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The effect of Kapitza pendulum and price equilibrium

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Abstract

Periodical perturbations of market dynamics are analyzed using a method of time scales separation similar to the approach that is usually applied for the analysis of Kapitza pendulum. It is shown that if the perturbations are fast enough then the market oscillates around a new equilibrium price that is shifted comparing to the equilibrium price of the unperturbed system. The shift is proportional to the difference $D''(p) - S''(p)$ between the curvature of demand and supply functions. It follows that periodical perturbations will increase the equilibrium price of a typical market. Numerical simulations are in a good agreement with analytical results.

Keywords: Fast oscillations; Time scale separations; Market dynamics

1. Introduction

During the last few years there have been performed a lot of studies of economical and social systems using tools of statistical physics, nonlinear dynamics and time series analysis (for a review see Ref. [1]). Common examples are investigations of scaling phenomena [2] and volatility clustering [3] for price changes or microscopic models that are developed to understand the behaviour of agents acting at the market [4] and in social groups [5]. The paper is devoted to the problem of disturbance of market by fast periodic oscillations where time scales separation techniques well known from the classical mechanics [6] can be applied. As far as we know this kind of approach has not been used for studies of economical models.

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2. Kapitza pendulum—changing the mechanical equilibrium by fast periodic oscillations

One of classical examples of the Control System Theory is the problem of the so-called inverted pendulum [7]. The goal is to keep a mass attached to a rigid rode in the *inverted* vertical position. The task can be solved due to many *feedback* methods, i.e., by measurements of a temporary pendulum position (or measurements of corresponding forces/torques) and by introducing appropriate rode movements to balance the system. However, there exists also another, simple and spectacular method when the pendulum can be stabilized without any feedback but only due to *fast* vertical oscillations of the pendulum support [6] (see Fig. 1). In fact, using parameters as depicted in Fig. 1 and introducing an appropriate lagrangian [6] we get the following equation of motion for the angle φ describing the pendulum position:

$$ml^2\ddot{\varphi} = -U'(\varphi) + f(\varphi,t) , \tag{1}$$

where $U(\varphi) = -mgl\cos\varphi$, $f(\varphi,t) = -mla\omega^2\cos(\omega t)\sin\varphi$, a is the amplitude of the support oscillations, ω is its frequency and g is the gravitation acceleration. A natural approach is to separate time scales by writing the variable $\varphi(t)$ as the sum of a slow part $\phi(t)$ and a fast oscillating part $\xi(t)$ where $|\xi(t)| \ll |\varphi(t)|$ and expanding the r.h.s. of Eq. (1) into series of powers of the variable ξ

$$ml^2(\ddot{\phi} + \ddot{\xi}) = -U'(\phi) + f(\phi,t) - \xi U''(\phi) + \xi f'(\phi)\ldots \tag{2}$$

Now slow and fast terms can be separated and the equation for the fast variable follows as $ml^2\ddot{\xi} = f(\phi,t)$ with the solution

$$\xi(t) = (a/l)\cos(\omega t)\sin\phi . \tag{3}$$

Putting Eq. (3) into Eq. (2) and performing the time average one gets the equation for the slow variable as $ml^2\ddot{\phi} = -U'_{eff}(\phi)$ where

$$U_{eff}(\phi) = mgl\left(-\cos\phi + \frac{a^2\omega^2}{4gl}\sin^2\phi\right) . \tag{4}$$

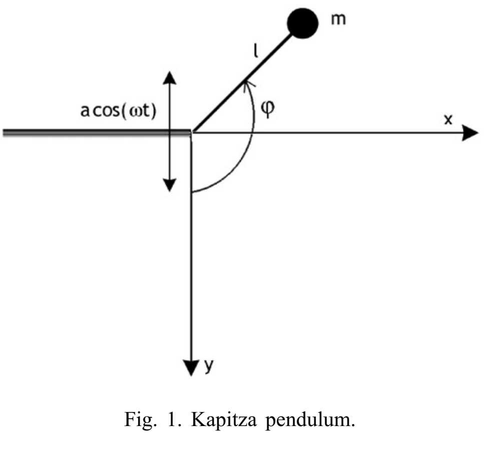

Fig. 1. Kapitza pendulum.

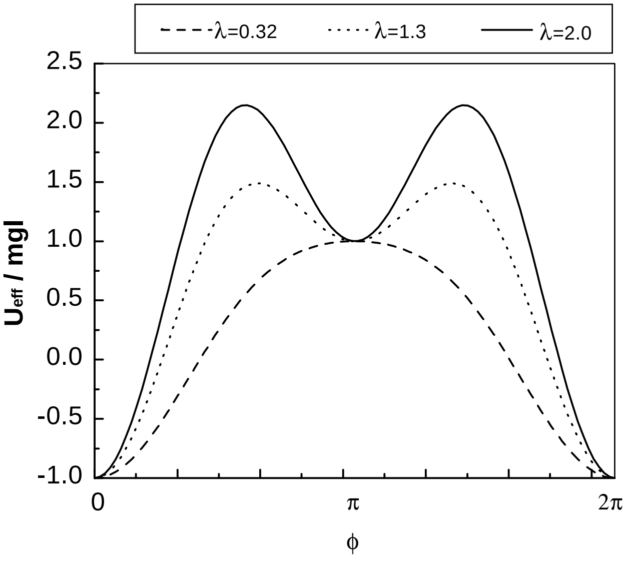

Fig. 2. The effective potential $U_{eff}(\phi)$ in the model of Kapitza pendulum for different values of the characteristic parameter λ.

One can see that if $\lambda = a^2\omega^2/(4gl) > 1/2$ then the potential $U_{eff}(\phi)$ possesses a minimum for $\phi = \pi$ (Fig. 2) that corresponds to the *inverted position* of the pendulum. It follows that as the effect of a fast oscillating force a new equilibrium position of the perturbed system can appear. One needs to stress that the potential Eq. (4) was received using the perturbation theory and the exact dynamics of the system Eq. (1) is more complex. If one increases the amplitude a then the inverted state undergoes a cascade of resurrections, i.e., it becomes stabilized after its instability, destabilizes again, and so forth ad infinitum [8].

3. Changing equilibrium of the market

Now let us consider a market with a commodity traded at the price $p(t)$. We assume that time changes of $p(t)$ are proportional to the temporal difference between the demand D and the supply S where the both functions depend on the actual price $p(t)$

$$\dot{p} = j[D(p(t)) - S(p(t))] . \tag{5}$$

If the price dynamics is described by Eq. (5) then the price tends to its equilibrium value $p = p^*$ where $D(p^*) = S(p^*)$. Now let us assume that the market is perturbed by the presence of a periodic influence that can occur due to additional time-dependent changes of supply or demand (e.g. the demand starts to depend on day time or a day of a week). For simplicity we start from the simplest case when the periodic perturbations lead to the following equation

$$\dot{p} = j[D(p(t)) - S(p(t))] + a(p)\sin(\omega t)\,. \tag{6}$$

Here the constant ω corresponds to the frequency of market perturbations while $a(p)$ describes their price dependent amplitude. We assume that the period of market perturbations $T = 2\pi/\omega$ is much small comparing to the characteristic time T_0 describing the relaxation speed of the price towards its equilibrium value p^*. Now, similarly as in the case of the Kapitza pendulum we split the price $p(t)$ into its slow/large component $\phi(t)$ and the fast/small component $\xi(t)$ and expand the r.h.s. of Eq. (6) into the series of powers of ξ. As result we get

$$\begin{aligned}\dot{\phi} + \dot{\xi} &= F(\phi) + F'(\phi)\xi + \tfrac{1}{2}F''(\phi)\xi^2 \cdots \\ &\quad + \left[a(\phi) + a'(\phi)\xi + \tfrac{1}{2}a''(\phi)\xi^2 + \cdots\right]\,\sin(\omega t)\end{aligned} \tag{7}$$

where $F(\phi) = j[D(\phi) - S(\phi)]$. Taking into account the leading perturbation term at the r.h.s. of Eq. (7) we get the following equation for the fast variable:

$$\dot{\xi} = a(\phi(t))\sin(\omega t). \tag{8}$$

Since the variable $\phi(t)$ changes in time very slowly comparing to the variable $\xi(t)$ and since the last variable should vanish for $a = 0$ we take the solution of Eq. (8) in the form

$$\xi(t) = -a(\phi(t))\omega^{-1}\cos(\omega t)\,. \tag{9}$$

Now the equation for the slow variable evolution can be received by putting the solution Eq. (9) into Eq. (7) and taking the time average. As a result we get

$$\langle \dot{p} \rangle = \dot{\phi} = j[D(\phi) - S(\phi)] + \tfrac{1}{4}j[D''(\phi) - S''(\phi)]a^2(\phi)\omega^{-2}. \tag{10}$$

It follows that the influence of fast periodic perturbations on the slow price component is dependent on *curvatures* of demand and supply functions so it disappears when both functions are linear.

It is easy to find the shift of the mean equilibrium price due to the presence of the fast perturbations. Expanding the equation for the price dynamics (10) into the series of powers around the value p^* we get

$$\dot{\phi} = j[D'(\phi) - S'(\phi)](\phi - \phi^*) + \cdots + \frac{j[D''(\phi) - S''(\phi)]a^2(\phi)}{4\omega^2}\,. \tag{11}$$

It follows that in the effect of fast oscillations the new equilibrium price $\langle p \rangle^*$ is shifted from the old equilibrium by

$$\Delta p^* = \langle p \rangle^* - p^* = \frac{a^2[D''(p) - S''(p)]}{4\omega^2[S'(p) - D'(p)]}. \tag{12}$$

Now let us assume that the demand and the supply fulfill the following relations: $D'(p) < 0$, $D''(p) > 0$, $S'(p) > 0$ and $S''(p) < 0$. The relations are frequently met

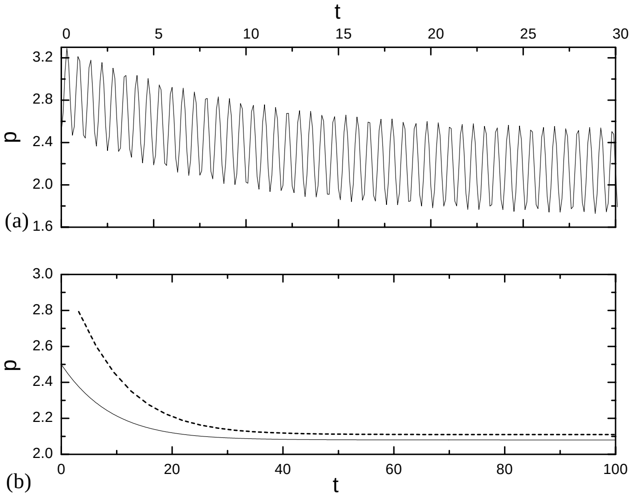

Fig. 3. Influence of fast periodic oscillations on the model Eq. (13) with parameters $A = 12$, $B = 4$, $a_2 = -1$, $b_2 = 0.5$, $j = 0.03$, $a = 4.08$ and $\omega = 10$. (a): price changes $p(t)$ for the model, (b): evolution of the slow variable $\phi(t)$ taken as the time average of $p(t)$ - dashed curve and the unperturbed price dynamics—full curve.

and mean that the demand $D(p)$ decreases slower than linearly with increasing price p while the supply $S(p)$ increases slower than linearly with the increasing price p. The situation corresponds to the presence of saturation effects in demand and supply characteristics. It follows that in such as case the effective influence of the fast variable $\xi(t)$ on the slow variable $\phi(t)$ is to *increase* the average equilibrium price $\langle p(t)\rangle$ since $\Delta p^* > 0$.

The result Eq. (12) gives the absolute value for the equilibrium price shift. Corresponding relative changes can be defined as the ratio of Δp^* to the equilibrium price p^* or to the amplitude of price changes δp_{osc} where the last value follows from Eq. (9), $\delta p_{osc} = a/\omega$.

For numerical illustration of above results we have chosen a simple characteristics for the demand and supply behaviour:

$$D(p) = A p^{a2}, \quad S(p) = B p^{b2} , \tag{13}$$

where $A > 0$, $B > 0$, $a_2 < 0$, $b_2 > 0$ are model constants. The equilibrium price is then given by

$$p^* = (A/B)^{1/(b_2 - a_2)} . \tag{14}$$

Fig. 3 shows results of numerical simulations for the model Eq. (13). One can observe that in the presence of periodic oscillations the equilibrium price $\langle p^* \rangle$ is higher (for parameters used in this simulation). Combining Eqs. (12) and (13) we get the following

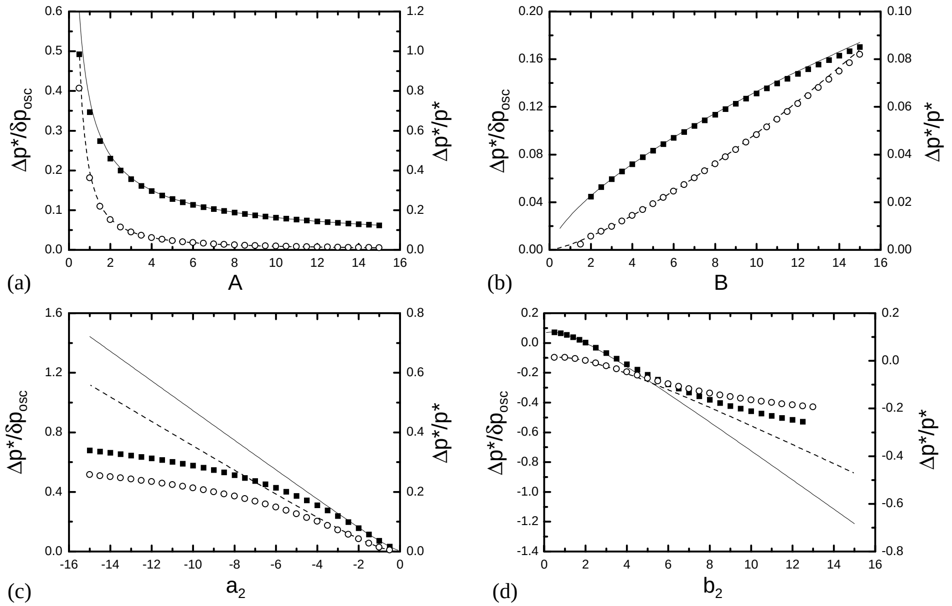

Fig. 4. (a–d) Relative changes of the equilibrium price $p*$ as functions of system parameters for the model 13. Left scale: $\Delta p^*/\delta p_{osc}$, full squares (numerical values) and full lines (theory). Right scale: $\Delta p^*/p^*$, open circles (numerical values) and dashed lines (theory). Theoretical values were obtained using Eq. (15).

relation for the price shift in the model (13):

$$\Delta p^* = \frac{a^2[Aa_2(a_2-1) - Bb_2(b_2-1)p^{b_2-a_2}]}{4\omega^2[Bb_2 p^{b_2-a_2+1} - Aa_2 p]} \ . \qquad (15)$$

Fig. 4 shows relative price changes $\Delta p^*/p^*$ and $\Delta p^*/\delta p_{osc}$ as functions of different parameters of this model. One can see that the price shift Δp^* can be comparable to the amplitude of periodic perturbations δp_{osc} and can be of order of magnitude of the original equilibrium price p^*. Negative values of price shift Δp^* observed in Fig. 4d occur for $b_2 > 1$, i.e., when $S''(p) > 0$. There is a large disagreement between analytical and numerical results observed for some regions of parameters a_2 and b_2 at Figs. 4c–d. This disagreement follows from the fact that for these values of model parameters our analytical theory (basing on the approach of fast oscillations) is invalid since the period of external oscillations T becomes comparable or larger than the characteristic relaxation time T_0 describing the speed of price relaxation towards its equilibrium value. The latter period is given by

$$T_0^{-1} = j\left\{S'(\langle p\rangle^*) - D'(\langle p\rangle^*) + \frac{a^2}{4\omega^2}[S'''(\langle p\rangle^*) - D'''(\langle p\rangle^*)]\right\} \qquad (16)$$

and it is dependent on model parameters.

In reality market perturbations are more complex and instead of one periodic force as in (6) there is a sum of many periodic contributions

$$\dot{p} = j[D(p(t)) - S(p(t))] + \sum_i c_i(p)\sin(\omega_i t + \theta_i) . \tag{17}$$

One can easily find that if the frequencies ω_i and their differences $|\omega_j - \omega_j|$ are large comparing to the system characteristic frequency $\Omega_0 = 2\pi/T_0$ then the corresponding value for the equilibrium price shift is

$$\Delta p^* = \langle p \rangle^* - p^* = \frac{D''(p) - S''(p)}{S'(p) - D'(p)} \sum_i \frac{c_i^2}{4\omega_i^2} . \tag{18}$$

4. Conclusions

In conclusion we have shown that fast periodic perturbations of the market lead to a shift of resulting equilibrium price. The value of this shift has been calculated by a method similar to the approach known from the analysis of Kapitza pendulum. Since the shift is proportional to the difference $D''(p) - S''(p)$ between the curvature of demand and supply functions thus it is positive for a typical market. Numerical simulations are in a good agreement with the analytic theory.

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References

[1] R.N. Mantegna, H.E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge, UK, 1999;
J.-P. Bouchaud, M. Potters, Theory of Financial Risk, Cambridge University Press, Cambridge, UK, 1999;
S. Moss de Oliveira, P.M.C. de Oliveira, D. Stauffer, Evolution, Money, War and Computers, Teubner, Stuttgart-Leipzig, 1999.
[2] R.N. Mantegna, H.E. Stanley, Nature 376 (1995) 46 Nature 383 (1996) 587;
S. Ghashghaie, et al., Nature 381 (1996) 767;
L.A. Nunes Amaral, et al., Phys. Rev. Lett. 80 (1998) 1385.
[3] Y. Liu, et al., Phys. Rev. E 60 (1999) 1390;
P. Gopikrishnan, et al., Phys. Rev. E 60 (1999) 5305;
P. Gopikrishnan, et al., Physica A 287 (2000) 362.
[4] J. Cremer, Physica A 246 (1977) 377;
G. Iori, Int. J. Modern Phys. C 10 (1999) 1149;
D. Chowdhury, D. Stauffer, Eur. Phys. J. B 8 (1999) 477;
T. Kaizoji, Physica A 287 (2000) 493;
D. Sornette, Physica A 284 (2000) 355;

B.M. Roehner, D. Sornette, Eur. Phys. J. B 16 (2000) 729;
S. Bornholdt, Int. J. Mod. Phys. C 12 (2001) 667;
S. Solomon, G. Weisbuch, L. De Arcangelis, N. Jan, D. Stauffer, Physica A 277 (2000) 239–247;
A. Krawiecki, J.A. Hołyst, D. Helbing, Phys. Rev. Lett. 15 (2002) 158701.
[5] K. Kacperski, J.A. Hołyst, J. Stat. Phys. 84 (1996) 169;
J.A. Hołyst, K. Kacperski, F. Schweitzer, Ann. Rev. Comput. Phys. 9 (2001) 253–273.
[6] L.D. Landau, E.M. Lifshitz, Mechanics, Pergamon, New York, 1976;
V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer, New York, 1978.
[7] A. Bradshaw, J.D. Shao, Robotica 14 (1996) 397–405;
S.J. Huang, C.L. Huang, IEEE Trans. Ind. Appl. 36 (2000) 452–458;
Y.C. Xu, M. Iwase, K. Furuta, J. Dyn. Systems Meas. Control-Trans. ASME 123 (2001) 518–527;
http://www.engin.umich.edu/group/ctm/examples/pend/invpen.html;
http://www.dynamical-systems.org/mathieu.
[8] S.T. Kim, B.B. Hu, Phys. Rev. 58 (1998) 3028–3035;
E.I. Butikov, Phys. A—Math. General 35 (2002) 6209–6231.