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# Phase transitions in social impact models of opinion formation

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**Abstract**

We study phase transitions in models of opinion formation which are based on the social impact theory. Two different models are discussed: (i) a cellular-automata-based model of a finite group with a strong leader where persons can change their opinions but not their spatial positions, and (ii) a model with persons treated as active Brownian particles interacting via a communication field. In the first model, two stable phases are possible: a cluster around the leader, and a state of social unification. The transition into the second state occurs for a large leader strength and/or for a high level of social noise. In the second model, we find three stable phases, which correspond either to a "paramagnetic" phase (for high noise and strong diffusion), a "ferromagnetic" phase (for small nose and weak diffusion), or a phase with spatially separated "domains" (for intermediate conditions). 

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## 1. Introduction

During the last years there has been a great interest in applications of statistical physics in social science [1–3]. Usually, economical models are studied using the

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techniques of stochastic dynamics [4], percolation theory [5] or the chaos paradigm [6]. Another important subject of this kind is the process of *opinion formation* treated as a collective phenomenon. On the "macroscopic" level, it can be described using the master equation or Boltzmann-like equations for global variables [7–12], but microscopic models are constructed and investigated as well [13–15] using standard methods of statistical physics. A quantitative approach to the dynamics of opinion formation is related to the concept of *social impact* [16–22], which enables to apply the methods similar to the cellular automata approach [14,23].

The aim of this work is to study various kinds of phase transitions in two models based on the social impact theory. In Section 2 we consider phase transitions in a social impact model that can occur in a finite group in the presence of a *strong individual* (a leader) [24–26]. As two special cases, we discuss a purely deterministic limit and a noisy model. Section 3 is devoted to an extension of social impact models to include phenomena of migration, memory effects and a finite velocity of information exchange. Here the concepts of *active Brownian particles* [27–29] and the communication field [30] will be applied.

## 2. Phase transitions in the presence of a strong leader

### 2.1. The model

Our system consists of $N$ individuals (members of a social group); we assume that each of them can share one of two opposite opinions on a certain subject, denoted as $\sigma_i = \pm 1$, $i = 1, 2, \ldots, N$. Individuals can influence each other, and each of them is characterized by the parameter $s_i > 0$ which describes the strength of his/her influence. Every pair of individuals $(i, j)$ is ascribed a distance $d_{ij}$ in a social space. The changes of opinion are determined by the *social impact* exerted on every individual

$$I_i = -s_i\beta - \sigma_i h - \sum_{j=1, j\neq i}^{N} \frac{s_j \sigma_i \sigma_j}{g(d_{ij})}, \qquad (1)$$

where $g(x)$ is an increasing function of social distance. $\beta$ is a so-called self-support parameter reflecting the inclination of an individual to maintain his/her current opinion. $h$ is an additional (external) influence which may be regarded as a global preference towards one of the opinions stimulated by mass-media, government policy, etc.

Opinions of individuals may change simultaneously (synchronous dynamics) in discrete time steps according to the rule

$$\sigma_i(t+1) = \begin{cases} \sigma_i(t) & \text{with probability } \dfrac{\exp(-I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \\[2ex] -\sigma_i(t) & \text{with probability } \dfrac{\exp(I_i/T)}{\exp(-I_i/T) + \exp(I_i/T)} \end{cases} . \qquad (2)$$

Eq. (2) is analogous to the Glauber dynamics with $-I_i\sigma_i$ corresponding to the local field. The parameter $T$ may be interpreted as a "social temperature" describing a degree of randomness in the behaviour of individuals, but also their average volatility (cf. [14,15]). The impact $I_i$ is a "deterministic" force inclining the individual $i$ to change his/her opinion if $I_i > 0$, or to keep it otherwise. The model is a particular case of the system considered in Ref. [19].

We assume that our social space is a 2D disc of radius $R \gg 1$, with the individuals located on the nodes of a quadratic grid. The distance between nearest neighbours equals 1, while the geometric distance models the social immediacy. The strength parameters $s_i$ of the individuals are positive random numbers with probability distribution $q(s)$ and the mean value $\bar{s}$. In the centre of the disc there is a *strong individual* (who we will call the "leader"); his/her strength $s_L$ is much larger than that of all the others ($s_L \gg s_i$).

## 2.2. Deterministic limit

Let us first recall the properties of the system without noise, i.e., at $T = 0$ [24,25]. Then, the dynamical rule (2) becomes strictly deterministic:

$$\sigma_i(t+1) = -\mathrm{sign}(I_i\sigma_i)\,. \tag{3}$$

Considering the possible stationary states we find either the trivial unification (with equal opinion $\pm 1$ for each individual) or, due to the symmetry, a circular cluster of individuals who share the opinion of the leader. This cluster is surrounded by a ring of their opponents (the majority). These states remain stationary also for a small self-support parameter $\beta$; for sufficiently large $\beta$ any configuration may remain "frozen".

Using the approximation of continuous distribution of individuals (i.e., replacing the sum in (1) by an integral), one can calculate the size of the cluster, i.e., its radius $a$ as a function of the other parameters. In the case of $g(r) = r$ and $\bar{s} = 1$ we get from the limiting condition for the stationarity $I = 0$ at the border of the cluster:

$$a \approx \tfrac{1}{16}\left[2\pi R - \sqrt{\pi} \pm \beta - h \pm \sqrt{(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L}\,\right]\,. \tag{4}$$

This is an approximate solution valid for $a \ll R$, but it captures all the qualitative features of the exact one which can be obtained by solving a transcendent equation (cf. Fig. 1). Here and in the next section we assume that the leader's opinion is $\sigma_L = +1$, but the analysis is also valid for the opposite case if $h \to -h$.

The branch with the "$-$" sign in front of the square root in Eq. (4) corresponds to the stable cluster. The one with "$+$" corresponds to the unstable solution which separates the basins of attraction of the stable cluster and unification (cf. Fig. 1). Owing to the two possible signs at the self-support parameter $\beta$ in (4), the stable and unstable solutions are split and form in fact two bands. The states within the bands are "frozen" due to the self-support which may be regarded as an analogy of the dry friction in mechanical systems. This way also the unstable clusters can be observed for $\beta > 0$ and appropriately chosen initial conditions.

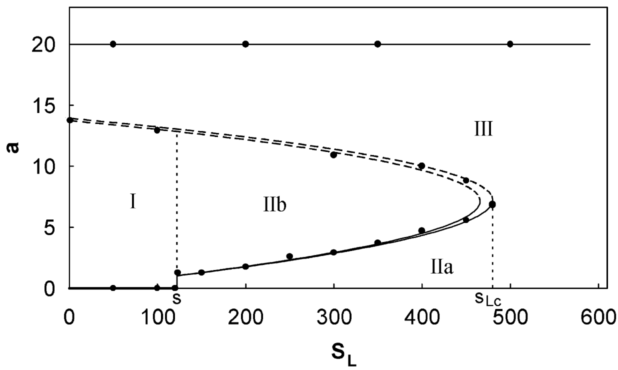


Fig. 1. Cluster's radius $a$ vs. leader's strength $s_L$ – phase diagram for circular social space. Interactions proportional to inverse of mutual distance ($I \propto 1/r$). Lines correspond to analytical results, points to computer simulations.

According to Eq. (4) real solutions corresponding to clusters exist provided

$$(2\pi R - \sqrt{\pi} \pm \beta - h)^2 - 32 s_L \geqslant 0 . \tag{5}$$

Otherwise, the general acceptance of the leader's opinion (unification) is the only stable state. When, having a stable cluster, the condition (5) is violated by changing a parameter, e.g. $s_L$ or $h$, one can observe a discontinuous phase transition: *cluster* $\rightarrow$ *unification*.

On the other hand, if the leader's strength is too weak, it may be impossible for him/her not only to form a cluster but also to maintain his/her own opinion. The limiting condition for the minimal leader's strength $s_{L\min}$ to resist against the persuasive impact of the majority can be calculated from the limiting condition $I_L = 0$ ($I_L$ – the impact exerted on the leader):

$$s_{L\min} = \frac{1}{\beta}(2\pi R - \sqrt{\pi} - h) . \tag{6}$$

Fig. 1 shows a phase-diagram $s_L$-$a$ for $h = 0$. All the plots are made for a space of radius $R = 20$ (1257 individuals) and $\beta = 1$ unless stated otherwise. Points in Fig. 1 are obtained by numerical simulations of (3) while the curves are solutions of a transcendent equation following from the stationary condition $I(a) = 0$. Solid lines represent stable fixed points – attractors (they correspond to the solution (4) with "–" sign before the square root); dashed lines represent unstable repellers (corresponding to "+" in (4)).

We find two kinds of attractors: (i) unification ($a = R$ when the leader's opinion wins, $a = 0$ when it ceases to exist) and (ii) a stable cluster resulting from a solution of (4). In the $s_L$-$a$ space one can distinguish between three basins of attraction. Starting from a state in the area denoted as *I*, the time evolution leads to unification with $a = 0$. The stable cluster attractor divides its basin of attraction into the areas *IIa* and *IIb*. All states from *III* will evolve to unification with $a = 20$. Owing to the two possible signs of self-support parameter $\beta$ in (4), the attractor and repeller are split. The space between their two parts enclose the "frozen" states that do not change in the course of time. These states correspond to local equilibria of the system dynamics similar to spin-glass states. Thus, as a result of self-support, even repeller states can be stabilized. As one can see, the results of computer simulations fit the calculated curves very well.

Taking into account conditions (5), (6) and the two possible opinions of the leader one can draw a phase-diagram $h - s_L$ distinguishing the regions where different system states are possible [24,26]. Apparently, the system shows *multistability* in a certain range of $s_L$ and $h$. It depends on the history which of the states is realized, so we can observe a *hysteresis* phenomenon [24,26]. Moving in the parameter space $s_L - h$, while starting from different configurations one can have many possible scenarios of phase transitions [26].

### 2.3. Effects of social temperature

It is obvious that the behaviour of an individual in a group depends not only on the influence of others. There are many more factors, both internal (personal) and external, that induce opinion formation and should be modeled somehow. In our model, we do this by means of a noisy dynamics, i.e., we use Eq. (2) with the parameter $T > 0$. In the presence of noise, the marginal stability of unstable clusters due to the self-support is suppressed and they are no longer the stationary states of the system. The borders of the stable clusters become diluted, i.e., individuals of both opinions appear all over the group. Our simulations [24,26] prove that the presence of noise can induce the transition from the configuration with a cluster around the leader to the unification of opinions in the whole group. There is a well-defined temperature $T_c$ that separates these two phases. To estimate the dependence of $T_c$ on other system parameters analytically, one can use a mean field approach, like methods developed in [24,26]. The two limiting cases of such an approach correspond to low- and high-temperature approximations and are discussed in the following.

### 2.4. Low-temperature mean-field approximation

For *low temperatures* $T$, i.e., for a small noise level, the cluster of leaders followers is only slightly diluted and its *effective radius* $a(T)$ can be treated as an order parameter. One can then calculate the impact $I(d)$ acting on the group member inside $(d < a)$ and outside $(d > a)$ the cluster respectively [24]:

$$I_i(d) = -\frac{s_L}{d} - 8aE\left(\frac{d}{a}, \frac{\pi}{2}\right) + 4RE\left(\frac{d}{R}, \frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta\,, \tag{7}$$

$$I_o(d) = \frac{s_L}{d} + 8aE\left(\frac{d}{a}, \arcsin\frac{a}{d}\right) - 4RE\left(\frac{d}{R}, \frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta\,, \tag{8}$$

where $E(k, \varphi) = \int_0^{\varphi} (1 - k^2 \sin^2 \alpha)^{1/2} d\alpha$ is the elliptic integral of the second kind.

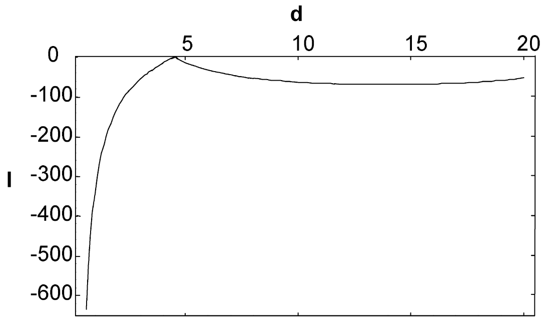


Fig. 2. Social impact $I$ as a function of distance $d$ to the leader. Leader's strength $s_L = 400$.

Both functions are plotted in Fig. 2 for $s_L = 400$. The system remains in equilibrium, therefore the impact on every individual is negative (nobody changes his/her opinion). It approaches zero at the border of the cluster which means that individuals located in the neighbourhood of that border are most sensitive to thermal fluctuations. We can however observe a significant *asymmetry* of the impact. It is considerably stronger inside the cluster. Individuals near the leader are deeper confirmed in their opinion, so they are also more resistant against noise in dynamics. When we increase the temperature starting from $T \simeq 0$, random opinion changes begin. Primarily, it concerns those near the border (the weakest impact). As a result, individuals with adverse opinions appear both inside and outside the cluster. They are more numerous outside because of the weaker impact (cf. Fig. 2).

Effectively, we observe the growth of a minority group. This causes the supportive impact outside cluster to become still weaker and the majority to become more sensitive to random changes. It is a kind of positive feedback. At certain value of temperature the process becomes an avalanche, and the former majority disappears. Thus, noise induces a jump from one attractor (cluster) to another (unification). Such a transition is possible for every non-zero temperature, but its probability remains negligible until the noise level exceeds a certain critical value that corresponds to our critical temperature $T_c$.

Using Eq. (2) and taking into account Eqs. (7) and (8), we can compute the probability $\Pr(\sigma = 1)(r)$ that an individual at the distance $r$ from the leader, shares opinion $+1$, which is assumed as the opinion of the leader. Then, the mean number of all individuals with opinion $+1$ may be calculated by integrating this probability multiplied by the surface density (equaling 1) over the whole space:

$$\overline{n(\sigma = 1)(T)} = \int_0^R \Pr(\sigma = 1)(r) 2\pi r \, \mathrm{d}r \, . \tag{9}$$

This number equals the effective area of the circular cluster, so its radius is

$$\overline{a(T)} = \sqrt{\frac{\overline{n(\sigma = 1)(T)}}{\pi}} \, . \tag{10}$$

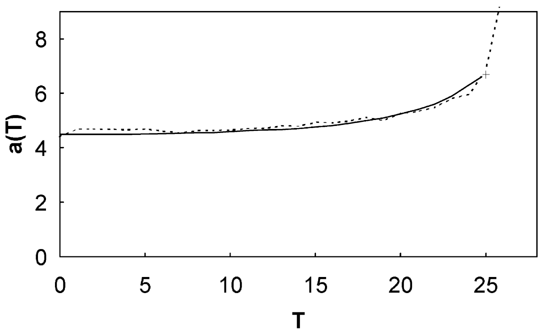


Fig. 3. Mean cluster radius $a$ vs. temperature $T$; $s_L = 400$. Results of calculation (solid) and computer simulation (dotted).

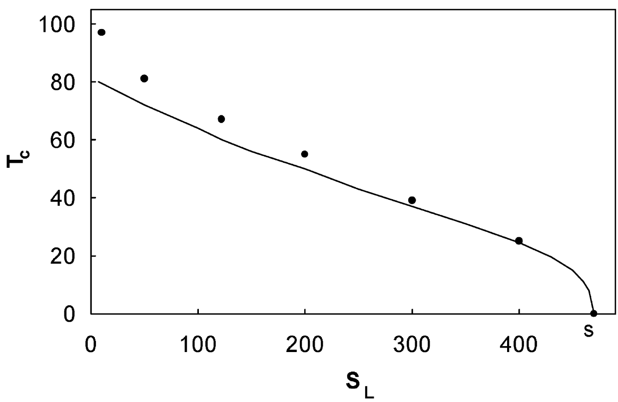


Fig. 4. Critical temperature $T_c$ (above which no stable cluster exists) vs. leader's strength $s_L$. Leader's opinion fixed (independent of the group). Line – calculations (Eq. (10)), points – simulations.

Eq. (10) is a rather involved transcendent equation for $a(T)$ (it appears on the right-hand side in $I_i(r)$ and $I_o(r)$). For low temperatures $T$ it has three solutions $a(T)$ corresponding to a stable cluster, an unstable cluster and a social homogeneous state. The numerical solution for the radius of stable cluster is presented in Fig. 3 together with results of computer simulations. One should point out that the radius of the cluster $a$ is an increasing function of the temperature $T$ for the reasons discussed above. At some critical temperature, a pair of solutions corresponding to the stable and the unstable cluster coincide [24,26]. Above this temperature, there exists *only* the solution corresponding to the social homogeneous state. Fig. 4 shows the plot of the critical temperature $T_c$ obtained from (10) as the function of the leader strength $s_L$ together with results of computer simulations.

## 2.5. High-temperature mean-field approximation

For high temperatures or small values of the leader's strength $s_L$, the cluster around the leader is very diluted and it is more appropriate to assume that there is a *spatially homogeneous mixture* of leaders followers and opponents, instead of a *localized cluster* with a radius $a(T)$. It follows that at each site there is *the same* probability $0 < p(T) < 1$ to find an individual sharing the leaders opinion, and $p(T)$ plays the role of order parameter. Neglecting the self-support ($\beta = 0$) one can write the social impact acting on a opponent of the leader at place $x$ as [26]

$$I(x) = \frac{s_L}{g(x)} + (2p - 1)\rho\bar{s}J_D(x) + h\,, \tag{11}$$

$J_D(x) = \int_{D_R} 1/g(|\mathbf{r} - \mathbf{x}|)\,\mathrm{d}^2\mathbf{r}$ is a function which depends only on the size of the group and the type of interactions. After a short algebra one gets the following equation for the probability $p(T)$ [26]:

$$p = \frac{1}{\pi R^2 \rho}\int_0^R \rho\,\mathrm{Pr}(r) 2\pi r\,\mathrm{d}r = \frac{1}{R^2}\int_0^R \frac{\exp[I(r,p)/T]}{\cosh[I(r,p)/T]} r\,\mathrm{d}r \equiv f(p)\,, \tag{12}$$

where $I(x,p)$ is given by (11). Similar to Eq. (10) obtained for low temperatures, there are three solutions of Eq. (12): the smallest one corresponds to the stable cluster around the leader, the middle one to the unstable cluster which, in fact, is not observed, and the largest one to the unification. The size of the stable cluster grows with increasing temperature up to a critical value $T_c$ when it coincides with the unstable solution. At this temperature, a transition from a stable cluster to unification occurs [26]. For $T > T_c$, unification is the only solution, but it is no longer a perfect unification because due to the noise individuals of the opposite opinion appear. When the temperature increases further, $p(T)$ tends to $\frac{1}{2}$ which means that the dynamics is random and both opinions appear with equal probability.

## 3. Modelling opinion dynamics by means of active Brownian particles

### 3.1. The model

There are several basic disadvantages of the model considered in the previous chapter. In particular, it assumes, that the impact on an individual is updated with infinite velocity, and no memory effects are considered. Further, there is no migration of the individuals, and any "spatial" distribution of opinions refers to a "social", but not to the physical space.

An alternative approach [30] to the social impact model of collective opinion formation, which tries to include these features is based on *active Brownian particles* [27–29,31,32], which interact via a *communication field*. This scalar field considers the spatial distribution of the individual opinions, further, it has a certain life time, reflecting a collective memory effect and it can spread out in the community, modeling the transfer of information.

The spatio-temporal change of the communication field is given by the following equation:

$$\frac{\partial}{\partial t} h_\sigma(\mathbf{r},t) = \sum_{i=1}^{N} s_i \delta_{\sigma,\sigma_i} \delta(\mathbf{r}-\mathbf{r}_i) - \gamma h_\sigma(\mathbf{r},t) + D_h \Delta h_\sigma(\mathbf{r},t)\,. \tag{13}$$

Every individual contributes permanently to the field $h_\sigma(\mathbf{r},t)$ with its opinion $\sigma_i$ and with its personal strength $s_i$ at its current spatial location $\mathbf{r}_i$. Here, $\delta_{\sigma,\sigma_i}$ is the Kronecker Delta, $\delta(\mathbf{r}-\mathbf{r}_i)$ denotes Dirac's Delta function used for continuous variables, $N$ is the number of individuals. The information generated by the individuals has a certain average life time $1/\gamma$, further it can spread throughout the system by a diffusion-like process, where $D_h$ represents the diffusion constant for information exchange. If two different opinions are taken into account, the communication field should also consist of two components, $\sigma = \{-1,+1\}$, each representing one opinion.

In this model, the scalar *spatio-temporal communication field* $h_\sigma(\mathbf{r},t)$ [30], plays in part the role of social impact $I_i$ used in [24,26]. Instead of a social impact, the communication field $h_\sigma(\mathbf{r},t)$ influences the individual $i$ as follows: At a certain location $\mathbf{r}_i$, the individual with opinion $\sigma_i = +1$ is affected by two kinds of information: the information resulting from individuals who share his/her opinion, $h_{\sigma=+1}(\mathbf{r}_i,t)$, and the information resulting from the opponents $h_{\sigma=-1}(\mathbf{r}_i,t)$. Dependent on the *local* information, the individual reacts in two ways: (i) it can *change its opinion*, (ii) it can *migrate* towards locations which provide a larger support of its current opinion. These opportunities are specified in the following.

We assume that the probability $p_i(\sigma_i,t)$ to find the individual $i$ with the opinion $\sigma_i$ changes in the course of time due to the master equation (the dynamics is continuous in time):

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i(\sigma_i,t) = \sum_{\sigma_i'} w(\sigma_i|\sigma_i') p_i(\sigma_i',t) - p_i(\sigma_i,t) \sum_{\sigma_i'} w(\sigma_i'|\sigma_i)\,, \tag{14}$$

where rates of transition probability are described in a similar way to Eq. (2)

$$w(\sigma_i'|\sigma_i) = \eta \exp\{[h_{\sigma'}(\mathbf{r}_i,t) - h_\sigma(\mathbf{r}_i,t)]/T\} \quad \text{for } \sigma \neq \sigma' \tag{15}$$

and $w(\sigma_i|\sigma_i) = 0$. The movement of the individual located at space coordinate $\mathbf{r}_i$ is described by the following overdamped Langevin equation:

$$\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \alpha_i \left.\frac{\partial h_e(\mathbf{r},t)}{\partial r}\right|_{\mathbf{r}_i} + \sqrt{2D_n}\,\xi_i(t)\,. \tag{16}$$

In the last term of Eq. (16), $D_n$ means the spatial diffusion coefficient of the individuals. The random influences on the movement are modeled by a stochastic force with a $\delta$-correlated time dependence, i.e., $\xi(t)$ is white noise with $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$. The term $h_e(\mathbf{r},t)$ in Eq. (16) means an *effective* communication field which results from $h_\sigma(\mathbf{r},t)$ as a certain function of both components, $h_{\pm 1}(\mathbf{r},t)$ [30]. Parameters $\alpha_i$ are individual response parameters. In the following, we will assume $\alpha_i = \alpha$ and $h_e = h_\sigma$.

### 3.2. Critical conditions for spatial opinion separation

The spatio-temporal density of individuals with opinion $\sigma$ can be obtained as follows:

$$n_\sigma(\mathbf{r},t) = \int \sum_{i=1}^{N} \delta_{\sigma,\sigma_i}\delta(\mathbf{r}-\mathbf{r}_i)P(\sigma_1,\mathbf{r}_1,\ldots,\sigma_N,\mathbf{r}_N,t)\,\mathrm{d}\mathbf{r}_1,\ldots,\mathrm{d}\mathbf{r}_N\,, \tag{17}$$

$P(\underline{\sigma},\underline{r},t)=P(\sigma_1,\mathbf{r}_1,\ldots,\sigma_N,\mathbf{r}_N,t)$ is the canonical $N$-particle distribution function which gives the probability to find the $N$ individuals with the opinions $\sigma_1,\ldots,\sigma_N$ in the vicinity of $\mathbf{r}_1,\ldots,\mathbf{r}_N$ on the surface $A$ at time $t$. The evolution of $P(\underline{\sigma},\underline{r},t)$ can be described by a master equation [30] which considers both Eqs. (15) and (16). Neglecting higher-order correlations, one obtains from Eq. (17), the following reaction–diffusion equation for $n_\sigma(\mathbf{r},t)$ [28,30]:

$$\begin{aligned}\frac{\partial}{\partial t}n_\sigma(\mathbf{r},t) &= -\nabla[n_\sigma(\mathbf{r},t)\alpha\nabla h_\sigma(\mathbf{r},t)] + D_n\Delta n_\sigma(\mathbf{r},t)\\ &\quad - \sum_{\underline{\sigma'\neq\sigma}}[w(\sigma'|\sigma)\,n_\sigma(\mathbf{r},t) - w(\sigma|\sigma')\,n_{\sigma'}(\mathbf{r},t)]\end{aligned} \tag{18}$$

with the transition rates given by Eq. (15). Eq. (18) together with Eq. (13) form a set of four equations describing our system completely.

Now, let us assume that the spatio-temporal communication field *relaxes faster* than the related distribution of individuals to a quasi-stationary equilibrium. The field $h_\sigma(\mathbf{r},t)$ should still depend on time and space coordinates, but, due to the fast relaxation, there is a fixed relation to the spatio-temporal distribution of individuals. Further, we neglect the independent diffusion of information, assuming that the spreading of opinions is due to the migration of the individuals. Using $\dot{h}_\sigma(\mathbf{r},t)=0$, $s_i=s$ and $D_h=0$ we get

$$h_\sigma(\mathbf{r},t) = \frac{s}{\gamma}n_\sigma(\mathbf{r},t)\,. \tag{19}$$

Inserting Eq. (19) into Eq. (18) we reduce the set of coupled equations to two equations.

The homogeneous solution for $n_\sigma(\mathbf{r},t)$ is given by the mean densities

$$\bar{n}_\sigma = \frac{\bar{n}}{2} \quad \text{where } \bar{n} = \frac{N}{A}\,. \tag{20}$$

Under certain conditions however, the homogeneous state becomes unstable and a spatial separation of opinions occurs. In order to investigate these critical conditions, we allow small fluctuations $\delta n_\sigma \sim \exp(\lambda t + \mathrm{i}\mathbf{kr})$ around the homogeneous state $\bar{n}_\sigma$ and perform linear stability analysis [30]. The resulting dispersion relations read

$$\begin{aligned}&\lambda_1(\mathbf{k}) = -k^2C + 2B, \qquad \lambda_2(\mathbf{k}) = -k^2C\,,\\ &B = \frac{\eta s\bar{n}}{\gamma T} - \eta, \qquad C = D_n - \frac{\alpha s\bar{n}}{2\gamma}\,.\end{aligned} \tag{21}$$

It follows that stability conditions of the homogeneous state, $n_\sigma(\mathbf{r},t) = \bar{n}/2$, can be expressed as

$$T > T_1^c = \frac{s\bar{n}}{\gamma}, \qquad D > D_n^c = \frac{\alpha}{2}\frac{s\bar{n}}{\gamma}\,. \tag{22}$$

If the above conditions are not fulfilled, the homogeneous state that corresponds to *paramagnetic phase* is unstable (i) against the formation of spatial "domains" where one of opinions $\sigma = \pm 1$ *locally dominates*, or (ii) against the formation of a *ferromagnetic* state where the *total numbers* of people sharing both opinions are not equal.

Case (i) can occur only for a systems whose linear dimensions are large enough, so that large-scale fluctuations with small wave numbers can destroy the homogeneous state [30]. In case (ii), each subpopulation can exist either as a *majority* or as a *minority* within the community. Which of these two possible situations is realized, depends in a deterministic approach on the initial fraction of the subpopulation. Breaking the symmetry between the two opinions due to *external influences* (support) for one of the opinions would increase the region of initial conditions which lead to a majority status. Above a critical value of such a support, the possibility of a minority status completely vanishes and the supported subpopulation will grow towards a majority, regardless of its initial population size, with no chance for the opposite opinion to be established [30].

## 4. Conclusions

This work discusses the possibilities of phase transitions in models of opinion formation which are based on the social impact theory (two opinions case). In the presence of a strong leader situated in the centre of a finite group, a transition can take place from a state with a cluster around the leader to a state of uniform opinion distribution where virtually all members of the group share the leader's opinion. The transition occurs if a leader's strength exceeds a well defined critical value or if the noise level ("social temperature") is high enough. The weaker the leader's strength is, the larger noise is needed. The value of the critical temperature can be calculated using mean field methods where either the existence of an effective value of the cluster radius (low-temperature method) or a spatially homogeneous mixture of both opinions (high-temperature method) is assumed. Numerical simulations confirm the analytic results.

The extension of the social impact model is based on the concept of active Brownian particles which communicate via a scalar, multi-component communication field. This allows us to take into account effects of spatial migration (drift and diffusion), a finite velocity of information exchange and memory effects. The reaction–diffusion equation for the density of individuals with a certain opinion is obtained. In this model, the transition can take place between the "paramagnetic" phase, where the probability to find any of opposite opinions is $\frac{1}{2}$ at each place (a high-temperature and a high-diffusion phase), the "ferromagnetic" phase with a global majority of one opinion (a low-temperature and a low-diffusion phase) and a phase with spatially separated "domains" with a local majority of one opinion (an intermediate phase).

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