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Opinion formation model with strong leader and external impact: a mean field approach

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Abstract

We study a model of opinion formation based on the theory of social impact and the concept of cellular automata. The case is considered when two strong agents influence the group: a strong leader and an external social impact acting uniformly on every individual. There are two basic stationary states of the system: cluster of the leader's adherents and unification of opinions. In the deterministic limit the variation of parameters like the leader's strength or external impact can change the size of the cluster or, when they reach some critical values, make the system jump to another phase. For a certain range of parameters multistability and hysteresis phenomena are observed. In the presence of noise (social temperature) the rapid changes can be regarded as the first-order phase transitions. When both agents are in a kind of balance, a second-order transition and critical behaviour can be observed. Another kind of noise-induced transitions are the inverses (flips) of the unified group opinion due to random flips of the leader's opinion. Analytical results obtained within a mean field approximation are well reproduced in computer simulations. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Interdisciplinary research has been drawing much attention in the last decades. Models and methods developed in theoretical physics proved to be fruitful in studying

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complex systems [1,2], composed of relatively simple mutually interacting elements, coming from domains as diverge as neural networks [3–5], disease spreading [6,7], population dynamics [8,9], etc. But the range of the investigations goes also beyond the natural sciences and includes problems from sociology or economy, like e.g. pedestrian motion and traffic [10–12], migrations [13–15], financial crashes [16,17].

Another important subject of this kind is the process of opinion formation in social groups or decision making, also at the level of whole countries. One way of its quantitative description consists in a macroscopic approach based on the master equation or the Boltzmann-like equations for global variables [13,14,18–21]. Alternatively, by making some sociologically motivated assumptions on the mechanisms of interactions between individuals "microscopic" models are constructed and investigated numerically and analytically by means of the methods known from statistical physics [22–24]. One concludes that the variety of the emerging collective phenomena has much in common with the complex social processes.

One of the examples is the class of models based on the concept of cellular automata [25,26] and the theory of *social impact* formulated by Latané [27], and conformed in a number of sociological studies [28–30]. Different variants of the model were explored numerically [28,31], and many of the observations were than explained in the framework of a mean field approach [32] and recently the Landau theory [33]. An extention of the model introducing the time variance of the social strengths of individuals according to some learning rule has also been studied [34]. The essential outcome of both the theory and simulations is the onset of clusters of minority that can survive within the majority holding the opposite opinion and be persistent throughout long periods of dynamics. It has been indicated that strong individuals play an important role in the formation and stability of the clusters. Motivated by this, in [35,36] we considered a particular case of the model, namely when a strong individual (a leader) is present in the social group. Multistability and hysteresis phenomena, as well as different kinds of rapid changes (phase transitions) in the distribution of opinions were encountered.

In this paper the effects of competition between the strong leader and an *external influence* or *preference* acting homogeneously within the group are studied in terms of a mean field approach. We show that such antagonism can lead not only to sudden changes of opinion, but also to the critical behaviour. The high "social temperature" reveals the dominance of the stronger agent. After introducing the model (Section 2) and recalling some results for the deterministic case (Section 3) we investigate in detail the noise-induced transitions giving rise to the critical behaviour (Section 4.1). We also report a new kind of sudden changes in the system due to noise induced flips of the leader's opinion (Section 4.2).

2. Model of a social group

Our system consists of N individuals (members of a social group); we assume that each of them can share one of two opposite opinions on a certain subject, denoted as $\sigma_i = \pm 1$, $i = 1, 2, \ldots, N$. Individuals can influence each other, and each of them are characterised by the parameter $s_i > 0$ which describes the strength of his/her influence. Every pair of individuals (i, j) is ascribed a distance d_{ij} in a social space. The changes of opinion are determined by the *social impact* exerted on every individual:

$$I_i = -s_i\beta - \sigma_i h - \sum_{j=1,\, j\neq i}^{N} \frac{s_j \sigma_i \sigma_j}{g(d_{ij})}\,, \tag{1}$$

where $g(x)$ is an increasing function of social distance, β is a so-called self-support parameter reflecting the inclination of an individual to maintain his/her current opinion, and h is an additional (external) influence which may be regarded as a global preference towards one of the opinions stimulated by mass–media, government policy, etc.

Opinions of individuals change simultaneously (synchronous dynamics) in discrete time steps according to the rule

$$\sigma_i(t+1) = \begin{cases} \sigma_i(t) \text{ with probability } \dfrac{\exp(-I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \\[2ex] -\sigma_i(t) \text{ with probability } \dfrac{\exp(I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \end{cases} \tag{2}$$

analogous to the Glauber dynamics with $-I_i\,\sigma_i$ corresponding to the local field. The parameter T may be interpreted as a "social temperature" describing a degree of randomness in the behaviour of individuals, but also their average volatility (cf. [24]). The impact I_i is a "deterministic" force inclining the individual i to change his/her opinion when $I_i > 0$ or to keep it otherwise.

Our model social space is a 2D disc of radius R with the individuals located in the nodes of quadratic grid; the distance between nearest neighbours equals 1, and each node is occupied with probability ρ which may be also regarded as a constant surface density of individuals. The geometric distance models the social immediacy. Strength parameters s_i of the individuals are positive random numbers with probability distribution $q(s)$ and the mean value $\bar{s}$. In the centre of the disc there is a *strong individual* (whom we will call the "leader"); his/her strength s_L is much greater than that of all the others ($s_L \gg s_i$).

3. Deterministic limit

Let us first recall the properties of the system without noise, i.e. at $T=0$ (see [35,36] for details). The dynamical rule (2) becomes then strictly deterministic: $\sigma_i(t+1) = -\text{sign}(I_i\sigma_i)$. Considering the possible stationary states we find the trivial unification (with equal opinion ± 1 for each individual) or, due to the symmetry, a circular cluster of individuals sharing the opinion of the leader surrounded by a ring of their opponents (the majority). These states remain stationary also for small self-support parameter β; for sufficiently large β any configuration may remain "frozen".

Using the approximation of continuous distribution of individuals (i.e. replacing the sum in (1) by an integral) we have calculated the size of the cluster, i.e. its radius a as a function of the other parameters. In the case of $g(r) = r$, $\rho = 1$ and $\bar{s} = 1$ we got from the limiting condition for the stationarity $I = 0$ at the border of the cluster

$$a \approx \frac{1}{16}[2\pi R - \sqrt{\pi} \pm \beta - h \pm \sqrt{(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L}]\,. \tag{3}$$

This is an approximate solution valid for $a \ll R$, but it captures all the qualitative features of the exact one which can be obtained by solving a transcendent equation. Here and in the next section we assume that the leader's opinion $\sigma_L = +1$, but the analysis is also valid for the opposite case if $h \to -h$.

The branch with the "−" sign in front of the square root corresponds to the stable cluster and the one with "+" to the unstable solution separating basins of attraction of the stable cluster and unification. Owing to the two possible signs at the self-support parameter β in (3), the stable and unstable solutions are split and form in fact two bands. The states within the bands are "frozen" due to the self-support which may be regarded as an analogy of the dry friction in mechanical systems. In this way also the unstable clusters can be observed at $\beta > 0$ and appropriately chosen initial conditions.

Eq. (3) has real solutions corresponding to clusters provided

$$(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L \geqslant 0\,. \tag{4}$$

Otherwise, the general acceptance of the leader's opinion (unification) is the only stable state. When, having a stable cluster, condition (4) is violated by changing a parameter, e.g. s_L or h one can observe a discontinuous phase transition: *cluster* → *unification.*

If, on the other hand, the leader is too weak it may be impossible for him not only to form a cluster but also to maintain his/her own opinion. The limiting condition for the minimal leader's strength $s_{L\min}$ to resist against the persuasive impact of the majority can be calculated from the limiting condition $I_L = 0$ (I_L – the impact exerted on the leader):

$$s_{L\min} = \frac{1}{\beta}(2\pi R - \sqrt{\pi} - h)\,. \tag{5}$$

So far we considered the stability condition at the border of the cluster. However, for sufficiently large external impact h it may happen that the individuals at the border of the whole group (at the distance R from the leader) begin to convert their opinions. Again, from the limiting condition $I = 0$ we get the critical value of h at which it happens causing the boundary-induced transition:

$$|h_c| = 4R - \sqrt{\pi} + \beta \pm \frac{s_L}{R}. \tag{6}$$

The sign "−" before s_L applies when the external impact is positive and favours the leader's opinion, then the transition *cluster* → *unification* occurs. The "+" corresponds

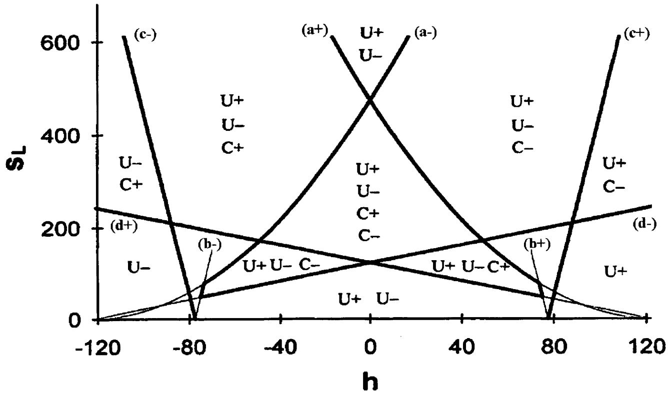

Fig. 1. $s_L - h$ phase diagram. Thick lines limit the areas where different phases are possible: "U+" – uniform opinion +1, "U−" – uniform opinion −1, "C+" – cluster of +1 adherents, "C−" – cluster of −1 adherents. The lines were obtained from limiting conditions for: (a) – stability at the border of the cluster Eq. (4), (b) – stability at the border of the group with the external impact favouring the leader Eq. (6), (c) – stability at the border of the group with the external impact against the leader Eq. (6), (d) – minimal leader's strength $s_{L\min}$ to resist against the persuasive impact of the majority Eq. (5). Parameter values: $R=20$, $\beta=1$, $\rho=1$, $\bar{s}=1$, $h=0$. The choice $\beta=\bar{s}$ means that the individual's own opinion is as important as that of his/her nearest neighbour.

to the opposite case [i.e. when $\mathrm{sign}(h) = -\mathrm{sign}(\sigma_L)$]; then we have the transition *unification* → *cluster* or, when the strength of the leader is below the minimal (5), the transition *unification* → *unification*, i.e. the inversion (flip) of opinions.

Taking into account conditions (4), (5) and(6) one can make a phase diagram $h - s_L$ distinguishing the regions where different states of the system are possible (Fig. 1). It is apparent that the system shows *multistability* in certain range of s_L and h. It depends on the history which of the states is realized, so we come to a *hysteresis* phenomenon (Fig. 2). Moving in the parameter space $s_L - h$ and starting from different configurations one can have many possible scenarios of phase transitions. Let us consider one of them (Fig. 2), namely when we go along the line $s_L = 300$ in Fig. 1. Assume that we start with a cluster $C+$ around the leader of the opinion +1 at $h = 0$. The increase of the external impact h causes growth of the cluster up to a point on curve (a+) in Fig. 1 where the transition to unification $U+$ occurs. Further increase of h will certainly not cause any more changes, but if its sign is reversed we can reach another critical point (line (c−) in Fig. 1) at which the transition: *unification* ($U+$) → *cluster* ($C+$) takes place. When we now go with h back to zero we come to our original state. But if the negative impact is increased beyond (d+), the leader alone is not able to sustain the persuasive impact and we have the transition *cluster* ($C+$) → *unification* ($U-$).

The approximation of the sum in (1) by an integral remains valid as far as the density of individuals is big enough regarding the size of the system so that $\rho \quad a^2 \gg 1$. Taking large R and small ρ we have the individuals randomly distributed in 2D space instead of somewhat artificial square grid geometry. This approximation implies also a complete rotational symmetry of the system. In the case of a square lattice the

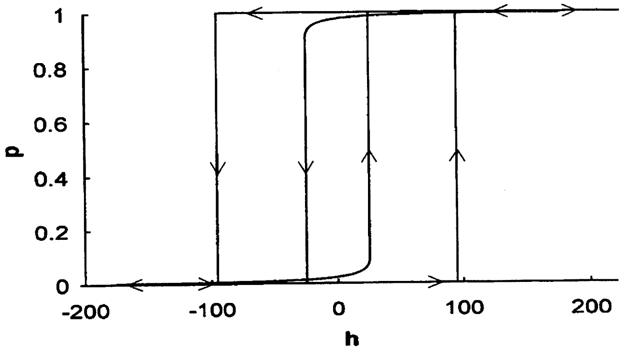

Fig. 2. Hysteresis in the dynamics of the system. The curves have been obtained from (3) for $s_L = 300$. Other parameter values as in Fig. 1.

symmetry is reduced to the 4-fold axis, and in effect the actually observed clusters are not exactly circular but rather square (smaller clusters) or octagonal (larger clusters). We did not observe other polygons but one can anticipate that they can appear for model parameters allowing still larger clusters.

Qualitatively, similar results were obtained for the mutual interactions of shorter range, namely when $g(r) = r^n$, $n = 2, 3, 4$. In this case, however, an individual mostly "feels" his/her nearest neighbours and their exact position becomes important. For this reason the results obtained within the approximation do not fit perfectly to those from computer simulations.

4. Noisy dynamics

In the presence of noise the marginal stability of unstable clusters due to the self-support is suppressed and they are no longer the stationary states of the system. Borders of the stable clusters become diluted, i.e. individuals of both opinions appear all over the group. Considering the dynamics (2) we can conclude that the influence of noise on a single individual depends on the ratio I_i/T. Because of the strong influence of the leader, the supportive (negative) impact is stronger inside the cluster than outside it. Thus, due to noise-induced flips of opinions the adherents of the leader appear more often outside the cluster (among the majority of opponents) than the opponents inside it. Moreover, the area outside is much greater than that of the cluster itself, so we observe the effective growth of the minority group. This causes that the supportive impact outside the cluster becomes still weaker and the majority becomes more sensitive to random changes; it is a kind of positive feedback. At a certain value of temperature the process becomes avalanche like and the former majority disappears. Thus noise induces a jump from one attractor (cluster) to another (unification). Such a transition is possible at every non-zero temperature but its probability remains very small until the noise level exceeds a certain critical value. Our simulations prove that it is indeed a well-defined temperature that separates two phases (i.e. two attractors). Similarly, the transition unification → cluster in the presence of external impact can be induced by noise.

4.1. Mean field approximation. Phase transitions, critical dynamics

We can derive analytically the stationary states of the system an nonzero temperature using a kind of mean field approximation. To do this let us calculate the impact on an individual at a distance x from the leader, and sharing the opposite opinion:

$$I(x) = \frac{s_L}{g(x)} + \rho\bar{s}\int_{D_R} Pr(r)\frac{1}{g(|\mathbf{r}-\mathbf{x}|)}\mathrm{d}^2\mathbf{r} - \rho\bar{s}\int_{D_R}(1-Pr(r))\frac{1}{g(|\mathbf{r}-\mathbf{x}|)}\mathrm{d}^2\mathbf{r} - \beta\bar{s} + h\,. \quad (7)$$

The integration is performed over the whole space D_R excluding the individual under consideration. $Pr(r)$ denotes the probability of finding a leader's follower at the distance r from the centre of the group; it is in fact determined by Eq. (2) from which it follows that it depends on the actual state of the system in the previous time step. We would like, however, to have a stationary function $I(x)$. This can be achieved relatively easy if we neglect the self-support term. Let us put $\beta = 0$ in the subsequent calculations and discuss the influence of nonzero self-support later.

Now we make an approximation replacing $Pr(r)$ by its stationary mean value p over D_R and putting it outside the integral. This is equivalent to the simple assumption that $Pr(r)$ is uniform. We can expect it to be valid for large temperatures when the dynamics is almost random, or for small leader's strength s_L (because it is in fact the large value of s_L that contributes most to the nonuniformity of $Pr(r)$). From (7) we get

$$I(x) = \frac{s_L}{g(x)} + (2p-1)\rho\bar{s}J_D(x) + h, \quad (8)$$

where $J_D(x) = \int_{D_R} 1/g(|\mathbf{r}-\mathbf{x}|)\,\mathrm{d}^2\mathbf{r}$ is a function dependent only on the size of the group and type of interactions. Note that the impact on an individual sharing the opinion of the leader would be the same as in (8) but with the opposite sign. Due to this fact we can easily derive the expression for the stationary probability $Pr(r)$ from the dynamical rule (2) which gives the transition probabilities. Then from the definition of the mean value p (the mean part of individuals sharing the opinion of the leader) it follows

$$p = \frac{1}{\pi R^2\rho}\int_0^R \rho\,\mathrm{Pr}(r)2\pi r\,\mathrm{d}r = \frac{1}{R^2}\int_0^R \frac{\exp[I(r,p)/T]}{\cosh[I(r,p)/T]}r\,\mathrm{d}r \equiv f(p), \quad (9)$$

where $I(x,p)$ is given by (8). This is an integral equation for p. Note that in the above derivation we set the leader's opinion fixed, independent of the influence of the group and the noise.

In Fig. 3 one can see the graphical solution of the Eq. (9) for certain set of parameters and $g(r) = r$. At low temperatures there are three solutions: the smallest one

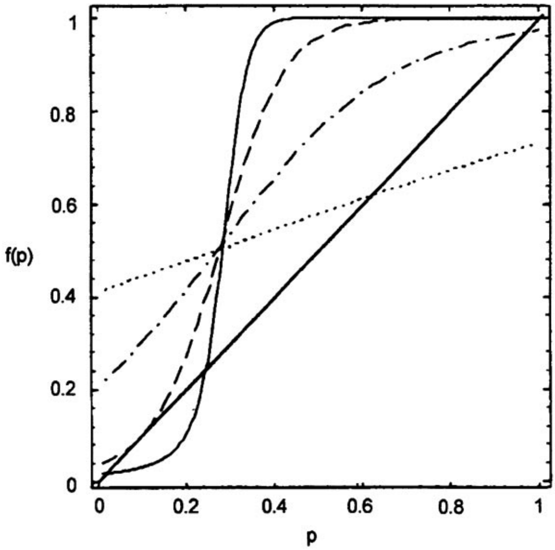

Fig. 3. Numerical solution of Eq. (9) for $s_L = 250$ and $h = 25$ (other parameters as in Fig. 1). $f(p)$ is the RHS of (9) plotted for different temperatures: solid line - $T = 10$, dashed - $T = 28$, dotted–dashed - $T = 80$, dotted - $T = 300$.

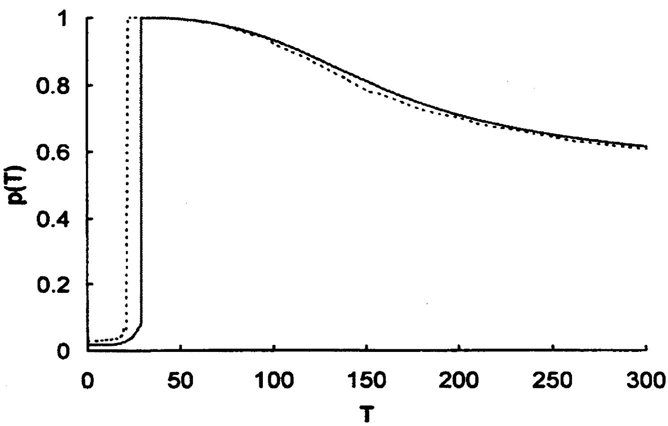

Fig. 4. Fraction p of leader's followers vs. temperature; $s_L = 400$, $h = 0$. Results of our calculations are represented by a solid line and those of computer simulations by a dotted line. Other parameter values as in Fig. 1.

corresponding to the stable cluster around the leader, the second – to the unstable cluster which, in fact, is not observed, and the biggest – to the unification. The size of the stable cluster grows with increasing the temperature up to a critical value T_{tr} when it coincides with the unstable solution. At this temperature a transition from a stable cluster to unification occurs (Fig. 4). For $T > T_{tr}$ unification is the only solution, but it is no longer a perfect unification because due to the noise individuals of the opposite opinion appear. When the temperature goes on growing the curve in Fig. 3 becomes more flat and p tends to $\frac{1}{2}$ what means that the dynamics is random and both opinions appear with equal probability.

Fig. 4 shows the function $p(T)$ compared with the results of computer simulations. The phase transition mentioned above can be observed. The analytical curve fits the results of simulations quite well, particularly at large temperatures for the reasons mentioned above. Other mean field approximations, e.g. the one used by us in [35] yield better results at low noise levels, specifically for the value of the transition temperature. They consist in taking another function (instead of constant) to replace $Pr(r)$ in (7) (e.g. a stepwise in [35]). In fact, the definition (2) can be used with I given by (8); this would be a "second-order" extension of our simple approach. Higher-order approaches can be constructed in a similar way; they would give more accurate results, however, for the prise of more complex equation (9) for p.

Let us thus remain at the "first-order" equation (9). Changing the external impact and/or the leader's strength results in a shift of the curves $f(p)$ along the $p-$axis. If we apply a large enough external impact against ($h<0$) the leader the curves $f(p)$ in Fig. 3 would be shifted to the right so that now, starting from the uniform opinion $p=1$ and exceeding a critical temperature we observe the transition *unification* → *cluster* which remains the only solution at high noise level.

At a certain value of $h=h_b<0$ the influences of the leader and the external impact are in a way balanced and the curve $f(p)$ becomes roughly symmetric with respect to the bisector p. Let us set the condition for the balance as $f(\frac{1}{2})=\frac{1}{2}$. With the use of Eqs. (9) and (8) it can be written as

$$\frac{1}{R^2}\int_0^R \frac{\exp[(\frac{s_L}{g(x)}+h_b)/T]}{\cosh[(\frac{s_L}{g(x)}+h_b)/T]}x\,\mathrm{d}x=\frac{1}{2}\,, \tag{10}$$

which gives implicitly h_b as a function of other parameters.

With the increase of temperature at $h=h_b$ (maintained by appropriate changes of s_L) the two stable solutions p_1 and p_2 (corresponding to cluster and unification, respectively) converge towards $p=\frac{1}{2}$ (the third, unstable solution). At some critical noise level T_c the solutions coincide and we have only $p=\frac{1}{2}$ which now becomes stable. The condition for this critical temperature can be written as $\frac{\mathrm{d}f(p)}{\mathrm{d}p}|_{p=1/2}=1$. Again using Eqs. (9) and (8) we get an implicit integral equation for T_c:

$$\frac{2}{T_cR^2}\rho\bar{s}\int_0^R \frac{J(x)}{\cosh^2[(\frac{s_L}{g(x)}+h_b)/T_c]}x\,\mathrm{d}x=1 \tag{11}$$

with h_b given by (10).

In the case of $g(x)=x$ we found that the curve $f(p)$ is almost exactly symmetric with respect to the point $f(\frac{1}{2})$ and thus $p_2\approx 1-p_1$. Moreover, the balanced external impact h_b depends only weakly on the noise level, and the critical temperature T_c is almost independent of the leader's strength s_L.

Fig. 5 shows the temperature of the described above transitions *cluster* → *unification* (for $h>h_b$) and *unification* → *cluster* (for $h<h_b$) as a function of h. Both curves

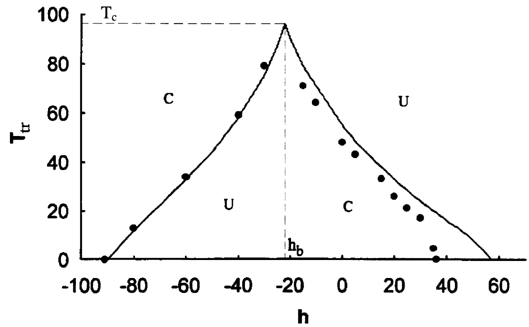

Fig. 5. Transition temperature T_{tr} vs. external impact h at $s_L = 250$ (other parameter values as in Fig. 1). Leader's opinion was fixed (independent of the group). Line corresponds to analytical results (Eq. (9)), points to the computer simulations of the model.

meet at the critical point $h = h_b$, $T = T_c$. When moving along both curves towards the critical point the magnitude of jump in the majority–minority proportion due to transition decreases and at (h_b, T_c) there is no jump at all.

In analogy to physical systems the transitions occurring while crossing the curves in Fig. 5 from below may be called first-order transitions. When the critical point (h_b, T_c), at which the difference between two different phases disappear, is crossed going from the region below the curves a second-order transition occurs. An example of the dynamics in the neighbourhood of this kind of transition is shown in Fig. 6. As the solutions p_1 and p_2 are approaching $\frac{1}{2}$ increasing fluctuations around them can be observed (Fig. 6(a)). At T close to T_c, noise-induced random jumps between p_1 and p_2 are possible (Fig. 6(b)). With $T \to T_c$ the average frequency of jumps increases (Fig. 6(c)), and at $T = T_c$ we observe the dynamics with $p = \frac{1}{2}$ and large fluctuations (Fig. 6(d)). When the temperature is further increased the amplitude of fluctuations decreases (Fig. 6(e)). Large fluctuations in the vicinity of the critical point are a general characteristic feature of critical phenomena in physical systems.

The above described second-order transition through critical point can also be observed in the absence of a leader and the external impact, i.e. when $s_L \approx \bar{s}$ and $h = 0$. Then the two phases corresponding to unifications $U+$ and $U-$ which are stable in low temperatures merge giving rise to the high-temperature phase – at average equal numbers of individuals sharing both opinions. In this case the model reduces to the Ising model with long-range interactions if we additionally put $s_i = \bar{s}$ for every i.

It should also be mentioned that the fluctuations in the vicinity of the critical point are relatively slow, e.g. in our simulations significant changes of p happen during a few tens of time steps. This phenomenon known as the critical slowing down is caused by local correlations of opinions. In the temperatures saliently greater than T_c the correlations disappear and we observe fast random fluctuations around $p = \frac{1}{2}$.

The second-order transition occurs also in the opposite direction, i.e. when the temperature is decreased starting from the $p = \frac{1}{2}$ phase at $T > T_c$. At the critical point there

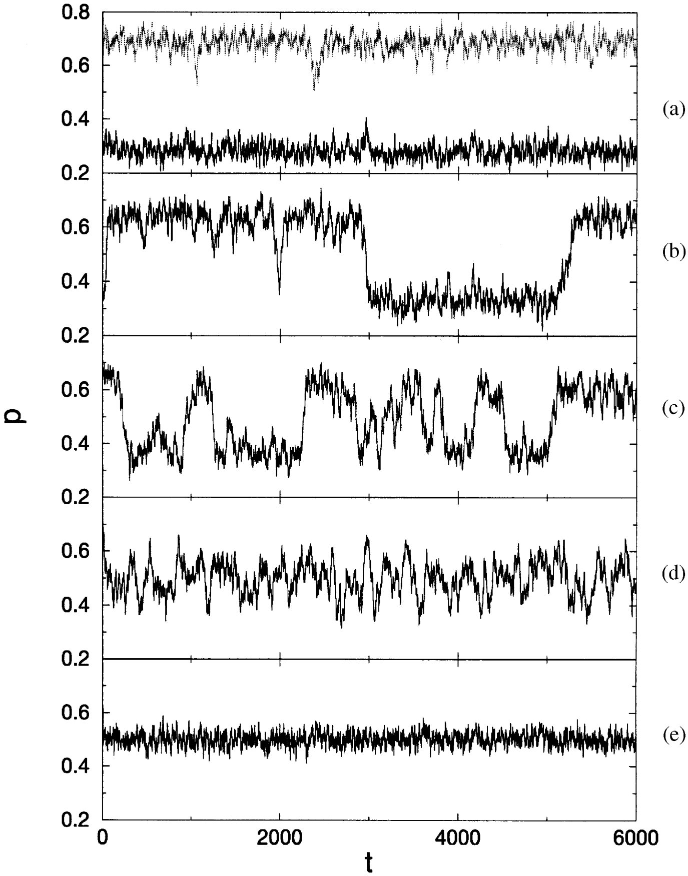

Fig. 6. Time evolution of the number of leader's followers in the vicinity of the critical point, $h=24.65 \approx h_b$, temperatures: (a) $T=94$, (b) $T=97$, (c) $T=99$, (d) $T=102 \approx T_c$, (e) $T=120$; other parameter values as in Fig. 1. The opinion of the leader has been fixed. Fig. (a) shows the results of two runs starting from initial conditions $p(0)>0.5$ (upper curve) and $p(0)<0.5$ (lower curve).

is a symmetry breaking; the choice between two symmetric phases depends sensitively on tiny deviations from the balanced external impact h_b and random fluctuations.

One could also describe the global dynamics of the system introducing an effective potential. At low temperatures it would have two minima corresponding to the stable solutions of Eq. (9) separated by a maximum corresponding to the unstable solution. For $h=h_b$ both minima would be approximately equal, but otherwise the one corresponding to the dominance of the stronger agent would be global while the other the local one with the corresponding state being *metastable*. However, as we have already mentioned, the probability of escape from the metastable state is very small for $T<T_{tr}$.

In the analytical calculations of this section we neglected the self-supportiveness of individuals putting $\beta=0$. Let us now briefly discuss the effect of a nonzero β. As it

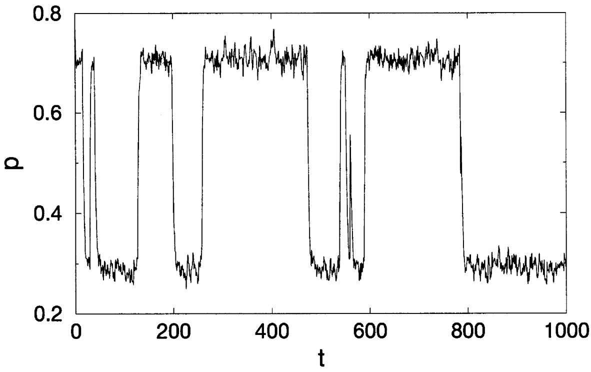

Fig. 7. Time evolution of the number of individuals holding "+1" opinion for $h=0$ and $T=150>T_{tr}$; other parameter values as in Fig. 1. Jumps due to the noise-induced flips of the leader's opinion can be seen.

was already mentioned, the self-support may be regarded as an analogy of dry friction in mechanical systems. It strives to maintain the current state of individuals. Writing formula (8) for the impact we assumed a stationary state and did not regard how it was achieved. We could do it, as well as to use the formula for the probability in (9) which is independent of the current state only in the case of $\beta=0$. Due to nonzero self-support the mean (stationary) value of the proportion p of the leader's adherents depends on the initial state. One can regard that the curve in Fig. 3 splits into a band and so do the appropriate solutions of the Eq. (9). It depends on the starting point on which side of the band the system settles. However, at high-temperatures noise-induced random walk within the band will finally make the system forget its initial conditions and the mean value for $\beta=0$ will be observed. Accordingly, the first-order transition occurs when the band looses tangency to the bisector and thus the temperature of transition will be higher than for $\beta=0$. We can also expect that the critical behaviour will be observed in a broader range of temperatures. Nevertheless, for small β (like e.g. $\beta=1$ in our simulations) the influence of the self-support is negligible and the results of our mean field theory remain valid.

4.2. *Flips of leader's opinion*

So far throughout this section we have kept the opinion of the leader fixed, independent of the impact and noise. But if we let it evolve according to (2), as for any other individual, another kind of rapid changes of opinion distribution are possible due to the fact that in the presence of noise there is a finite probability that the leader would inverse his/her opinion. It is more significant for large temperatures, so let us consider the situation above T_{tr}. When the leader changes his/her opinion, the whole majority group follows the change and we observe random global flips of opinion. Fig. 7 shows an example of such dynamics in the case when there is no external impact ($h=0$) and both phases "+1" and "−1" are symmetric. In this case the average residence times in each of them are equal. When external impact is present it favours one of the phases and we have two different average residence times $\overline{\tau_+}$ and $\overline{\tau_-}$. These averages are determined by the inverse of the probability $Pr(\sigma_L \to -\sigma_L)$ that the leader's opinion flips,

$$\bar{\tau} = \frac{1}{Pr(\sigma_L \to -\sigma_L)} . \tag{12}$$

The probability is given by the general dynamical rule (2) applied for the leader. The impact on the leader can be calculated as [cf. Eq. (7)]

$$I_L = -\rho \ \bar{s} \int_{D_R} Pr(r) \frac{1}{g(r)} \rho \, d^2\mathbf{r} + \rho \bar{s} \int_{D_R} (1 - Pr(r)) \frac{1}{g(r)} d^2\mathbf{r} - \beta s_L \pm h \, . \tag{13}$$

The signs "+" and "−" before h appear when the leader shares the opinion deterred or supported by the external impact, respectively. We apply again the simple mean field approximation replacing $Pr(r)$ by its mean value p which is the appropriate solution of the nonlinear equation (9) and we get

$$I_L = \rho \bar{s} J(0)(1 - 2p) - \beta s_L \pm h \, , \tag{14}$$

and the average residence times

$$\overline{\tau_{\mp}} = \frac{2\cosh(I_L(T)/T)}{\exp(I_L(T)/T)} \, . \tag{15}$$

The impact I_L acting on the leader depends on temperature through p. However, in the case of long-range interactions $g(x) = x$ one can see from Fig. 4 that for $T \gtrsim T_{tr}$ the dependence is weak so one can neglect it putting $I_L(T) \approx I_L(T_{tr})$ in (15) which is approximately $I_L(p = 1)$, provided we are not close to the critical point. Finally, we get

$$\overline{\tau_{\mp}} \approx \exp\left(\frac{-2\rho \bar{s} J(0) + \beta s_L \pm h)}{T} \right) . \tag{16}$$

The plot of (15) in appropriately chosen coordinates (Fig. 8) proves that the approximation (16) remains valid in quite large range of temperatures. Results of analytical calculations in the framework of the mean field approximation are in good coincidence with the average times between flips measured in computer simulations. Notwithstanding, for smaller temperatures we observe that the residence times actually measured in the simulation are slightly larger than those predicted by (15). One may expect that the divergence would grow for still lower temperatures because the probability distribution of opinions becomes more nonuniform, in contrast to our approximation $Pr(r) = p$ in (13). Another words, the cluster around the leader becomes tighter and shields the influence of noise.

Assuming the probabilities of the leader's opinion flips in consecutive time steps to be independent we find that the residence times have the geometrical probability distribution

$$Pr(\tau_{\pm} = k) = \frac{1}{\overline{\tau_{\pm}}} \left(1 - \frac{1}{\overline{\tau_{\pm}}} \right)^{k-1} . \tag{17}$$

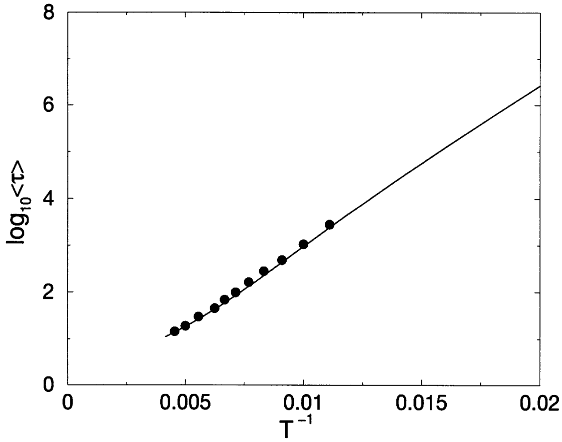

Fig. 8. The average time between flips as a function of the inverse temperature for $h = 0 (T_{tr}^{-1} \approx 0.039)$ calculated from (15) (solid line); other parameter values as in Fig. 1. Points indicate the appropriate values measured in computer simulations.

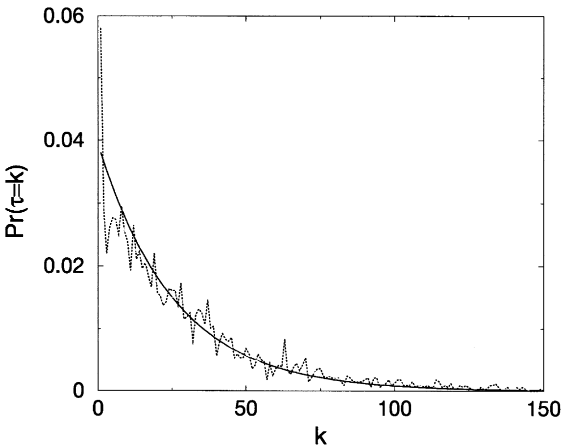

Fig. 9. The histogram of times between flips for $T = 180$ and $h = 0$ (dotted line) measured in a 10^5 time steps long computer simulation. Other parameter values as in Fig. 1. The mean value $\bar{\tau} = 26.3$. The dotted line shows the geometrical distribution (17) for this value.

Its plot is shown in Fig. 9 compared with the results of computer simulation of the model.

5. Summary

Let us summarize the outcomes of our analysis. The influence of two agents on the group: the strong leader and an external impact gives rise to multistability in certain ranges of parameters, hysteresis and discontinuous changes in the distribution of opinions in the deterministic case. The situation looks similar in the presence of noise which models the complexity and indeterminism of the process of opinion formation at the level of a particular individual. The noise level ("social temperature") is an additional transition inducing parameter. We have shown that the noise favours the stronger agent; With growing temperature it needs smaller prevalence to convert the majority to its opinion. In the case of a balance (symmetry) between the two agents the appropriate phase transition becomes continuous and we observe characteristic critical behaviour (large fluctuations, critical slowing down).

The described phenomena can be understood and described quantitatively in terms of a mean field approach. In general, a hierarchy of approximations can be constructed, but our calculations and computer simulations of the model show that already in the first order we get quite good quantitative results.

The model presented here, though presumably not directly applicable to the description of a real social group, may still be useful in explaining the mechanisms underlying often very complicated social processes. Being aware of their complexity we are circumspectful in drawing far reaching sociological conclusions from our results. Let us point out just one general outcome which seems reasonable: the rapid changes of opinions under the influence of strong leaders and some general preferences or prejudices are more probable during the times of big social transformations or turbulence, when people are much confused in their views (high social temperature). Then the opinion supported by the stronger agent prevails and may remain dominant later on in a more quiet period, even after the influence of the agent decreases or even ceases to exist.

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