Dynamics of the classical Heisenberg ferromagnet

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Dynamical properties of a classical anisotropic Heisenberg ferromagnet are analyzed by means of the stereographic representation for spin variables. The Gilbert damping term is included, and the failure of the complex time rescaling to account for damped solutions is shown. It is argued that the stereographic projection approach allows for formulating the damped Heisenberg ferromagnet dynamics within the mixed canonical-dissipative method. Finally the Benjamin-Fair instability of the finite-amplitude spin waves in the anisotropic Heisenberg chain is discussed.

The continuum limit of a classical low-dimensional spin system is a convenient proving ground for a variety of modern theoretical concepts, particularly those related to nonlinear dynamics. The analysis of the behavior of such a system is usually greatly facilitated by the proper choice of dynamical variables. For a one-dimensional chain one has a variety of possible choices of such variables: conventional polar angles,[1] scattering data,[2] or less familiar curvature-torsion ones.[3] Th latter variables are particularly useful in the analysis of an isotropic Heisenberg chain. For easy axis or easy plane chains curvature-torsion variables seem to be of little use.

All these variables, except the polar angles, neither generalize easily to more than one spatial dimension nor are they useful for the description of *damped* systems, at least within the Gilbert-Landau-Lifshitz theory.

In a recent very interesting Letter Lakshmanan and Nakamura[4] suggested that the stereographic projection of spin onto a complex plane may be the optimal choice. The aim of this report is to look more closely at this suggestion, to point out certain weak points in the Lakshmanan and Nakamura analysis, and to show explicitly how the description by means of stereographic projection variables works in practice. We shall show that those variables allow us to formulate the dynamics of a system described by the Gilbert-Landau-Lifshitz equation within the scope of the so-called mixed canonical-dissipative formulation of the many-body physics[7] and, henceforth, allow for a novel statistical mechanical description. Details of that aspect of the theory are planned to be published elsewhere.[6] We shall also show how the stereographic variables allow us to find new exact solutions for anisotropic chain dynamics and to assess their stability.

We begin with recollection of some points from Ref. 4. Consider a classical spin field $\mathbf{S}(\mathbf{R},t)$, where $\mathbf{R}$ is a vector in d-dimensional space and t stands for the time variable. We shall assume, without the loss of generality, that $|\mathbf{S}(\mathbf{R},t)|=1$. The Gilbert-Landau-Lifshitz equation can now be written as

$$\partial_t \mathbf{S}(\mathbf{R},t)=\mathbf{S}\times\mathbf{B}_{\text{eff}}-\lambda\mathbf{S}\times(\mathbf{S}\times\mathbf{B}_{\text{eff}}) , \quad (1)$$

where $\mathbf{B}_{\text{eff}}$ is an effective field felt by the spin at the point $\mathbf{R}$, and λ is the Gilbert damping parameter $\lambda\geq 0$.

For systems we are interested in $\mathbf{B}_{\text{eff}}$ can be derived from a Hamiltonian $H\{\mathbf{S}\}$; $\mathbf{B}_{\text{eff}}=-\delta H\{\mathbf{S}\}/\delta\mathbf{S}(\mathbf{R},t)$. For the sake of definiteness we shall consider systems for which the exchange and the local anisotropy energies are the only contributions to H. Thus we write

$$H\{\mathbf{S}\}=\tfrac{1}{2}\int d^dR\{\nabla\mathbf{S})^2+2A(\mathbf{S}\cdot\mathbf{n})\} , \quad (2)$$

where A is the anisotropy coefficient and $\mathbf{n}$ is the anisotropy direction which we choose to coincide with the z axis of the coordinate system.

Following Lakshmanan and Nakamura[4] we introduce complex field $\sigma(\mathbf{R},t)$ via which spin field can be described according to the stereographic projection rules. Thus,

$$\sigma(\mathbf{R},t)=(S^x+iS^y)/(1+S^z) . \quad (3)$$

One sees from Eq. (3) that the σ field shares some similarity with the bosonic field operator known from the continuum limit of the Holstein-Primakoff representation. Using the representation of Eq. (3), Lakshmanan and Nakamura have found, that the Gilbert-Landau-Lifshitz equation can be rewritten as

$$\partial_t\sigma=i(1-i\lambda)\{\nabla^2\sigma-[2\sigma^*(\nabla\sigma)^2 -2A\sigma(1-|\sigma|^2)]/(1+|\sigma|^2)\} . \quad (4)$$

Note, that for small $|\sigma|$, i.e., for large polarization of the system along the z axis one easily obtains from Eq. (4) the spectrum of damped spin waves

$$\omega(\mathbf{k})=(\mathbf{k}^2-2A)(1-i\lambda) . \quad (5)$$

The simple, multiplicative dependence of the frequency on the rescaling factor $(1-i\lambda)$ is in fact misleading. In the general case[5] one *does not* obtain the solution of Eq. (4) from those with $\lambda=0$ by simple rescaling of the time variable $t\to t(1-i\lambda)$. The reason for that is that the right-hand side of Eq. (4) depends in a nonanalytic fashion on the σ variable. To see this more clearly, write down equations for both σ and σ^* and consider both of them as independent (canonical) variables (see below). Now, of course, both equations are perfectly analytic, however, one can see that the damping factor enters those equations differently. Thus there is no unique rescaling for both canonical variables, and that makes the scheme impractical. (The same is true for simple equation $\dot{\sigma}=i\sigma^*$ which real time solutions do not fulfill, after rescaling, the complex time equation.)

In spite of this, Eq. (4) is very useful also for damped systems, i.e., for $\lambda\neq 0$. To see this, consider first the undamped case. Using Eqs. (2) and (3) one rewrites the Hamiltonian H as a functional of the field σ:

$$H\{\sigma\}=\int d^dR\,[2\,|\nabla\sigma|^2+A(1-|\sigma|^2)^2]/(1+|\sigma|^2)^2 \,. \tag{6}$$

Making use of the well-known Poisson brackets (PB) for spin variables, i.e.,

$$\{S^{\alpha}(\mathbf{R},t),S^{\beta}(\mathbf{R}',t)\}_{\rm PB}=\delta(\mathbf{R}-\mathbf{R}')\epsilon^{\alpha\beta\gamma}S^{\gamma}(\mathbf{R},t)\,, \tag{7}$$

we calculated the Poisson brackets for the fields σ:

$$\{\sigma(\mathbf{R},t),\sigma^*(\mathbf{R}',t)\}_{\rm PB}=-\frac{i}{2}\delta(\mathbf{R}-\mathbf{R}')(1+|\sigma|^2)^2 \,. \tag{8}$$

All the other brackets do vanish.

Having Eqs. (6) and (8) we can reformulate dynamics of the spin system entirely in terms of the σ field. Indeed

$$\begin{aligned}\dot{\sigma}(\mathbf{R},t)&=\{\sigma(\mathbf{R},t),H\}_{\rm PB}\\ &=i\nabla^2\sigma-i\,[2\sigma^*(\nabla\sigma)^2\\ &\qquad -2A\sigma(1-|\sigma|^2)]/(1+|\sigma|^2),\end{aligned} \tag{9}$$

i.e., $\lambda=0$ version of Eq. (4).

Now, the form of the Gilbert dissipation term to be added to Eq. (9) can be established on the basis of purely geometrical considerations. Indeed, writing $\mathbf{f}_C=\mathbf{S}\times\mathbf{B}_{\rm eff}$ and $\mathbf{f}_D=\mathbf{S}\times\mathbf{f}_C$ we note that vectors $\mathbf{S}$, $\mathbf{f}_C$, and $\mathbf{f}_D$ form a local triad of mutually orthogonal vectors (a local frame). Since the right-hand side of Eq. (1) equals $f_C-\lambda f_D$ one expects that $\dot{\sigma}=g_C-\lambda g_D$, where g_C stands for the right-hand side of Eq. (9).

Using the chain rule of differentiation and the orthogonality of the local frame vectors we easily find that $g_D=i\epsilon g_C$ with real ϵ. Since $|\mathbf{f}_C|=|\mathbf{f}_D|\Longrightarrow|\epsilon|=1$. The sign of ϵ follows from the physical requirement that

$$\frac{d}{dt}H\{\sigma\}<0 \,. \tag{10}$$

Equation (10) implies that

$$0>-\lambda\epsilon\int\left|\frac{\delta H}{\delta\sigma}\right|^2(1+|\sigma|^2)^2d^dR\,, \tag{11}$$

thus $\epsilon=1$. Therefore $g_D=ig_C$.

The above fact allows us to bring Eq. (4) into the general scheme of the so-called mixed canonical-dissipative formulation of the many-body dynamics.[7] Denoting $\psi=\binom{\sigma}{\sigma^*}$ we write Eq. (9) as

$$\partial_t\psi_A(\mathbf{R},t)=\int d^dR'L_{AB}(\mathbf{R},\mathbf{R}')\delta H\{\psi\}/\delta\psi_B(R')\,, \tag{12}$$

where $L_{AB}(\mathbf{R},\mathbf{R}')=\{\psi_A(\mathbf{R}),\psi_B(\mathbf{R}')\}_{\rm PB}$ is the antisymmetric matrix with off-diagonal matrix elements given by Eq. (8).

Now, the dissipative term can be added to Eq. (12) by replacing the kernel L_{AB} by $D_{AD}=L_{AB}-\lambda D'_{AB}$ with D'_{AB}being a symmetric matrix equal to

$$D'_{AB}=i\tau^3_{AC}L_{CB}\,, \tag{13}$$

where τ^3_{AC} is the zth Pauli matrix.

Following the general canonical-dissipative formulation[7(a)] we can develop statistical mechanics for the Gilbert-damped Heisenberg ferromagnet along the line analogous to that used in hydrodynamics.[7(b),7(c)] In a forthcoming publication we shall present a more detailed account of that problem. In particular, we shall address the problem of how the mode-mode coupling modifies the value of the Gilbert-damping coefficient, and how that depends on the system spatial dimensionality.[6]

Equation (9) can also be useful in finding and analyzing exact solutions for the anisotropic Heisenberg chain. It is well known that the isotropic Heisenberg chain supports, in addition to the nontopological envelope solitons, also finite-amplitude spin waves. It has been shown previously[8] that those waves exhibit the Benjamin-Fair instability which has close relations to the Fermi-Pasta-Ulam recurrence.

Here, we will show that a similar situation exists in the anisotropic chain, but now the instability depends on the sign of the anisotropy coefficient A.

Consider now Eq. (9) and write $\sigma(R,t)=\exp\{\theta(R,t)+i\phi(R,t)\}$ with both θ and ϕ real. In one spatial dimension it follows from Eq. (9) that θ and ϕ obey equations:

$$\phi_t=-\tanh\theta(\theta_R^2-\phi_R^2)+\theta_{RR}-2A\tanh\theta\,, \tag{14a}$$

$$\theta_t=2\tanh\theta(\theta_R\phi_R)-\phi_{RR}\,. \tag{14b}$$

The finite-amplitude spin waves are the solutions of Eqs. (14) with $\theta=\theta_0=$const and $\phi(R,t)=kR-\omega t$.

It follows then from Eq. (14) that ω and k obey the dispersion relation

$$k^2+\omega\coth\theta_0=2A\,. \tag{15}$$

This obviously generalizes the $\lambda=0$ case of dispersion relation (5). It is actually instructive to see that $\theta=\theta_0$, $\phi=kR-\omega t(1-i\lambda)$ is not an exact solution for the full Eq. (4). For $A=0$, we recover from Eq. (15) the isotropic chain result.

To assess the stability of our solutions we perform linear stability analysis by writing

$$\begin{aligned}\phi(R,t)&=\phi_0(R,t)+\epsilon_\phi\exp[i(qR-\Omega t)]\,,\\ \theta&=\theta_0+\epsilon_\theta\exp[i(qR-\Omega t)]\,.\end{aligned} \tag{16}$$

On substituting Eqs. (16) into Eqs. (4) and neglecting terms with higher powers of ϵ_ϕ and ϵ_θ we obtain the stability condition in the form

$$(\Omega+2kq\tanh\theta_0)^2=q^2[q^2+(2A-k^2){\rm sech}^2\theta_0]\,. \tag{17}$$

One sees from Eq. (17) that the existence of the Benjamin-Fair instability[8,9] depends on the sign and magnitude of the one-site chain anisotropy A. For easy plane anisotropy, when $A>0$, the finite-amplitude spin waves are stable for the long wavelength when $k<\sqrt{2A}$. When $k^2>2A$ the Benjamin-Fair instability sets in and we should see here similar effects as in the isotropic chain.

For easy axis anisotropy, i.e., when $A<0$ even $k\to 0$ waves are unstable for $q<q_C=\sqrt{2|A|}\,\mathrm{sech}\theta_0$. The discrete map analysis of Eq. (9) should reveal whether the Benjamin-Fair instability shown above, and the related Fermi-Pasta-Ulam recurrence lead to the chain chaotic behaviors.

In conclusion we have shown that the stereographic projection of the spins proposed by Lakshmanan and Nakamura can be a very useful tool in the description of nonlinear dynamics of the anisotropic Heisenberg chain and also in the formulation of the statistical mechanics for Heisenberg ferromagnets. We have illustrated above an example of the stability analysis for finite-amplitude spin waves and the mixed canonical-dissipative formulations of the dynamics.

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