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Phase transitions as a persistent feature of groups with leaders in models of opinion formation

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Abstract

A class of models of opinion formation based on the concept of cellular automata and the theory of social impact is studied, in particular the case when a strong leader and external impact are present. The rapid changes of the opinion distribution with a continuous change of a system parameter, which was previously observed for the model with geometric structure, prove to be present also for much larger class of mutual interaction architectures. We study random connections with different probability distributions. The theoretical results obtained in the framework of mean field approximation are confirmed by the numerical simulations of the model. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Social impact model; Opinion formation; Phase transition; Critical phenomena; Mean field approximation

1. Introduction

Quantitative methods in social science are gaining a growing popularity recently [1–16]. The developed models not only give a new insight into the complex social phenomena, but are often interesting examples of systems exhibiting complex behaviour. The model of opinion formation studied in this paper is based on the sociological theory of social impact introduced by Latané [17–19]. It gave rise to a class of models providing quantitative description of interpersonal influences underlying the process of opinion formation [20]. A few basic concepts constitute the crucial link

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between sociological and physical description. Let us briefly recall these concepts (see Ref. [21] for details).

Two-state opinion is assumed, i.e., we restrict to questions on which only two opposite answers are possible (yes–no, for–against). The opinion of an individual is determined by the *social impact* – a kind of force pushing towards the particular judgement, and being a sum of influences from all other individuals, as well as the influence of mass-media, prejudices, etc. The strength of influence of an individual j on i (the contribution to the impact I_i) is proportional to the social *immediacy* from i to j, and the persuasiveness (relevant for convincing others to change their mind) or supportiveness (ability to support others who share the same opinion) of j. There is, in general, some randomness in the model accounting for the fact that decision-making or opinion formation is a very complicated process influenced not only by the "external world" but also personal experience, mental processes, etc., factors that are extremely difficult to model in a deterministic way.

In the simulations of social impact models [20] it has been usually assumed that individuals are placed in a 2D Euclidian space which models the social space, and the distance between them (inverse of the immediacy) is just the geometrical distance or an increasing function of it. There was some empirical evidence justifying this assumption, and we also assumed it in our previous models with strong leader and external field [22–24]. Here we generalise the results obtained there for the particular geometry of social space. Instead of assuming that the individuals are placed in a geometrical 2D space we now consider the immediacies to be random numbers with some probability distribution. Such an approach is more realistic; actually the social distance does not have to fulfill the axioms of geometric distance, e.g. the triangle inequality. While an individual i is close to j and j is close to k, it can happen that i is far from k or does not know him/her at all so there is no direct exchange of information between them (think e.g. about the set of your friends). Similarly, mutual symmetry of the immediacy, especially between individuals of different personal influence or position in hierarchy of the group can be broken.

It turns out that in such a randomly connected model, the main features of the "geometric" model with a strong leader, namely the rapid changes of majority–minority proportion with continuous changes of system parameters ("social phase transitions") are preserved. They are, to large extent, independent of the particular type of social connections or the distribution of strength of individuals – the parameters that might be very difficult to assess for real social groups.

2. Model

The model system consists of N elements (individuals) described by constant in time strengths s_i and the state variables $\sigma_i = \pm 1$. Every individual j influences the opinion of other individual i with a magnitude proportional to their immediacy m_{ij}; it does not need to be equal to m_{ji}. The values of all σ_i are updated synchronously in discrete time steps due to the dynamical rule

$$\sigma_i(t+1) = \begin{cases} 1 & \text{with probability } \dfrac{\exp(I_i/T)}{\exp(-I_i/T)+\exp(I_i/T)}, \\ -1 & \text{with probability } \dfrac{\exp(-I_i/T)}{\exp(-I_i/T)+\exp(I_i/T)}, \end{cases} \tag{1}$$

where I_i is a social impact experienced by the ith individual and T is the social temperature – the degree of randomness in decision-making. The impact is defined as

$$I_i = \sum_{j=1}^{N} s_j m_{ij} \sigma_j + h, \tag{2}$$

where h is an external impact acting uniformly on all individuals (generalisations to non-uniform external impacts are of course possible) and favouring one of the opinions; it may describe e.g. the influence of mass-media, general prejudices, etc. The quantity introduced here relates to the one defined in Refs. [22–24] via the simple transformation $I_i \to -\sigma_i I_i$. Further on we assume the strength parameters s_i to be random variables with the mean value $\bar{s}$; one of the individuals, however, has the strength s_L much larger than all the others, $s_L \gg s_i \ \forall i$; we will call this individual the *leader*. The immediacies are also assumed to be random variables with the distribution $\rho_m(m_{ij})$ with the mean value $\bar{m}$. The diagonal terms m_{ii} describe the self-supports of individuals, i.e., a kind of social inertia – inclination to preserve their present opinion.

3. Stationary states: mean field approximation

3.1. Noiseless limit

In the limit $T \to 0$ we get from (1) a deterministic rule

$$\sigma_i(t+1) = \mathrm{sign}[I_i(t)]. \tag{3}$$

Typically, starting from a random initial distribution of opinions the system will evolve to a stationary state which depends of course on the particular realisation of the connections m_{ij}, strength parameters s_j, and the initial states $\sigma_i(0)$. We are interested in the question: how does the average number pN of the leader's adherents depend on the other system parameters?

The only element breaking the symmetry (in the stochastic sense) of the system is the leader. To make it simpler we shall keep his/her opinion constant, which is equivalent to putting the self-supportiveness of the leader m_{LL} very large (infinite). To be specific we take $\sigma_L = 1$, the opposite case is symmetric. One can expect that the individuals with large immediacy m_{iL} to the leader will share his/her opinion while those who do not feel his/her strong influence may hold the opposite opinion. For the individuals with moderate immediacy to the leader, the influences of the leader and his/her adherents approximately compensate with that of their opponents so that the impact on them is close to zero. These individuals change their opinions in the course of time evolution until Eq. (3) gives no more changes. Following this intuition we make an approximation assuming that in the stationary state there exists some m such that

$$\sigma_i = 1 \quad \forall i\colon m_{iL} > m, \quad \text{and} \quad \sigma_i = -1 \quad \forall i\colon m_{iL} < m \tag{4}$$

and consequently the impact on the individuals is positive and negative respectively, while the impact on the individuals whose immediacy to the leader equals m is equal to zero. The average of latter quantity over the quenched disorder (m_{ij} and s_i) and the possible realisations of σ_i can be calculated for large N:

$$I(m) = s_L m + \left\langle \sum_{i \neq L} s_j m_{ij} \sigma_j \right\rangle + h = s_L m - (1 - 2p) N \bar{m}\bar{s} + h\,, \tag{5}$$

where p is the fraction of individuals sharing the opinion of the leader. It can be, on the other hand, also expressed as

$$p = \mathrm{P}(m_{ij} > m) = \int_m^{m_{\max}} \rho_m(m')\,\mathrm{d}m'\,. \tag{6}$$

From the condition $I(m) = 0$ we get using (5)

$$p = \frac{1}{2} - \frac{h}{2N\bar{m}\bar{s}} - \frac{s_L}{2N\bar{m}\bar{s}} m\,. \tag{7}$$

Resolving (6) and (7) one can get the somehow artificially introduced parameter m and the fraction p of leader's adherents which is actually the interesting quantity. Note that in the absence of external impact ($h = 0$) the impact defined by (5) is always positive for $p > \frac{1}{2}$, so the condition $I(m) = 0$ cannot be fulfilled. It means that the approximation does not capture the trivial unification state $p = 1$. Similarly, the case $p = 0$ is not obtained as a solution of (6) and (7).

The RHS of Eq. (6) is a continuous (but not necessary differentiable) function of m decreasing from 1 at $m = m_{\min}$ to 0 at $m = m_{\max}$. If the immediacy distribution ρ_m is continuous and differentiable the number of possible solutions depends on its monotonicity. If ρ_m is monotonous, at most two solutions are possible. If it has one maximum or minimum there may be three solutions. In general, the maximal number of solutions equals $2 + k$ where k is the total number of maxima and minima of the ρ_m.

3.2. Noisy dynamics

Eq. (1) defines a discrete Markov process $\sigma(t)$. Instead of deriving its probability distribution we are again interested in the question how, starting from some initial distribution of opinions, the number of individuals pN sharing the opinion of the leader will evolve. As in the noiseless case, we shall assume m_{LL} very large (infinite) in order to keep the opinion of the leader constant, specifically we take $\sigma_L = 1$. The only element breaking the symmetry of the system is the leader. One may, therefore, order the individuals according to their immediacy to the leader m_{iL} and express the fraction of the leader's adherents as

$$p = \int_{m_{\min}}^{m_{\max}} \mathrm{P}(\sigma_i = \sigma_L | m_{iL} = m')\rho_m(m')\,\mathrm{d}m' . \tag{8}$$

Having the opinion of the leader fixed, the conditional probability under the integral is given directly by the dynamical rule (1) with the impact (2) depending on the current state of the system. In order to proceed we make a mean field approximation and replace the actual impact from all individuals except the leader by its mean value over the quenched disorder (m_{ij} and s_i) and the actual state σ_i. The impact on an individual i whose immediacy to the leader is $m_{iL} = m$ is then given by (5). In the stationary state, i.e., when p remains constant we obtain from (8), (1) and (5) an equation for this equilibrium value:

$$p = \int_{m_{\min}}^{m_{\max}} \frac{\exp[(s_L m' - (1-2p)N\bar{m}\bar{s} + h)/T]}{2\cosh[(s_L m' - (1-2p)N\bar{m}\bar{s} + h)/T]}\rho_m(m')\,\mathrm{d}m' \equiv f(p) . \tag{9}$$

Given a specific immediacy distribution ρ_m the solutions can be found at least numerically. The RHS of (9), $f(p)$, is a positive and increasing function of p; $f(0) \geqslant 0$ and for $p \to 1$ it tends to a value smaller than 1 so Eq. (9) has at least one solution. At some parameter values $f(p)$ may be tangent to the diagonal at some p^*; then a pair of solutions appears or disappears according to the direction of parameter changes. These points correspond to phase transitions – rapid changes of the number of leader's adherents. The conditions for the occurrence of such points are

$$p^* = f(p^*), \quad \frac{\mathrm{d}}{\mathrm{d}p} f(p)|_{p=p^*} = 1 . \tag{10}$$

In the following we shall analyse a few particular examples of immediacy distributions.

3.3. Examples – noise-free limit

3.3.1. Uniform immediacy distribution

Consider the uniform distribution of immediacies $\rho_m(m_{ij}) = 1/(2\bar{m})$ for $m_{ij} \in [0, 2\bar{m}]$ and zero elsewhere. Inserting this into (6) we get

$$p = 1 - \frac{1}{2\bar{m}} m . \tag{11}$$

The set of Eqs. (11) and (7) can give one solution

$$p_u = \frac{1 - 2\tilde{s}_L - \tilde{h}}{2(1 - \tilde{s}_L)} , \tag{12}$$

where rescaled variables

$$\tilde{s}_L = s_L/(N\bar{s}), \quad \tilde{h} = h/(N\bar{s}\bar{m}) \tag{13}$$

have been introduced. Additionally, the conditions $p_u > 0$ and $p_u < 1$ have to be imposed leading to a phase diagram (Fig. 1) distinguishing regions in $\tilde{s}_L$–$\tilde{h}$ space where different final states are possible.

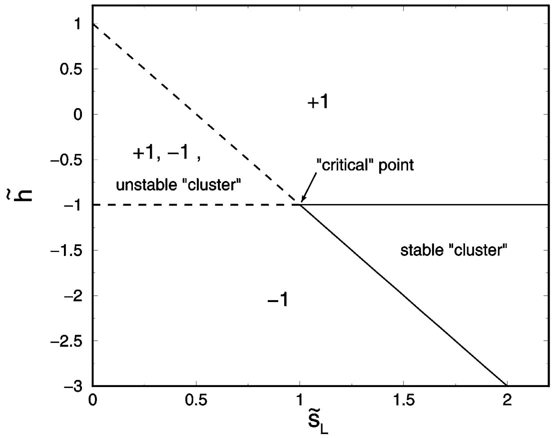


Fig. 1. $\tilde{s}_L$–$\tilde{h}$ phase diagram for uniform distribution of immediacies within the mean field approximation. The possible phases are indicated in the corresponding parts of the diagram. When crossing the dashed lines discontinuous phase transitions can occur.

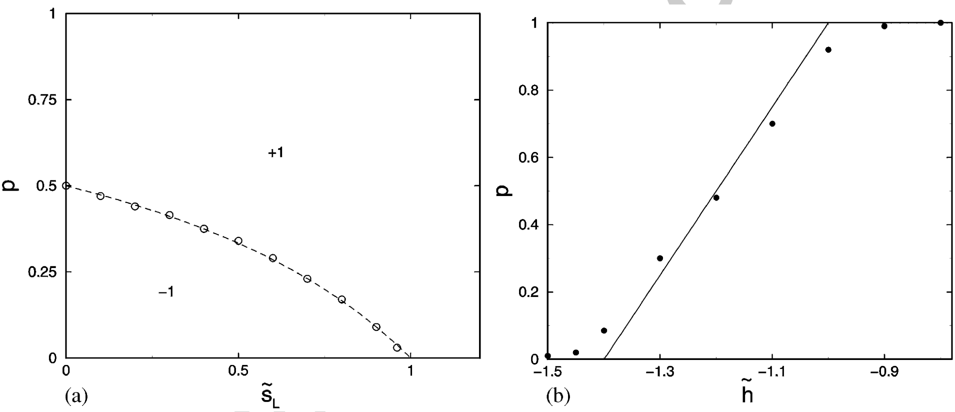


Fig. 2. Unstable (a) and stable (b) solution (12) of mean field equations for uniform immediacy distribution. In (b) the final states obtained in computer simulations of the model for $N = 200$, $s_L = 240$, $\bar{m} = 0.1$ and averaged over quenched disorder and different initial conditions are displayed as points. In (a) the open circles indicate the initial states p_0 (at $h = 0$) for which the final state was on average 0.5 (cf. Fig. 3(a)), so they can be considered as the separatrix between the states $+1$ and 1.

In the upper left triangle we have the solution p_u corresponding to the unstable state (cluster) separating the stable unification states $p = 0$ and 1, i.e., starting from an initial state with $p_0 < p_u$ ($p_0 > p_u$), the system, on average, evolves to unification $p = 0$ ($p = 1$) (see Fig. 2(a)). We use here the notion "cluster" for the group of the adherents of the leader by analogy to the "geometric" model [22–24], although now they are only relatively close to the leader, but not necessary to one another, since the social network does not have a geometric structure. The unifications themselves are not captured by Eq. (7) since it assumes balance of impacts ($I = 0$) which is not

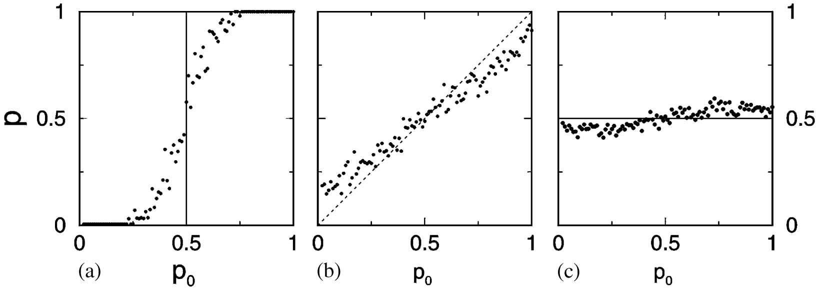


Fig. 3. Fraction of leader's adherents p in the final stationary state as a function of its initial value p_0 for the uniform immediacy distribution: (a) $\tilde{s}_L = 0.75$, $\tilde{h} = 0.75$, (b) $\tilde{s}_L = 1$, $\tilde{h} = 1$ (critical point), (c) $\tilde{s}_L = 1.25$, $\tilde{h} = 1.25$. The points have been obtained in simulations for $N = 200$, $\bar{m} = 0.1$ by averaging over the quenched disorder and initial conditions. Solid line in (a) shows stable states +1 and 1 separated by the unstable solution $p_u = 1/2$, in (c) the stable solution $p_u = 1/2$ (cf. Eq. (12)). The dashed line in (b) corresponds to the typical behaviour at the critical point.

the case when all individuals share the same opinion. If we start from a $+1$ (-1) unification state in the upper left triangle in Fig. 1 and move out of it by changing $\tilde{s}_L$ or/and $\tilde{h}$ to the -1 $(+1)$ region, we observe a discontinuous phase transition $+1 \to -1$ $(-1 \to +1)$ crossing the border of the triangle. In the lower right wedge solution (12) corresponds to a stable cluster. It converges to 1 (-1) on the borders with the $+1$ (-1) regions, respectively. Thus, crossing the wedge, e.g. upward, the state of the system changes continuously from -1 to $+1$ via the stable solution (12) (see Fig. 2(b)). The point $(\tilde{s}_L = 1,\ \tilde{h} = -1)$ corresponds to the case when the lines (7) and (11) coincide, so formally infinitely many solutions exist. It is a kind of "cold" double critical point where every state is marginally stable so the evolution of the system is hard to predict. Typically, due to non-zero self-support $(m_{ii} > 0)$ any initial state remains stationary. The qualitative change of the stationary state when crossing the critical point is illustrated in Fig. 3. The predictions of the mean field approximation are in a good agreement with the results of computer simulations. Fig. 2 shows examples of the $p_u(s_L)$ and $p_u(h)$ curves from the regions of unstable and stable solutions, respectively.

3.3.2. *Exponential immediacy distribution*

As the second example we take the exponential distribution of immediacies $\rho_m(m_{ij} = m) = (1/\bar{m})\exp(-m/\bar{m})$ for $m_{ij} > 0$. It means that every individual has relatively few "close friends" with high immediacies and many "acquaintances" who influence him/her only weakly. Similarly as in the previous example we can substitute the distribution into (6), analyse the solutions of Eqs. (7) and (6) taking into account the unification states and the condition $p \in [0, 1]$, and obtain a $\tilde{s}_L$–$\tilde{h}$ phase diagram (Fig. 4).

There are at most two non-trivial solutions corresponding to the stable cluster and the unstable cluster separating the stable one from the $+1$ unification (the triangular region). When the upper curve $\tilde{h} = \tilde{s}_L \ln(\tilde{s}_L/2) + 1 - \tilde{s}_L$ (dotted line) is reached both solutions collide and disappear (saddle-node bifurcation); if the system was in

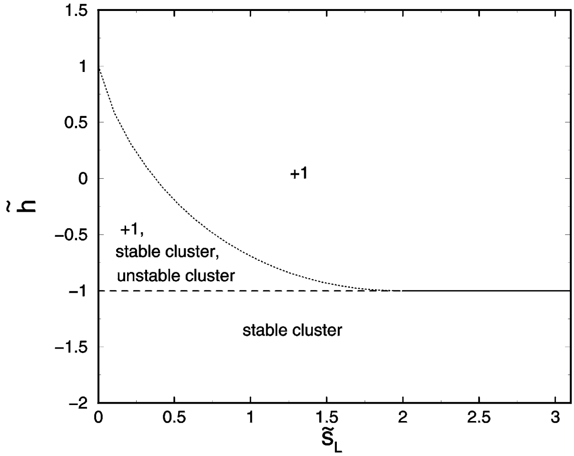


Fig. 4. $\tilde{s}_L$–$\tilde{h}$ phase diagram for exponential distribution of immediacies within the mean field approximation.

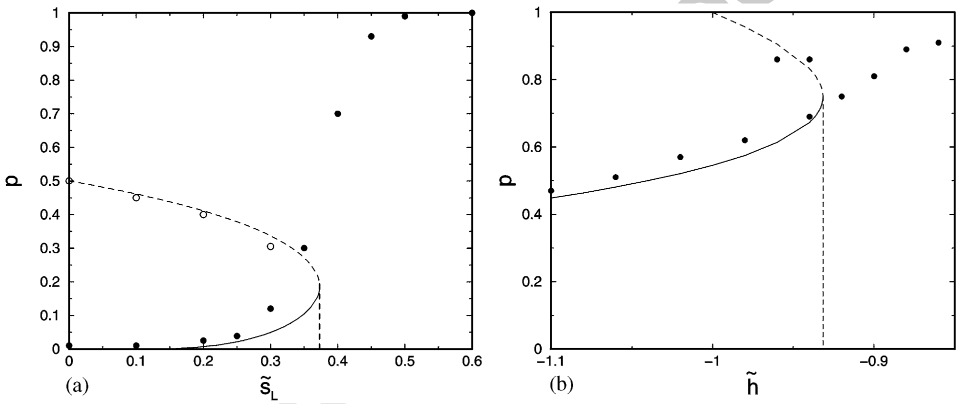


Fig. 5. Numerical solution of the mean field equations for exponential immediacy distribution for $\tilde{h} = 0$ (a) and $\tilde{s}_L = 1.2$ (b), solid lines – stable clusters, dashed lines – unstable clusters. Points correspond to simulations as in Fig. 2. Dashed lines indicate the border of basins of attraction of the stable cluster and $+1$ unification.

the state of stable cluster a discontinuous phase transition to $+1$ unification occurs (cf. Fig. 5(a)).

The line $\tilde{h} = -1$ for $\tilde{s}_L < 2$ indicates the place where the unstable cluster collides with the $+1$ unification and thus the stable cluster becomes the only stationary state for $\tilde{h} < -1$ (see Fig. 5(b)). If the system was in the $+1$ unification we observe the phase transition unification $\rightarrow$ stable cluster. For $\tilde{s}_L > 2$ the stable solution $p \rightarrow 1$ as $\tilde{h} \rightarrow -1^{(-)}$ so the transition stable cluster $\rightarrow$ unification in this case is smooth (see Fig. 5(b)). The point ($\tilde{s}_L = 2$, $\tilde{h} = -1$) is again a kind of critical point where the first-order phase transitions become the second order. Now, however, in contrast to the "double critical point" from the previous example the $+1$ unification is the only

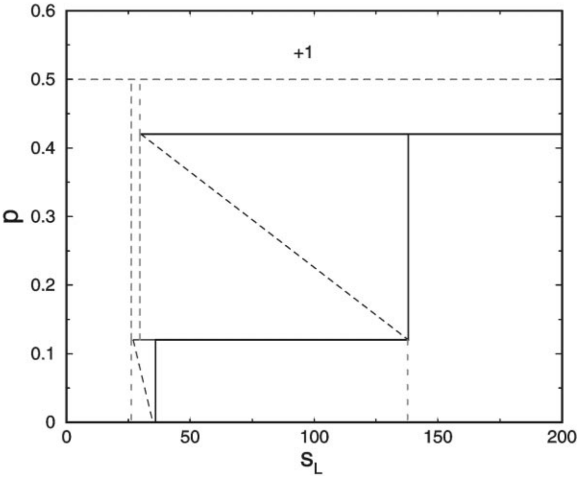


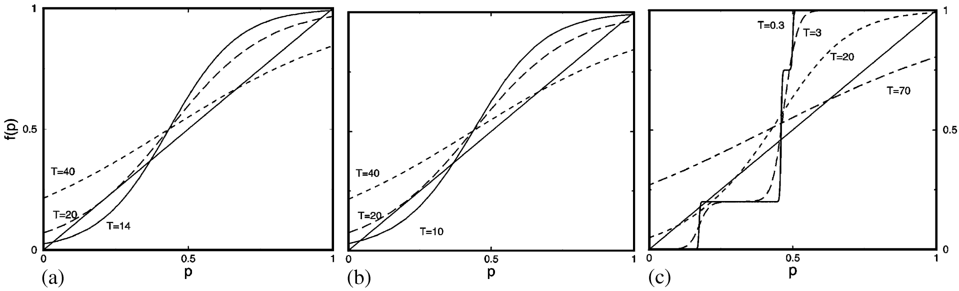
Fig. 6. Fraction of leader's adherents as a function of the leader's strength for discrete multimodal immediacy distribution: $N = 200$, $h = 0$, $m_1 = 0.8$, $m_2 = 0.1$, $a_1 = 0.2$, $a_1 = 0.55$; solid lines – stable clusters, dashed lines – unstable clusters (basin boundaries).

marginally stable state while all the others are unstable so the system would evolve towards $p = +1$ although the rate of convergence drops to zero as $p \to +1$. As in the previous case, the analytic results obtained within the mean field approximation are well reproduced in computer simulations. However, for the region in Fig. 5 where bistability appears, "interference" between the stable states is observed.

3.3.3. Discrete multimodal immediacy distribution

Let us finally consider a discrete distribution of immediacies assuming that each individual i has on average a_1N "close friends" – whose immediacy to him/her is $m_{ij} = m_1$, and a larger group of a_2N "acquaintances" ($a_2 > a_1$) with respective immediacy $m_{ij} = m_2$ ($m_2 < m_1$) who influence him/her relatively weakly. The rest of the $(1 - a_1 - a_2)N$ individuals in the group do not influence i directly at all. The distribution can be written as $\rho_m(m_{ij}) = (1 - a_1 - a_2)\delta(m_{ij}) + a_1\delta(m_{ij} - m_1) + a_2\delta(m_{ij} - m_2)$. It can be also considered as a limit case of a continuous distribution of immediacies with two local maxima, or an extremely reduced real situation where we have actually a set of discrete immediacies rather than a continuous spectrum since the group has a finite size. We can follow the same way as previously and obtain a phase diagram, now however, it would be more complicated since more parameters come into play. In general, two pairs of stable and unstable solutions are possible, leading to different transition scenarios as parameters are changed. Let us show just one example (Fig. 6).

In the absence of external impact, starting from $p \approx 0$ (leader alone), as the strength of the leader is increased first the group of "close friends" is convinced causing a discontinuous transition to small cluster. Further the stable cluster solution collides with an unstable one and disappears in a saddle-node bifurcation, and a transition to



(a) (b) (c)

Fig. 7. Numerical solutions of Eq. (9) for uniform (a), exponential (b), and discrete multimodal (c) immediacy distribution: $N = 200$, $h = 0$, $\bar{m} = 0.1$, $s_L = 40$ (a,b); $m_1 = 0.8$, $m_2 = 0.1$, $a_1 = 0.2$, $a_1 = 0.55$, $s_L = 35$ (c).

large cluster is observed. Since the rest of the group do not feel directly the influence of the strong leader, no more changes appear with the increase of s_L at $T = 0$. For other values of parameters different transition scenarios are possible.

3.4. Examples – noisy dynamics

Inserting the appropriate immediacy distributions into (9) we get the equation for the fraction p of the leader's adherents in stationary states in the presence of noise; it can be solved numerically. Now, the unification states are also included since no assumption on impact balance has been used in the derivation of Eq. (9). Fig. 7 shows some examples of the numerical solutions for the distributions defined above.

One can see that the sigmoidal shape of the function $f(p)$ in (9) for $T > 0$ is similar for all the cases. An increase of the parameters s_L and h essentially shifts the curves to the right, a decrease – to the left. An increase of temperature makes them more flat. These changes may lead to disappearance of pairs of stable and unstable solutions and to discontinuous phase transitions. One example is shown in Fig. 8. Eq. (9) gives us the dependence $p(T)$ as well as the transition temperature using conditions (10). Both quantities are in a good agreement with respective averaged values obtained in computer simulations (see Figs. 8 and 9).

Note moreover, that for all the examples shown in Fig. 7 the transition temperatures calculated for quite different immediacy distributions, but with (nearly) the same mean value $\bar{m}$ and number of individuals N are quite similar ($T_{tr} \approx 20$), which indicates that the actual distribution $\rho_m(m)$ in (9) does not influence much the properties of the function $f(p)$. One can conclude that in the presence of noise, at least in appropriately high temperatures, the average behaviour of the system is universal, only weakly dependent on a particular distribution of connections in the social network.

Finally, let us point out that, similarly as in the "geometric" case [24] in high temperature the phase favoured by the stronger agent, the leader or external impact, is realized. For each s_L one can find a "balanced" value of external impact h_b for which the curve $f(p)$ becomes (almost) symmetric with respect to the diagonal

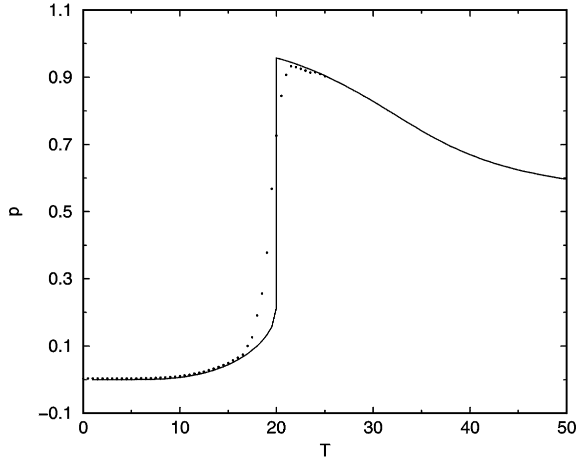


Fig. 8. Average fraction of leader's adherents vs. the noise level for uniform distribution, $N = 300$, $s_L = 40$, $h = 0$. Solid line is the numerical solution of Eq. (9), points – results of computer simulations. Rapid transition: stable cluster $\rightarrow$ (nearly) uniformity can be seen at $T = 20$.

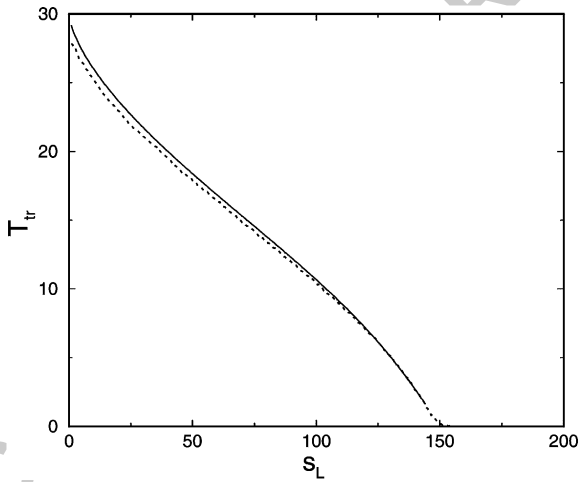


Fig. 9. Transition temperature vs. the leader's strength obtained from (9) for uniform distribution: $N=300$, $h = 0$ (solid line). The dotted line shows the results of related computer simulations. Starting from a cluster state the temperature was increased quasistatically by 0.2, T_{tr} was defined as the temperature at which the transition occurred within less than 200 time steps. The value was averaged over 100 runs with different realisations of s_i, m_{ij}, and initial conditions $\sigma_i(0)$.

(cf. Fig. 7) when the stable solutions converge symmetrically to the unstable one $p = 1/2$ as the temperature grows, and collide at a critical temperature T_c. Then the first-order (discontinuous) phase transitions become second order and typical critical behaviour is observed.

4. Conclusions

We have studied a class of models of opinion formation with random mutual connections (immediacies), which seems to be a realistic assumption from the point of view of real social systems. Despite qualitatively different phase diagrams in the noise-free limit, the models exhibit discontinuous phase transitions – rapid jumps in the majority–minority proportion for a large class of immediacy distributions. The behaviour becomes still more universal in the presence of noise. The phase transitions are a characteristic and persistent feature of systems with strong individuals – leaders and/or external impact, they may be considered as an extreme case of a "staircase dynamics" [21] caused by the quenched disorder. We have been studying the stationary states of the system for some parameter values changed "by hand". In real systems such changes may occur naturally due to different socio-historical processes and lead to rapid (revolutionary) changes of attitudes.

The models capture just one aspect of the process of opinion formation, namely the cooperative phenomenon due to instantaneous mutual influence of individuals. There are of course a number of other factors that play a role in the process like the historical experience and memory, influence of particular events, etc. They might be, to some extent, modelled through the appropriately time-dependent external impact, but many of such factors seem to be very difficult to quantify. In many systems these factors can be in fact crucial and completely conceal the mechanism treated by the model, so the model alone is not relevant for such situation. The process of opinion formation is a complicated one and the theory of social impact describes only one of its aspects. Models of social impact correspond to an imaginary situation when only the impact with some degree of noise determines the opinion and no other factors are involved. Nevertheless, the models supply a tool for description of an important mechanism inducing changes of attitudes – a cooperative phenomenon, which could be difficult to explain in the traditional descriptive sociological language.

References

[1] H. Haken, Synergetics. An Introduction, Springer, Heidelberg, New York, 1983.
[2] H. Haken, Advanced Synergetics, Springer, Heidelberg, New York, 1983.
[3] W. Weidlich, G. Haag, Concepts and Models of Quantitatively Sociology, Springer, Berlin, 1983.
[4] W. Weidlich, Phys. Reports 204 (1991) 1.
[5] D. Helbing, Physica A 193 (1993) 241.
[6] D. Helbing, J. Math. Sociol. 19 (1994) 189.
[7] D. Helbing, Quantitative Sociodynamics, Kluwer Academic, Dordrecht, 1995.
[8] R.R. Vallacher, A. Nowak (Eds.), Dynamical Systems in Social Psychology, Academic Press, San Diego, 1994.
[9] R.R. Vallacher, A. Nowak (Eds.), Dynamical Social Psychology, The Guilford Press, New York, 1998.
[10] N. Gilbert, J. Doran (Eds.), Simulating Societies, UCL Press, London, 1994.
[11] J. Fort, V. Méndez, Phys. Rev. Lett. 82 (1999) 867.
[12] W. Weidlich, J. Math. Sociol. 18 (1994) 267.
[13] S. Galam, Physica A 230 (1996) 174.
[14] S. Galam, Physica A 238 (1997) 66.

[15] E. Tarnow, Behav. Sci. 41 (1996) 136.
[16] D.B. Bahr, E. Passerini, J. Math. Sociol. 23 (1998) 1, 29.
[17] B. Latané, Am. Psychol. 36 (1981) 343.
[18] E.L. Fink, J. Commun. 46 (1996) 4.
[19] B. Latané, J. Commun. 46 (1996) 13.
[20] A. Nowak, J. Szamrej, B. Latané, Psychol. Rev. 97 (1990) 362.
[21] M. Lewenstein, A. Nowak, B. Latané, Phys. Rev. A 45 (1992) 763.
[22] K. Kacperski, J.A. Hołyst, J. Stat. Phys. 84 (1996) 169.
[23] K. Kacperski, J.A. Hołyst, Leaders and clusters in a social impact model of opinion formation: the case of external impact, in: F. Schweitzer (Ed.), Self-Organization of Complex Structures: From Individual to Collective Dynamics, Vol. II, Gordon and Breach, Amsterdam, 1997.
[24] K. Kacperski, J.A. Hołyst, Physica A 269 (1999) 511.