PHYSA 5632

ELSEVIER

Physica A 000 (2000) 000–000

PHYSICA A

www.elsevier.com/locate/physa

Chaos control in economical model by time-delayed feedback method

Janusz A. Hołyst*, Krzysztof Urbanowicz

Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland

Received 26 June 2000; received in revised form 10 July 2000

Abstract

A two-dimensional map describing chaotic behaviour of an economic model has been stabilized on various periodic orbits by the use of Pyragas time-delayed feedback control. The method avoids fancy data processing used in the Ott–Grebogi–Yorke approach and is based solely on the plain measurement and time lag of a scalar signal which in our case is a value of sales of a firm following an active investment strategy (Behrens–Feichtinger model). We show that the application of this control method is very straightforward and one can easily switch from a chaotic trajectory to a regular periodic orbit and simultaneously improve the system's economic properties. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 05.45.Gg; 87.23.Ge

Keywords: Chaos control; Time-discrete dynamics; Microeconomy

1. Introduction

It is generally accepted that economy belongs to extremely complex systems [1,2] and from the physics point of view both deterministic [3,4] and stochastic [5,6] descriptions are needed to describe main features of its dynamics. On the other hand, controlling of at least some economical processes seems to be one of the most important and challenging tasks facing the economists and politicians responsible for economical policy. In the present paper we make use of the theory of chaos control [7–9] to

* Corresponding author. Tel.: +48-22-6607133; Fax: +48-22-6282171.
E-mail address: jholyst@if.pw.edu.pl (J.A. Hołyst).

0378-4371/00/$ - see front matter
PII: S0378-4371(00)00395-2

show how the method of time-delayed feedback control can be applied to suppress deterministic chaos in a simple micro-economical model of two competing firms.

We start our paper with a short presentation (Section 2) of the so-called OGY and Pyragas methods of chaos control. Section 3 is devoted to properties of the chaotical Behrens–Feichtinger model while Section 4 includes results of numerical simulations of chaos control by the use of several time-delayed feedback control methods for this model. In Section 5 we present the theoretical analysis of our numerical results.

2. Principles of chaos control

In the paper [7] Ott, Grebogi and Yorke (OGY) introduced a method of chaos control that started a new direction of research for physicists working on chaotic systems [8]. The OGY-method makes use of the observation that a chaotic solution possesses in its immediate neighborhood infinite number of *unstable periodic orbits* (UPO-s). Although the OGY-method is well understood from the theoretical point of view its experimental implementations are seriously limited by the fact that all quantities needed to calculate values of a system control parameter are not directly given in an experimental data chain and to perform the control one needs to apply a complex data analysis. In the papers [10,11], it has been presented how the OGY method can be applied to suppress chaos in a simple microeconomical model [12].

In contrast to the OGY method the method of chaos control introduced by Pyragas [9] can be easily applied to experimental systems where the equations of motion are not known. The theoretical background for this method can be found in [13–15]. Let us assume that equation of motion for a continuous in time dynamical system $\mathbf{r}(t)$ subjected to control force $k(t)$ is

$$\dot{\mathbf{r}} = \mathbf{h}(\mathbf{r}(t), k(t)) \tag{1}$$

and in the absence of control, $k \equiv 0$, there is UPO of the period τ. Then one can use the following control force to stabilize this UPO

$$k(t) := K \sum_{v=0}^{\infty} s_v \{g[\mathbf{r}(t - v\tau)] - g[\mathbf{r}(t - (v+1)\tau)]\} \,. \tag{2}$$

Here $g[\mathbf{r}(t)]$ is a scalar quantity that can be obtained from a measurement of the system state $\mathbf{r}(t)$, while s_v are control parameters. The parameter K can be considered as the total control amplitude. The important feature of the control force given by (2) (it is a generalized version of the so-called extended time delayed autosynchronisation method [16]) is the property that for properly chosen coefficients s_v the control force $k(t)$ tends to zero when the system reaches the desired UPO. For the original method proposed by Pyragas [9] there is $s_v = \delta_{v,0}$ and the control amplitude K is the only parameter that is needed to be fixed. One can show however [14] that in such a case the effective control is limited to UPO-s that fulfill the relation $\lambda\tau < c$ where λ is the largest Lyapunov exponent of the desired UPO and c is a constant of the order of 2.

This limitation can be overcome if there are more nonzero control coefficients, for example $s_v = R^v$ [14]. Another limitation of this method is caused by the Lyapunov exponents that correspond to stable directions of the uncontrolled UPO but they may bifurcate and become positive due to the nonzero control term [15].

Various discrete versions of the control force (2) have been studied in the literature, e.g. [17–21]. Similarly as for time-continuous systems [13–15] analytical results for this kind of control base on the Floquet theory and it has been found that the limitation of the control is connected with the number of real Floquet multipliers (corresponding to the desired UPO) which are greater than unity [17,18,20]. Moreover, several types of bifurcation diagrams including Hopf bifurcations have been observed in such delayed-feedback control systems [21].

3. Behrens–Feichtinger model

We will show the effects of chaos control by time-delayed feedback using a simple microeconomical model [12] of two firms X and Y competing on the same market of goods. The firms perform active investment strategies, i.e., their temporary investments depend on their relative position on the market. The strategies are asymmetric: the firm X invests more when it has an advantage over the firm Y while the firm Y invests more if it is in a disadvantageous position compared to the firm X. One assumes that sales x_n, y_n of both firms measured in discrete time steps $n=1,2,3,\ldots$ evolve according to the equations

$$x_{n+1} = F^x(x_n, y_n) = (1-\alpha)x_n + \frac{a}{1+\exp[-c(x_n - y_n)]}\,, \tag{3}$$

$$y_{n+1} = F^y(x_n, y_n) = (1-\beta)y_n + \frac{b}{1+\exp[-c(x_n - y_n)]}\,. \tag{4}$$

The constants α and β (with $0<\alpha,\ \beta<1$) are the time rates at which the sales of both firms decay in the absence of investments while second terms on the r.h.s. of Eqs. (3) and (4) describe the influence of investments at time n on the sales at time $(n+1)$. Parameters a and b describe the investment effectiveness of both firms or scales of their investments while the parameter c is an "elasticity" measure of the investment strategies. Eqs. (3) and (4) form a two-dimensional map $\mathbf{r}_{n+1}=\mathbf{F}(\mathbf{r}_n)$, where $\mathbf{r}_n=[x_n, y_n]$ and $\mathbf{F}=[F^x, F^y]$, that completely defines the evolution of our discrete dynamical system. Depending on the specific values of parameters α, β, a, b and c solutions of (3) can be regular or chaotic [10–12]. An example of a chaotic attractor corresponding to the map (3) with parameters $a^0=0.16$, $b^0=0.9$, $c^0=105$, $\alpha^0=0.46$, $\beta^0=0.7$ is presented in Fig. 1. The model can also be considered as a nonlinear extension of the Richardson model of arms races [12].

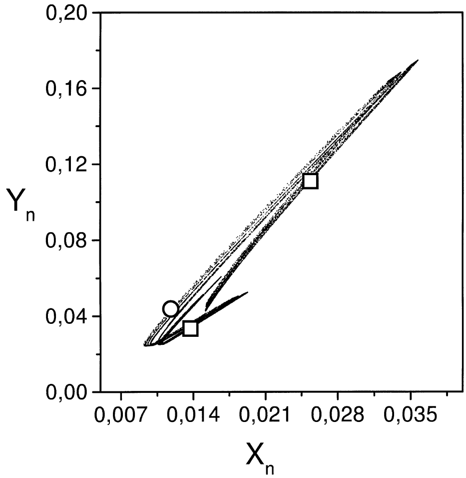

Fig. 1. Chaotic evolution of sales x_n and y_n in the Behrens–Feichtinger model. Unstable fixed point is depicted by the open circle and unstable period-two orbit is marked by the pair of open squares.

4. Numerical simulation of chaos control

4.1. Chaos control by changes of investment sizes

Let us assume that the firm Y would like to change the chaotic behaviour of its sales and tries to control the market by introducing additional amount of investments. To stabilize the market on a period-one or on a period-two orbit the firm needs to introduce at every moment n changes of its investments that are proportional to differences $y_n - y_{n-1}$ or $y_n - y_{n-2}$. In such a case, the equation describing the evolution of sales of this firm reads

$$y_{n+1} = F^y(x_n, y_n) + F_K \,. \tag{5}$$

Here F_K means a control force and we will assume that

$$F_K = K_{yy}(y_n - y_{n-m})\,, \tag{6}$$

where m is the delay lag that can be equal to the period of the controlled orbit, i.e., m=1 for the period-one orbit and m=2 the period-two orbit while K_{yy} is an appropriate control coefficient. Results of simulations of such a control are presented in Fig. 2. One can see that at $n = 44$ the firm Y switched on the control to stabilize the market at the period-one orbit. As the result sales x_n and y_n of both firms were fixed to constant values but unfortunately these values are below mean values of sales corresponding to uncontrolled, chaotic market. At the moment $n = 174$ the control stabilizing the period-one orbit was switched off and the market came back to the chaotic behaviour. At the moment n=308 the control of the period-two orbit was switched on. As the result sales x_n and y_n of both firms oscillated periodically between two values. However, in

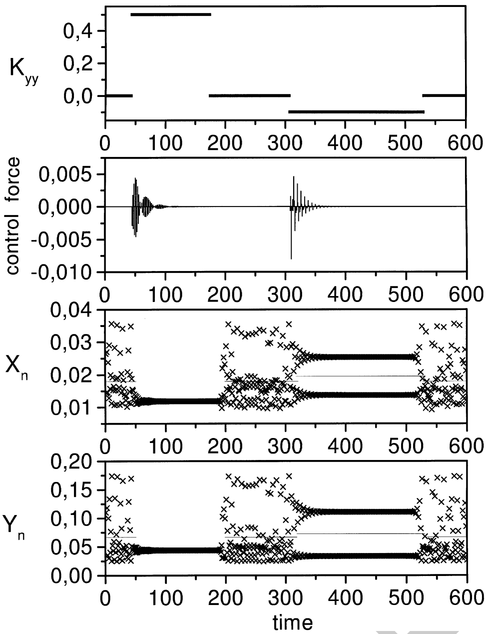

Fig. 2. Time dependence of the control amplitude K_{yy}, the control force F_K (Eq. (6)) and sales x_n, y_n. Results of stabilization procedure are clearly seen. Thin lines depict mean values of the sales in various time intervals.

such a case, corresponding mean values of sales were larger than means calculated for the uncontrolled, chaotic behaviour. It follows that in order to increase its sales (as compared to chaotic behaviour) the firm can stabilize the market on the period-two orbit. We stress that although the control of the market was introduced by the firm Y but actually the whole system was stabilized and consequently the sales x_n of the firm X became also periodic. It is interesting that the sales of the firm X decreased for the period-one orbit and increased for the period-two orbit similarly as the sales of the firm Y. This effect is of course at the cost of all other firms acting on the market and influencing the dynamics of the firms X and Y due to nonzero values of decay parameters α, β in Eqs. (3). The fact that the period-one orbit leads to the decrease of sales while period-two orbit leads to the increase of sales is a direct consequence of the particular position of these periodic orbits in the chaotic attractor (see Fig. 1).

4.2. Chaos control by many delay terms

Let us consider instead of Eq. (6) the following control term:

$$F_K = K_1(y_n - y_{n-1}) + K_2(y_n - y_{n-2}) \tag{7}$$

which is added to the r.h.s. of Eq. (4). Here constants K_1 and K_2 are control parameters. Such a control was discussed in [14] for time-continuous system and it was found that

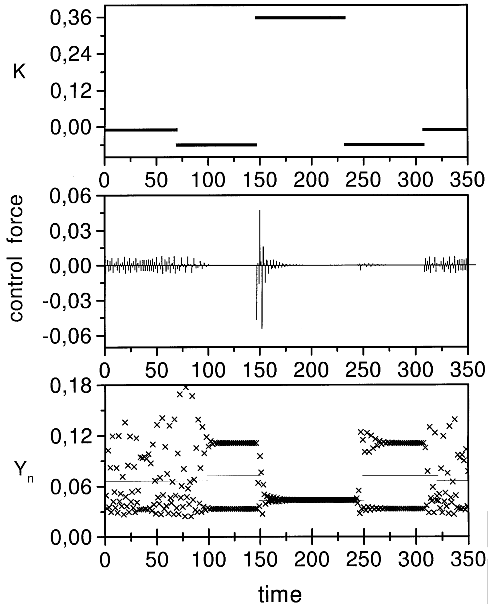

Fig. 3. Time dependence of the control amplitude K, the control force F_K (Eq. (7)) and sales y_n.

this control allows to stabilize UPO-s with longer periods or with larger Lyapunov exponents that the single delay term (6). Here we make use of the property that this control allows to switch between period-one and period-two orbits using changes of a single parameter. For example, if $K_1 = 1.4473K + 0.0868$ and $K_2 = K$ then as a result of changes of the parameter K we obtain the dynamics presented in Fig. 3. One has to point out that the value of the parameter K was changed in such a way that the term K_1 vanished during the intervals corresponding to the control of the period-two orbit otherwise the control force F_K would have not vanished during the stabilization of this orbit.

4.3. Chaos control by decrease of investment sizes

Although the control represented by Eq. (6) shifted the market towards the more optimal evolution one can observe in Fig. 2 that during several time steps the control force was positive, i.e., the firm Y had to *increase* its investments to perform this control. One can ask whether it is possible to stabilize the market just by *decreasing* investment values. This question is especially important if one considers force (6) not as changes of temporary investments but as changes of temporary *sales*. One can suppose that decreasing of sales is easy to introduce because it depends just on the firm Y but is much more difficult to cause the corresponding sales increase. Let us assume that instead of Eq. (6) the control force is given by

$$F_K = \mathrm{Min}[0, K_{yy}(y_n - y_{n-2})] \,. \tag{8}$$

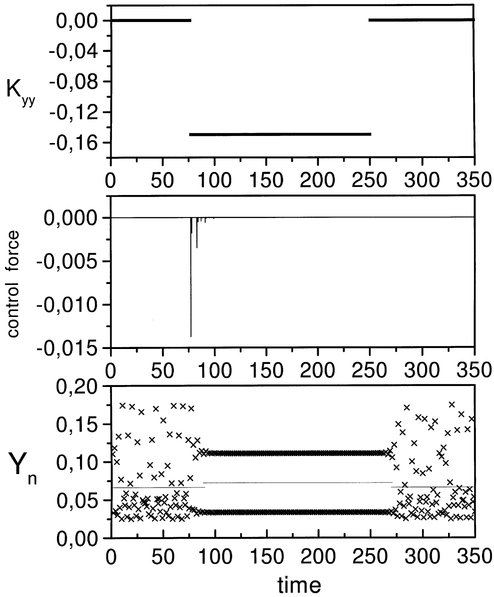

Fig. 4. Time dependence of the control amplitude K_{yy}, the control force F_K (Eq. (8)) and sales y_n.

Results of corresponding numerical simulations are presented in Fig. 4. One can see that the control was successful and the same period-two orbit was stabilized that was observed in Fig. 2. It is interesting that we were not able to perform the control of the period-one orbit in a similar way.

4.4. Chaos control by parameter changes

Both control methods discussed above made use of Eq. (5), i.e., we assumed that the control force F_K can be added to the *state variable*. A more sophisticated control method uses changes of system *parameters* (that can be easily accessible) instead of system variables. In our case such changes are very natural if they are introduced by the firm Y for the parameter b that represents its maximal investments values. Let us assume that Eq. (5) is changed to

$$y_{n+1} = (1-\beta)y_n + \frac{b + K_{yy}(y_n - y_{n-m})}{1+\exp[-c(x_n - y_n)]}\,, \tag{9}$$

where $m = 1, 2, 3, \ldots$. The results of numerical simulations for $m = 1$ and for $m = 2$ are presented in Fig. 5. One can see that the control of the period-one orbit ($m = 1$) and of the period-two orbit ($m = 2$) is possible by this method. For the first case, a constant value of the control parameter $K_{yy} = 14.7$ was used. However such a control was not possible for the period-two orbit and we had to use *time-dependent* control amplitude in the form $K_{yy} = K_{yy}(n) = -250.88 + 250(-1)^n$. In fact, for small values of the control

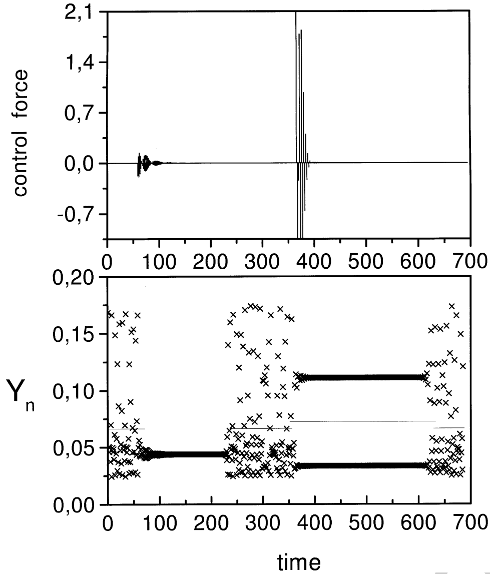

Fig. 5. Time dependence of the control force $F_K = K_{yy}(y_n - y_{n-m})$ for Eq. (9) and corresponding sales y_n.

forces one can perform linearization for Eq. (9) and as a result the equation can be reduced to the form of Eq. (5) with the control force given by

$$F_K = \frac{\partial y_{n+1}}{\partial b} K_{yy}(y_n - y_{n-2}) . \tag{10}$$

Because the partial derivative $(\partial y_{n+1})/\partial b$ calculated at $b = b^0$ depends on the variables x_n, y_n thus for the period-two orbit this derivative is a time-dependent coefficient. It follows that values of the control parameter K_{yy} should be suited to the temporal element of the periodical orbit.

5. Theoretical analysis of chaos control

To understand the results of our numerical simulations, we need to remind that controlling chaos is connected with stabilization of UPO-s. It follows that Lyapunov exponents of the stabilized orbit must be changed in such a way that all of them become negative. For systems with discrete-time dynamics Lyapunov exponents λ_i can be easily calculated as logarithms of absolute values of eigenvalues of the system Jacobian. The peculiar feature of such an analysis is the fact that the Jacobian of the controlled system $\hat{J}_K$ is a matrix, the dimension D of which is always higher than the dimension d of the Jacobian $\hat{J}_0$ corresponding to the uncontrolled system. This fact is connected with appearing of additional degrees of freedom that are effects of the control force. For example, if the control is performed for a period-one orbit according

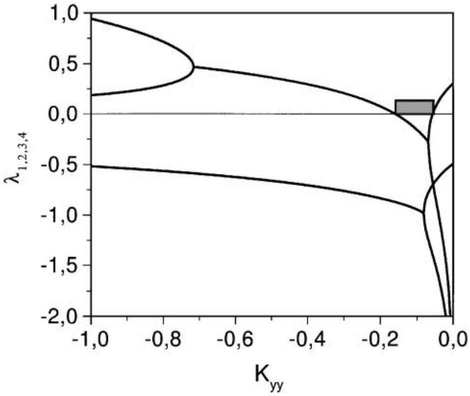

Fig. 6. Dependence of Lyapunov exponents on the control parameter K_{yy} in the case of the control of the period-two orbit using Eq. (6).

to Eq. (6) (with $m=1$) we can write the whole dynamics as

$$x_{n+1} = F^x(x_n, y_n)\,,$$

$$y_{n+1} = F^y(x_n, y_n) + K_{yy}(y_n - z_n)\,,$$

$$z_{n+1} = y_n\,. \tag{11}$$

In the above case, the dimension of controlled system equals to $D=3$. Similarly for the control of period-two orbit, i.e., when $m=2$ in Eq. (6), one obtains the system of four coupled equations. This fact leads to appearance of *new branches* of Lyapunov exponents in the controlled system. These additional Lyapunov exponents tend to minus infinity when all control parameters tend to zero however they may limit the possibility of the chaos control by time-delay feedback method similarly as it was found for time-continuous systems [13–15]. Observing changes of all Lyapunov exponents as a function of the control parameter one can find that the limitations come from two facts. First, there can exist a collision of a branch of Lyapunov exponent corresponding to the unstable direction of the uncontrolled system with an additional Lyapunov exponent. After the collision there exists a pair of two conjugated eigenvalues of the system Jacobian and further changes of the control parameter cause an increase of the resulting common Lyapunov exponent. Second, the Lyapunov exponents that correspond to stable directions of the uncontrolled system, i.e., that are negative in the absence of the control force, can become positive as the result of the control. This effect can also be connected with a collision of an old branch of stable Lyapunov exponents with a new branch coming from minus infinity. In Fig. 6, we present the plot of analytically calculated Lyapunov exponents for the control of the period-two orbit that was performed using Eq. (6). The grey box marks the region of the values of the control parameter K_{yy} when we observed the successful control. One can see that the region fits very well to the region where all Lyapunov exponents are negative. Fig. 7 shows the corresponding

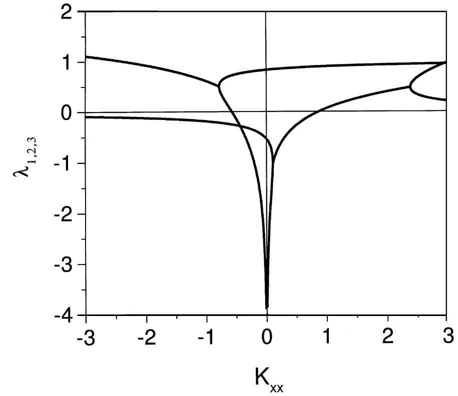

Fig. 7. Dependence of Lyapunov exponents on the control parameter K_{xx} in the case of the control of the period-one orbit using Eq. (12).

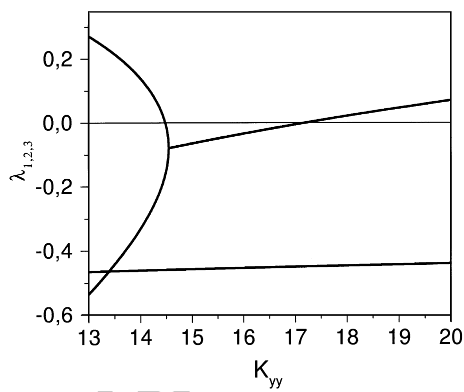

Fig. 8. Dependence of Lyapunov exponents on the control parameter K_{yy} in the case of the control of the period-one orbit using Eq. (9).

diagram for the control of the period-one orbit when the control force

$$F_K = K_{xx}(x_n - x_{n-1}) \tag{12}$$

was added to the r.h.s. of Eq. (3) and one can see that for all values of the control parameter K_{xx} at least one of the Lyapunov exponents is positive. In fact, we observed that it was not possible to stabilize the period-one orbit using this kind of control. We stress however that the control of the period-one orbit was possible when we used the control term in form (6) with $m = 1$ (see Fig. 2). It follows that the success of the control depends on the variable that is used for time-delayed feedback, and the delay lag.

A similar situation occurs when we apply the control by parameter changes. Fig. 8 shows the plot of analytically calculated Lyapunov exponents when the control method given by Eq. (9) is used for the period-one orbit (compare Fig. 5). One can see that there is a narrow region of the values of the parameter K_{yy} where the control is

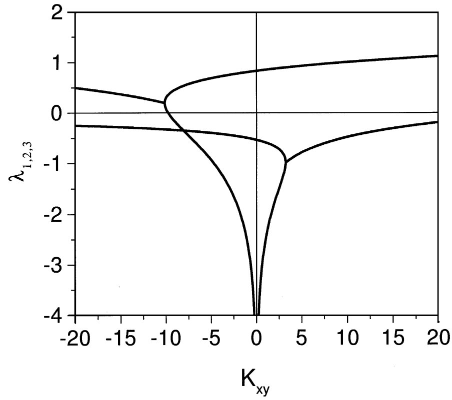

Fig. 9. Dependence of Lyapunov exponents on the control parameter K_{xy} in the case of the control of the period-one orbit using Eq. (13).

possible, i.e., when all Lyapunov exponents are negative. On the other hand, Fig. 9 corresponds to the control represented by the equations

$$x_{n+1} = (1-\alpha)x_n + \frac{a + K_{xy}(y_n - y_{n-1})}{1 + \exp[-c(x_n - y_n)]} ,$$

$$y_{n+1} = F^y(x_n, y_n) . \tag{13}$$

One observes that it is not possible to stabilize the period-one orbit using this kind of control.

6. Conclusions

We have shown that one can control the deterministic chaos in a simple microeconomical model describing competition of two firms and improve simultaneously the economic efficiency. The control consists of adding to system variables or to system parameters appropriate time-delayed feedback, i.e., control terms depending on differences between actual and past values of system variables. The success of such a chaos control is limited by the occurrence of additional Lyapunov exponents that represent new degrees of freedom and by the behaviour of Lyapunov exponents that correspond to the stable directions in the uncontrolled system. In general, the possibility of chaos control depends on the choice of system variables that are used as a time-delayed term and on the delay lag.

One can ask whether a control of this art is possible in real economical systems. In our opinion the method can be used and *it appears* in micro- and macro-economy. In fact, time-delayed control terms correspond to the behaviour of economical agents that is known as *rational expectations* [22,23], when firms change their policy taking into account differences between values of present and past sales or incomes. It follows that there can exist intrinsic market properties that suppress the chaotic behaviour.

References

[1] H. Haken, Synergetics. An Introduction, Springer, Heidelberg, 1983; Advanced Synergetics, Springer, Heidelberg, 1983.
[2] G.A. Cowan, D. Pines, D. Meltzer (Eds.), Complexity. Metaphors, Models, and Reality, Addison-Wesley, Santa Fe, 1994.
[3] H.W. Lorenz, Nonlinear Dynamical Equations and Chaotic Economy, Springer, Berlin, 1993.
[4] E. Moseklide, E.R. Larsen, System Dyn. Rev. 4 (1988) 131.
[5] R.N. Mantegna, H.E. Stanley, Introduction to Econophysics: Correlations Complexity in Finance, Cambridge University Press, Cambridge, 1999.
[6] D. Stauffer, D. Sornette, Physica A 271 (1999) 496.
[7] E. Ott, C. Grebogi, J.A. Yorke, Phys. Rev. Lett. 64 (1990) 1196.
[8] T. Kapitaniak, Controlling Chaos, Academic Press, New York, 1996.
[9] K. Pyragas, Phys. Lett. A 170 (1992) 421.
[10] J.A. Hołyst, T. Hagel, G. Haag, W. Weidlich, J. Evol. Econ., 6 1996 (31).
[11] J.A. Hołyst, T. Hagel, G. Haag, Chaos, Solitons Fractals 8 (1997) 1489.
[12] G. Feichtinger, in: G. Haag, U. Mueller, K.G. Troitzsch (Eds.), Economic Evolution and Demographic Change, Springer, Berlin, 1992.
[13] W. Just, T. Bernard, M. Ostheimer, E. Reibold, H. Benner, Phys. Rev. Lett. 78 (1997) 203.
[14] W. Just, E. Reibold, H. Benner, K. Kacperski, P. Fronczak, J. Hołyst, Phys. Lett. A 254 (1999) 158.
[15] W. Just, E. Reibold, K. Kacperski, P. Fronczak, J.A. Hołyst, H. Benner, Phys. Rev. E. 61 (2000) 5045.
[16] J.E.S. Socolar, D.W. Sukow, D.J. Gauthier, Phys. Rev. E 50 (1994) 2343.
[17] T. Ushio, IEEE CAS-I 43 (1996) 815.
[18] K. Konishi, H. Kokame, Phys. Lett. A 248 (1998) 359.
[19] M. Ishii, K. Konishi, H. Kokame, Phys. Lett. A 235 (1997) 603.
[20] H. Nakajima, Phys. Lett. A 232 (1997) 207.
[21] G. Chen, J. Lu, B. Nicholas, S.M. Ranganathan, Int. J. Bifurc. Chaos 9 (1999) 287.
[22] M.G. Kusch, B.E. Ydstie, Physica D 72 (1994) 309.
[23] H.W. Lorenz, M. Lohmann, Chaos, Solitons Fractals 7 (1996) 2135.