Eq. (1) has well-known[1-11] kink and antikink solutions (Fig. 2). These solutions can be treated as *domain walls* connecting two different *domains* of degenerated ground states of potential $U(\phi)$. From Eq. (2) we see that the kinks correspond to the motion of the particle between two degenerated maxima $V(\phi_1)$ and $V(\phi_3)$. We

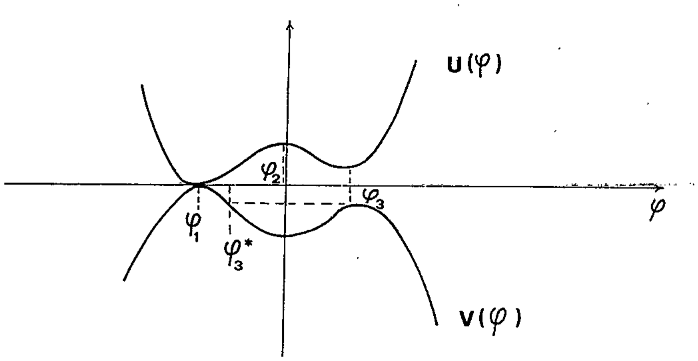

FIG. 1. Potentials $U(\phi)$ and $V(\phi)=-U(\phi)$.

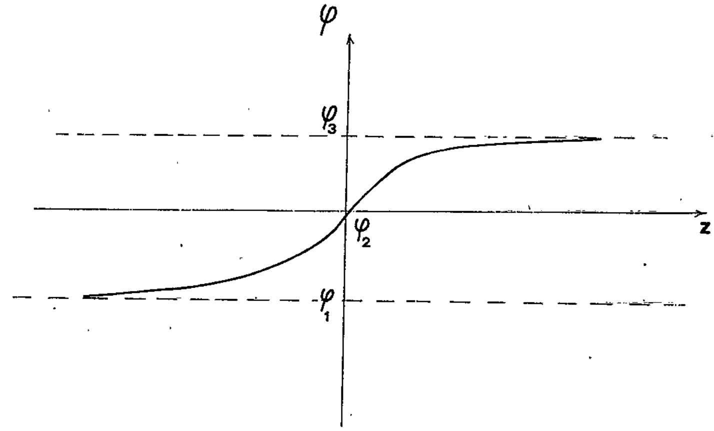

FIG. 2. Big kink solution of Eq. (1) for $V(\phi_1)=V(\phi_3)$, $R=0$.

shall call these kinks "big kinks" to distinguish them from other kinks ("little kinks") that can appear in damped systems. The big kinks can move with any constant velocity from the range $(-c,c)$.

Let us now consider the situation when $R=0$, but $V(\phi_1)>V(\phi_3)$. In our mechanical model described by Eq. (2) we need to consider separately both cases when the particle possessing zero *initial velocity* $[(d\phi/dz)_{z=-\infty}=0]$ starts its motion to the right from the point $\phi=\phi_1$ and when the particle starts its motion to the left from the point $\phi=\phi_3$. Obviously, in the first case the particle will come to the point $\phi=\phi_3$ with nonzero velocity and it will still move to the right with an increasing *velocity* $d\phi/dz$. This *particle trajectory* corresponds to the nonlocalized traveling-wave solution of Eq. (1) (see Fig. 3). It is easy to show that if one introduces a Hamiltonian corresponding to Eq. (1) then such a nonlocalized solution has an infinite energy; thus it possesses rather limited physical relevance. On the other hand, the particle starting its motion with the zero velocity from the point $\phi=\phi_3$ (and moving to the left) will come back to the same point and stop there. Such a particle trajectory corresponds to the interesting solution of Eq. (1) in the form of a solitary-wave pulse (Fig. 4). A "turning point" ϕ_3^* (see Fig. 1) describing the amplitude of this pulse can be determined from the condition $V(\phi_3^*)=V(\phi_3)$, $\phi_1<\phi_3^*<\phi_3$. The shape of the pulse $\phi_p(z)$ can be obtained from the first integral of Eq. (2),

$$\left|\frac{d\phi_p(z)}{dz}\right|=\{2[V(\phi_3)-V(\phi_p)]\}^{1/2} . \qquad (3)$$

The pulse can be treated as a bound pair kink-antikink. If we continuously symmetrize the form of the potential $V(\phi)$ such that the difference $V(\phi_1)-V(\phi_3)$ tends to zero then the *turning point* ϕ_3^* tends to ϕ_1, and the pulse dissociates into an uncoupled pair of infinitely separated kink and antikink. This is due to the fact that the integral

$$\int_{\phi_2}^{\phi_3^*}\frac{d\phi}{\{2[V(\phi_3)-V(\phi)]\}^{1/2}} \qquad (4)$$

tends to infinity when ϕ_3^* tends to ϕ_1 because, due to the analyticity of the potential $V(\phi)$ in this limit in the neighborhood of ϕ_3^*, the integrand diverges at least as $(\phi_1-\phi_3^*)^{-1}$. Pulses, similarly as kinks described before, can possess any velocity v from the range $(-c,c)$. For $V(\phi_1)<V(\phi_3)$ the shape of the pulse is inverted [Fig. 4(b)].

(2) Now we shall consider Eq. (1) with the parameter $R>0$. When the velocity v of the traveling wave is equal to zero, then the parameter γ in Eq. (2) is also equal to zero and the solutions of Eqs. (1) and (2) for this case are the same as the undamped cases analyzed earlier. When the velocity v is not equal to zero, then γ is also not equal to zero, and the "point particle" is damped or accelerated (depending on the signature of the velocity v of the corresponding traveling wave).

When the potential $V(\phi)$ is symmetric, e.g., $V(\phi_1)=V(\phi_3)$, then from the energy-conservation principle it is obvious that there are no solutions of Eq. (2) describing the motion of the particle starting with zero velocity from the point $\phi=\phi_1$ and stopping at the point $\phi=\phi_3$, or vice versa. It means that for symmetric potentials in the presence of dissipation there are no big kink solutions of Eq. (2) moving with a constant nonzero velocity. Of course, static big kinks are possible in this case.

Let us now suppose that parameter γ in Eq. (2) is positive, potential $V(\phi)$ is asymmetric, and $V(\phi_1)>V(\phi_3)$. If we consider the motion of the point particle moving with zero initial velocity from the point ϕ_1 to the right, depending on the value of the parameter γ for the fixed potential $V(\phi)$, there are six types of "particle trajectories," and there are six corresponding shapes of traveling wave.

(a) For small values of γ parameter the point particle from Eq. (2) will reach the lower minimum of potential $V(\phi)$ (e.g., $\phi=\phi_3$) with certain nonzero velocity $[(d\phi/dz)_{\phi=\phi_3}>0]$ and then it will move further to the right of this point. The corresponding shape of a traveling-wave solution is similar to that from Fig. 3 representing an unlocalized solution.

(b) For a certain value of parameter $\gamma=\gamma_k$ the particle will stop at the point $\phi=\phi_3$. Such a trajectory corresponds

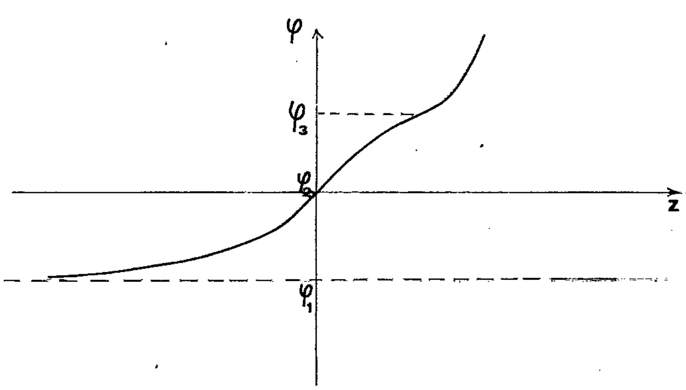

FIG. 3. Unlocalized traveling-wave solution for $V(\phi_1)>V(\phi_3)$, $R=0$.

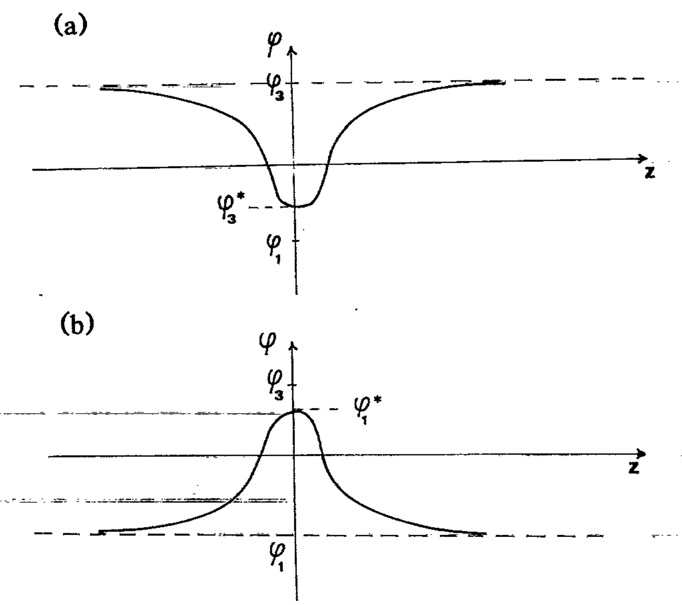

FIG. 4. Pulse solution of Eq. (1); (a) $V(\phi_1)>V(\phi_3)$, (b) $V(\phi_1)<V(\phi_3)$.

to the kink solution of Eq. (1) similar to that from Fig. 2. The value of the velocity v of this kink comes from the equation $\gamma_k = Rv[1-(v^2/c^2)]^{-1/2}$. From Eq. (2) it is easy to obtain the following relation[5,7] between the value of the parameter γ_k and the shape of the kink $\phi_k(z)$:

$$\gamma_k \int_{-\infty}^{+\infty} \left[\frac{d\phi_k}{dz} \right]^2 dz = V(\phi_1) - V(\phi_3) \ . \tag{5}$$

Suppose now that $V(\phi) = V_0(\phi) + \varepsilon V_1(\phi)$, where $V_0(\phi)$ is a symmetric potential [e.g., $V_0(\phi_1) = V_0(\phi_3)$], $V_1(\phi)$ is an asymmetric potential, and $\varepsilon \ll 1$. Using the perturbation scheme[5–8,11] one can rewrite Eq. (4) in the following form:

$$Rv = \pm \varepsilon [V_1(\phi_1) - V_1(\phi_3)] \Big/ \int_{\phi_1}^{\phi_3} [-2V_0(\phi)]^{1/2} d\phi \ , \tag{6}$$

which is the equation on the velocity v. We see that Eq. (6) has the form of Stoke's law for a particle driven by a constant effective force in a viscous medium.[5–8,10,11]

(c) For higher values of γ the particle will not reach the point ϕ_3 and it will oscillate with decreasing amplitude around the point ϕ_2. A linear stability analysis of Eq. (2) around this point leads to the eigenvalue equation $\lambda^2 + \gamma\lambda + \alpha = 0$, where $\alpha = (d^2V/d\phi^2)_{\phi=\phi_2}$. Thus for $\gamma^2 < 4\alpha$ the oscillations are damped but not overdamped. This motion of the point particle corresponds to the so-called[8,11] "semisoliton kink" and it is illustrated in Fig. 5(a).

(d) For still higher values of the parameter $\gamma (\gamma^2 > 4\alpha)$ the motion of the particle is overdamped. The corresponding kink solution is illustrated in Fig. 5(b). We see that this solution has a form of a kink similar to that from Fig. 2, but the topological charge of this kink (little kink) is equal to $\phi_3 - \phi_2$, while for the big kink from Fig. 1 it is equal to $\phi_3 - \phi_1$. Unlike the big kinks that, for a specified potential $V(\phi)$ and a specified value of a damping constant R, can move with only one velocity defined by Eqs. (5) or (6), little kinks can move with any velocity v for which the following inequality holds:

$$R^2 v^2 \Big/ \left[1 - \frac{v^2}{c^2} \right] > 4\alpha \ . \tag{7}$$

(e) There are also two possible point particle trajectories and two corresponding traveling-wave solutions that are analogous to semisolitons and small kinks from Figs. 5(a) and 5(b), respectively, but a starting point for these trajectories is not $\phi = \phi_1$, but $\phi = \phi_3$. Of course both semisolitons as well as little kink solutions are also possible for a symmetric type of potential $V(\phi_1) = V(\phi_3)$.

(f) Concerning the pulse soliton solutions described by Eq. (3), it is easy to show that in the presence of dissipation this kind of solution can be only possible as a static one.

We assumed that all traveling-wave solutions described in points (a)–(e) possessed positive velocities. Besides them there are also corresponding solutions moving with negative velocities. These solutions correspond to the motion of the *"accelerated point particles"* from Eq. (2). For example, for a kink $\phi_k(z)$ moving with a positive velocity v defined by Eq. (5) there exists antikink $\phi_{ak}(z) = \phi_k(-z)$ moving with the negative velocity $(-v)$.

The elementary stability analysis for all the solutions that we have discussed can be performed using, e.g., methods described in Refs. 7, 8, and 11. The final result of this analysis is that big kinks are stable against small perturbations both in the presence as well as in the absence of the dissipation. The zero frequency mode of small oscillations around the big kinks is the Goldstone mode connected with restoring the translational symmetry of the system broken by the kink existence. On the other hand, neither the pulse soliton nor the semisoliton solutions are stable. The highest eigenvalues of the stability operators for these solutions are positive in both cases. It is due to the fact that for these solutions there is such a point z at which the derivative $d\phi/dz$ vanishes.

In conclusion, we can state that the analysis of general properties of solitary waves that can appear in one-dimensional damped multistable systems can be illustratively performed when the mapping of the original problem onto a mechanical model of damped point particle motion is used. We have shown that for symmetric potentials and zero damping the only possible type of solitary wave solutions are big kinks. They are stable and they can move with the constant velocity from the range $(-c,c)$, where c is the characteristic velocity of the system. For asymmetric potentials and zero damping instead of big kinks, there are moving pulses that can be treated as bounded pairs kink-antikink.

The most interesting case is the problem of an asymmetric potential with damping included. In such a system, besides unlocalized traveling waves (that can appear in all cases), there are big kinks moving with a velocity whose value comes from the balance between the asymmetry of the potential and damping effects; moving little kinks and

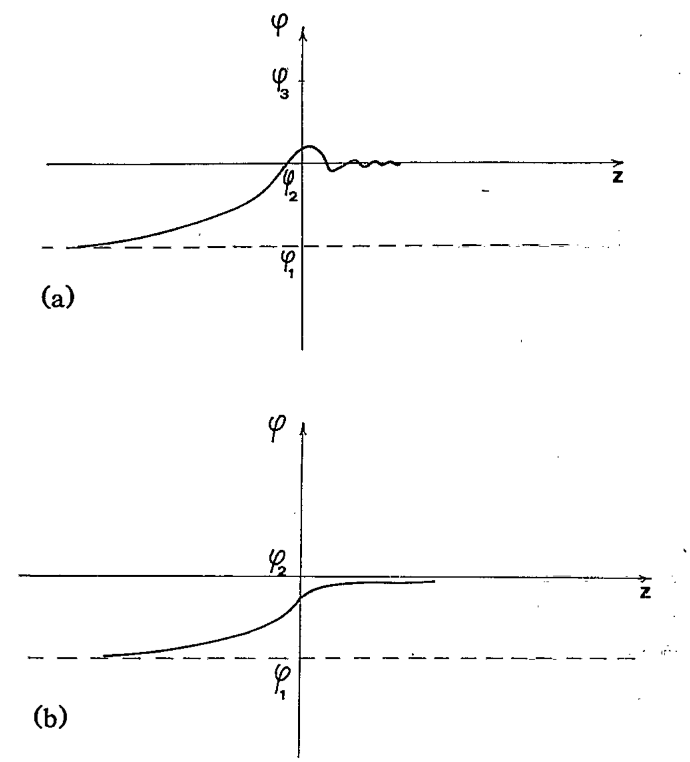

FIG. 5. (a) Semisoliton solution; $V(\phi_1) > V(\phi_3)$, $\gamma > 0$. (b) Little kink solution; $V(\phi_1) > V(\phi_3)$, $\gamma > 0$.

semisolitons as well as static pulses are also possible. In the case of a symmetric potential and nonzero damping the big kinks can be only static, and solutions in the form of pulses do not exist while the moving little kinks and semisolitons are still allowed.

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[1] *Solitons,* edited by R. K. Bullough and P. J. Caudrey (Springer, Berlin, 1980); *Solitons in Action,* edited by K. Longren and A. Scott (Academic, London, 1978).

[2] *Solitons (Mathematical Methods for Physicists),* edited by G. Eilenberger (Springer, Berlin, 1981); *Nonlinear Problems: Present and Future,* edited by A. Bishop, D. Campbell, and B. Nicolaenko (North-Holland, Amsterdam, 1982).

[3] A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D **1**, 1 (1980); *Solitons and Condensed Matter Physics,* edited by A. R. Bishop and T. Schneider (Springer, Berlin, 1978).

[4] *Physics in One Dimension,* edited by J. Bernasconi and T. Schneider (Springer, Berlin, 1981).

[5] R. Landauer, J. Appl. Phys. **51**, 5594 (1980); Phys. Rev. A **15**, 2117 (1977).

[6] M. A. Collins *et al.*, Phys. Rev. B **19**, 3630 (1979); J. F. Currie *et al.*, *ibid.* **19**, 3645 (1979).

[7] E. Magyari, Phys. Rev. B **29**, 7082 (1984); Phys. Rev. Lett. **52**, 767 (1984).

[8] S. Puri, Phys. Lett. **105B**, 443 (1984).

[9] M. Büttiker and H. Thomas, Phys. Lett. **77A**, 372 (1980); M. Büttiker and R. Landauer, Phys. Rev. A **23**, 1397 (1981).

[10] J. A. Hołyst, A. Sukiennicki, and J. J. Żebrowski, J. Magn. Magn. Mater. **54–57**, 865 (1986); Phys. Rev. B **33**, 3492 (1986).

[11] J. A. Gonzalez (unpublished).