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Anomalous oscillations of average transient lifetimes near crises

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Abstract

It is common that the average length of chaotic transients appearing as a consequence of crises in dynamical systems obeys a power law of scaling with the distance from the crisis point. It is, however, only a rough trend; in some cases considerable oscillations can be superimposed on it. In this Letter we report anomalous oscillations due to the intertwined structure of basins of attraction. We also present a simple geometrical model that gives an estimate of the period and amplitude of these oscillations. The results obtained within the model coincide with those yielded by computer simulations of a kicked spin model and the Hénon map. © 1999 Published by Elsevier Science B.V.

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1. Introduction

Among a large diversity of phenomena investigated in the scope of deterministic chaos in nonlinear dynamical systems one finds *crises* [1]. These are sudden changes in the structure of a chaotic attractor due to collision with an unstable periodic orbit when a system parameter p crosses some critical value p_c.

In the dynamics after crisis (for $p \gtrsim p_c$) characteristic transients appear; the system spends some time on the former (pre-crisis) attractor which is now a chaotic saddle. In the case of a *boundary crisis* the transient is followed by a definitive escape to some other attractor in the phase space while after an *interior crisis* transients are interrupted by (typically short) bursts to an extension of the pre-crisis attractor. After *attractor merging crisis* we have intermittent jumps between symmetric pre-crisis attractors. Such a behaviour is called *crisis-induced chaos–chaos intermittency*. For a large class of dynamical systems the time t that the system stays on the pre-crisis attractor has an exponential distribution $\Pr(t) = (1/T)\exp(-t/T)$ with a mean value T fulfilling a power scaling law,

$$T \sim (p - p_c)^{-\gamma}. \qquad (1)$$

For two-dimensional maps and three-dimensional flows, as well as in some cases of higher dimensional systems, the exponent $\gamma > 0$ can be expressed in terms of the eigenvalues of the periodic orbit involved in the crisis [2,3]. The power law (1) and the formulas for the critical exponent γ [2,3] have been confirmed in many numerical and experimental studies (see e.g. Refs. [4,6]). Throughout the recent years many other aspects of crises [7] as well as new

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types of crises [8] have been investigated.

However, it has also been noticed [2,9] that the power law (1) describes only the general tendency of the function $T(p-p_c)$, and one can observe some oscillations imposed on it. They can result from a ragged (fractal) measure of chaotic attractor colliding with its basin of attraction [10]; in the case of homoclinic crisis their period on a log–log plot of $T(p-p_c)$ is $|\log|\lambda_2||$, and their amplitude is large for small $|\lambda_2|$, where λ_2 is the contracting eigenvalue of the crisis orbit. These oscillations are indicated as a potential complication to verifying the scaling law (1) and determining the critical exponent γ.

In this paper we investigate another kind of oscillations that, in general, can be more pronounced, and arises from an intertwined structure of the basins of attraction [11–13]. Their particular feature is the existence of sections where the average characteristic time T grows up when moving off the crisis point, opposite to the general trend (1). Consequently, we call them *anomalous oscillations*. Oscillations of this kind have been observed in a 1D map describing a diode resonator [6] and presumably in many other systems (see e.g. Ref. [5]) but, to our knowledge, they were not explicitly pointed out. We estimate the period and calculate the amplitude of these basin-induced oscillations using a simple model of an attractor colliding with its basin boundary.

2. Crisis in a spin model, anomalous oscillations

First let us briefly introduce a spin model in which the anomalous oscillations can be explicitly observed. After Refs. [14,15], we consider a classical magnetic moment (spin) $\boldsymbol{S}, |\boldsymbol{S}| = S$ in the field of uniaxial anisotropy (z-easy/hard axis) with imposed transversal magnetic field $\tilde{B}(t)$ along the x-axis. The system can be described by the Hamiltonian $H = -A(S_z)^2 - \tilde{B}(t)S_x$, where A is the anisotropy constant. Such a model can describe a magnetic ion in a paramagnetic material or a single domain ferromagnetic sample. The motion of the spin is determined by the Landau–Lifschitz equation with damping term,

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \boldsymbol{S} \times \boldsymbol{B}_{\text{eff}} - \frac{\lambda}{S}\boldsymbol{S} \times (\boldsymbol{S} \times \boldsymbol{B}_{\text{eff}}) , \tag{2}$$

where $\boldsymbol{B}_{\text{eff}} = -\mathrm{d}H/\mathrm{d}\boldsymbol{S}$ is the effective magnetic field and $\lambda > 0$ is a damping parameter.

Taking the driving field in the form of periodic delta pulses of amplitude B and period τ: $\tilde{B}(t) = B\sum_{n=1}^{\infty}\delta(t-n\tau)$, and using the fact that $|\boldsymbol{S}|$ is constant, the equation of motion (2) can be transformed into a superposition of two 2D maps [14,15]: T_A describing the time evolution between kicks and T_B describing the effect of the kick. The complete dynamics is yielded as a composition of the two maps,

$$[S_z', \varphi'] = T_B[T_A[S_z, \varphi]] , \tag{3}$$

where φ is the angle between the axis x and the projection of the spin on the xy-plane. For different values of the parameters the system exhibits various types of dynamics including the periodic and chaotic ones [14,15] (see also Ref. [16]).

As an example, consider the crisis that occurs at $\tau_c = 2\pi$, $\lambda_c = 0.1054942$, $A_c = 1$ and $B_c = 1$ in which two symmetric chaotic attractors merge [14,15]. The attractors correspond to two Ising states (spin "up" and "down") existing in the absence of the external field. We take the amplitude of the driving field B as an accessible system parameter. For $B > B_c$ random jumps between the two, previously separate attractors can be observed.

Fig. 1 shows the dependence of the average time T between two subsequent switches on the distance $B - B_c$ from the crisis point in a log–log scale. We observe a linear trend according to (1) with the exponent $\gamma \approx 0.77$ (dashed line). This value coincides with $\gamma = 0.7703$ calculated from the formula $\gamma = (\log|\lambda_2|)/(2\log|\lambda_1\lambda_2|)$ for homoclinic crises [2] with the respective eigenvalues of the period three mediating orbit $\lambda_1 = 4.899$ and $\lambda_2 = 0.0108$ [15]. However, remarkably, roughly periodic oscillations around the trend line can also be seen. These oscillations include *anomalous* sections where T increases when moving away from the crisis point. The oscillations are due to an intertwined, fractal structure of the basins of attraction of the symmetric pre-crisis attractors. In fact, after crisis there is only one common attractor and almost all points of the phase space form its basin, but for $B \gtrsim B_c$ we can define *pseudo-basins* as sets of initial conditions evolving to the "upper" ($S_z > 0$) or the "lower" ($S_z < 0$) attractor respectively after M iterations, where $M \ll T$.

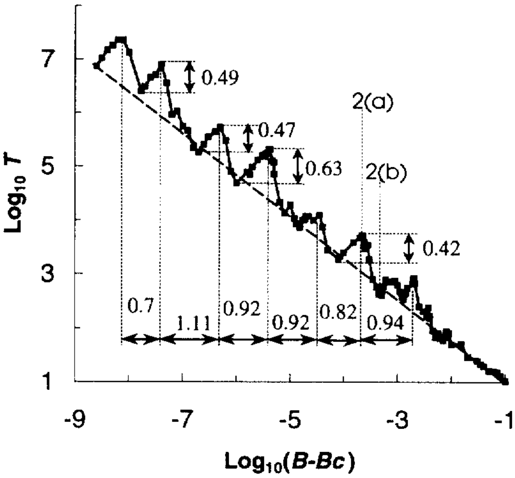

Fig. 1. The average time of residence on one of the pre-crisis attractors versus the distance from the crisis point $A_c = 1$, $\tau_c = 2\pi$, $\lambda_c = 0.1054942...$, $B_c = 1$ for the spin map (3). Standard deviation is of order or below the size of the plotted points. Two parameter values which correspond to Fig. 2 are marked.

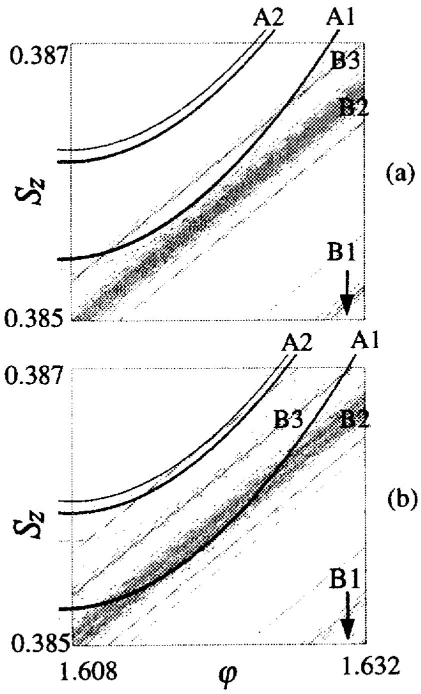

Fig. 2. Attractor of the spin map (3) creeping into the pseudo-basin of attraction of the other, symmetric attractor at $A = 1$, $\tau = 2\pi$, $\lambda = 0.1054942...$ and (a) $B = 1.0000032$, (b) $B = 1.000015$. Am ($m = 1, 2, ...$) denotes branches of the attractor, and Bm bands of the basin of attraction.

Now consider the situation depicted in Fig. 2 where a part of the attractor is plotted together with pseudo-basin of attraction (gray spots) of the other, symmetric attractor; here we took $M = 20$. The structure of pseudo-basins is much similar to the real basins before the crisis. In what follows, we shall refer to the post-crisis pseudo-basin of attraction of the other attractor as the basin of escape. The average transient time T is proportional to the inverse of the measure μ of the part of attractor overlapping with the basin of escape [3].

In Fig. 2a the branch A1 of the attractor crossed the band B3 of the basin of escape and is just before the next band B2. This corresponds to a local maximum in Fig. 1. When the parameter B is increased A1 enters the band B2, so μ grows and T decreases until A1 touches the lower edge of B2 (Fig. 2b). Then the total overlap μ begins to *decrease* and we observe an anomalous section in Fig. 1. But simultaneously another branch A2 enters the basin of escape, so the anomalous section is not as pronounced as in the "clean" case when A1 crosses B3.

If the band B3 is magnified by a factor of 0.125^{-1} a structure similar to that on Fig. 2 could be observed; the procedure could be repeated further on. This explains the fact that the mean period of oscillations in Fig. 1 is approximately $|\log_{10} 0.125| \approx 0.9$.

3. Simple model of anomalous oscillations

Motivated by the observed self-similarity we introduce a simple model of an attractor creeping into an intertwined basin of escape in order to assess the amplitude of anomalous oscillations. We define the basin in the vicinity of a collision point as a self-similar set $\mathfrak{B}$ of stripes of the width $\beta^i b_E$ accumulating at the line $y = 0$ (see Fig. 3),

$$\mathfrak{B} = \bigcup_{i=0}^{\infty} \{\{x, y\} : y > -\beta^i b \wedge y < -\beta^i b + \beta^i b_E\} . \tag{4}$$

$\beta^i b_R$ is the width of the gaps in the basin of escape that may be called "return regions". The parameters b, β, b_E and b_R fulfill the condition $b_E + b_R + \beta b = b$.

As a model attractor we take a single parabola $y = x^2 - r$. The parameter r corresponds to the distance

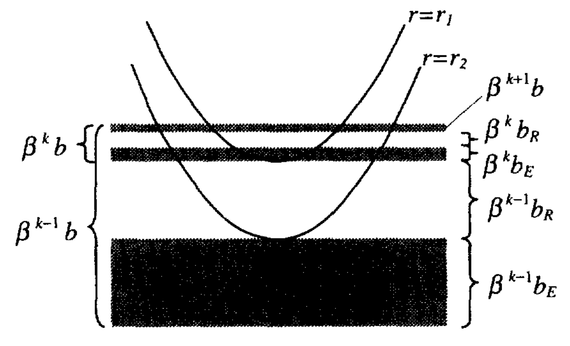

Fig. 3. Model of an intertwined basin boundary.

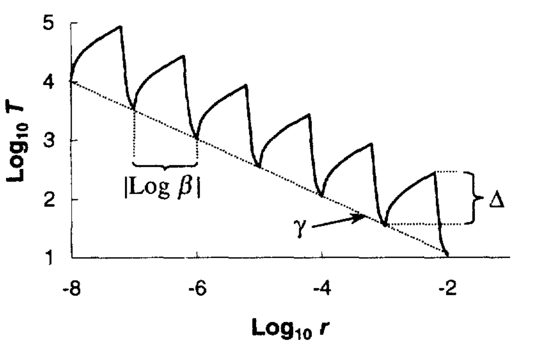

Fig. 4. Function $T(r)$ obtained when a model one-parabola attractor enters the model pseudo-basin of attraction $\mathfrak{B}$ after crisis.

$p-p_c$ from the crisis point in (1). At $r=0$ the attractor collides with the basin $\mathfrak{B}$ (crisis occurs). For $r>0$ there is a nonzero overlap of both sets. We assume the average transient time $T(r)$ proportional to the inverse of the measure $\mu(r)$ of the overlapping region (cf. Ref. [3]). Fig. 4 shows the function $T(r)$ in log–log scale obtained from the described model. Due to self-similarity of the model basin $\mathfrak{B}$, we observe regular oscillations with the period $\log\beta$ superimposed on the linear trend described by (1).

Our aim is to calculate the amplitude Δ of the anomalous oscillations measured as the difference between a maximum and the preceding minimum, see Fig. 4. Consider one kth period of the oscillation. By our definition

$$\Delta = \log T(r_2) - \log T(r_1) = \log\frac{\mu(r_1)}{\mu(r_2)}, \qquad (5)$$

where $r_1=\beta^k b$, a $r_2=\beta^k b+\beta^{k-1}b_R$ (cf. Fig. 3).

For small r, the length of parabola in the half-plane $y<0$ is proportional to $\sqrt{r}$. Using this approximation the measure of the overlap μ at points r_1 and r_2 can be written as

$$\mu(r_1) \sim \sqrt{\beta^k b_E},$$
$$\mu(r_2) \sim \sqrt{\beta^{k-1}b_R+\beta^k b_E} - \sqrt{\beta^{k-1}b_R}. \qquad (6)$$

Here we neglected all the strips with $i>k$ in (4). From (5) and (6) we get

$$\Delta = (1/\ln q)\sinh^{-1}\sqrt{\mathrm{RE}/\beta}, \qquad (7)$$

where $\mathrm{RE}=b_R/b_E$ and q is the base of the logarithm in the plot $\log T$ versus $\log r$ (in this Letter we use $q=10$).

One can also calculate the slope of the linear trend, getting $\gamma=\frac{1}{2}$ because of a 1D model-attractor. Analytical and numerical calculations considering other model-attractors show that the fractal structure of the basin of escape does not influence the slope γ.

4. Numerical examples: spin map and Hénon map

The formula (7) has been verified by comparing the calculated Δ with the amplitudes of anomalous oscillations measured on the log–log charts $T(p-p_c)$, obtained from computer simulations of the spin map (3). We investigated the intermittent dynamics near four different crisis points of the same kind as the one described above, and we made respective plots $T(B-B_c)$, similar to Fig. 1. The results are gathered in Table 1. The values of β and RE used in (7) have been measured employing a series of consecutive magnifications of the basin boundary at $B=B_c$ (similar to Fig. 2). Much the same values of β can be obtained as $10^{-\bar{\theta}}$, where $\bar{\theta}$ is the average period of anomalous oscillations measured as in Fig. 1.

Note, that the amplitudes of anomalous oscillations calculated from (7) are close to the maximal amplitudes observed in the simulations of the map (3). This can be understood noting that the maximal amplitude corresponds to a "clear" situation when a branch of attractor which may be considered as one parabola (its fractal fine structure may be ignored) leaves a band of the basin while no other branch collides with any other basin band. It is just the case considered in our model

Table 1
Amplitudes of anomalous oscillations of average characteristic times T in the vicinity of four different crisis points in the spin map (3) at $A = 1$. The average (Δ_{av}) and maximal (Δ_{max}) amplitudes are compared to the values Δ_{th} obtained from (7) with respective β and RE.

Crisis parameters			Model parameters		Amplitude of oscillations		
B	τ	λ	β	RE	Δ_{av}	Δ_{max}	Δ_{th}
1	2π	0.10549	0.125	0.5	0.502	0.63	0.627
0.92	2π	0.08964	0.08	0.95	0.7	0.8	0.847
0.92	5.8866	0.08	0.16	0.15	0.325	0.37	0.373
1.2	2π	0.1437	0.124	0.077	0.26	0.31	0.314

crisis. Conversely, if one branch of attractor leaves a basin band and simultaneously another branch enters the same or some other band, both effects can cancel and no pronounced peak is observed. Thus, (7) gives the *maximal* amplitude of anomalous oscillations.

We also studied a homoclinic boundary crisis that occurs in the Hénon map, $x_{n+1} = p - x_n^2 - Jy_n$, $y_{n+1} = x_n$ at $J_c = 0.3$ and $p_c = 2.124672450...$ [2,10]. If we make the $\log_{10}$-scale plot of $T(p - p_c)$ (here T is the length of chaotic transient before the escape to infinity, averaged over a set of initial conditions) measuring T with appropriate accuracy, and marking subsequent points dense enough we can see tiny anomalous peaks with average period 0.97 and average amplitude (defined as on Fig. 1) 0.1. These oscillations are superimposed on the linear trend (1) and, here dominating, oscillations due to the ragged measure of the chaotic attractor. A few consecutive magnifications of the fractal basin boundary gives the model parameter values RE ≈ 0.0167 and $\beta \approx 0.107$. The formula (7) then gives $\Delta = 0.168$. This value is again close to the maximal amplitude observed (≈ 0.16). Note, that the average period of anomalous oscillations is approximately $|\log_{10}\beta|$.

5. Discussion and summary

The presented model used to obtain the maximal amplitude of anomalous oscillations is a simple one. In real basins of attraction, every strip of $\mathfrak{B}$ may have its own fine structure and the chaotic attractor is locally an infinite fractal set of parabolas. In general we meet a problem of the overlap of two fractal sets. A similar question has been studied by Newhouse [18] who introduced a concept of the thickness of a fractal set and used it to formulate the condition under which two Cantor sets of Lebesgue measure zero have a nonempty overlap, persistent against small perturbation. In our case the situation is different because the basin of attraction is a set with nonzero Lebesgue measure. In addition it may also be locally a self-similar family of stripes accumulating at the tangency point, without a fractal structure.

In Ref. [17] we develop a model of chaotic attractor, similar to that of the basin of attraction, which lets us calculate the amplitude of "normal" oscillations caused by the fine structure of the attractor as well as to consider the general case when the fractal basin is penetrated by the fractal attractor.

Nevertheless, the model presented in this paper comprises the main properties of both sets responsible for the anomalous oscillations and gives a proper estimate of their maximal amplitude.

Anomalous oscillations seem to be quite common in nonlinear dynamical systems, as do the fractal, intertwined basin boundaries that give rise to them. They are, however, easy to confuse with statistical dispersion of the mean times T, especially when one has a small number of points on $T(p - p_c)$ plots. The presence of anomalous oscillations can be a complication to verify the scaling law (1), but if one is able to detect and measure them, they can give additional information on the structure of basins of attraction near crises. Note, that both Δ and β can be, in principle, assessed from the plot (like in Fig. 1) and Eq. (7) can be used to derive RE.

Concluding, we have observed anomalous oscillations of average characteristic crisis-induced transient times, and explained them in terms of a simple, geometric model, which gives the period of the oscillations and enables to calculate with a fair accuracy their maximal amplitude. The model is universal, and the results are applicable to a large class of 2D systems undergoing different types of crises. It could be also customized for some special nongeneric cases.

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