Noisefree stochastic multiresonance near chaotic crises

A. Krawiecki,[1] S. Matyjaśkiewicz,[1,2] K. Kacperski,[3,4,1] and J. A. Hołyst[1,2,4]
[1]*Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland*
[2]*Institute of Physics, Humboldt University at Berlin, Invalidenstraße 110, D-10115 Berlin, Germany*
[3]*Department of Physics, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, United Kingdom*
[4]*Max Planck Institute for the Physics of Complex Systems, Nöthnitzerstraße 38, D-01187 Dresden, Germany*
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We report on the phenomenon of noisefree stochastic multiresonance that appears in a natural way in systems where the threshold crossing probability has a nonmonotonous derivative with respect to the control parameter. In particular, we consider periodically driven chaotic dynamical systems above crisis threshold where the nonmonotonicity is caused by the fractal structure of precritical attractors and, possibly, their basins of attraction. The spectral power amplification as a function of the control parameter can be easily obtained from the postcritical average transient times, and the heights of its multiple maxima can be estimated on the basis of simple geometric models.

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Stochastic resonance (SR) [1–3] appears in periodically driven systems with noise. Noisefree SR [4,5] is a related phenomenon observed in chaotic systems where the internal chaotic dynamics plays the role of noise. The essence of these phenomena is that the transmission of a periodic signal through the system is optimum for nonzero intensity of the stochastic or internal noise, respectively. It has also been noticed that in some systems the signal transmission is maximized for several different noise levels; this phenomenon has recently been given a name stochastic multiresonance (SMR) [6]. In this paper we show that in a large class of systems SMR appears as a natural consequence of their dynamical properties.

Among systems with SR the threshold crossing (TC) systems form an important class [2,7]. We consider the case of discrete dynamics with a control parameter q modulated by the input signal, $q(n)=q_0+q_1\cos(\omega_0 n)$, where n is the iteration number. The parameter q_0 controls the average level of internal noise. The output signal is defined as 1 when a selected system variable crosses a given threshold, and 0 otherwise. As a measure of SR we take the spectral power amplification (SPA) $\sigma=|P_1|^2/q_1^2$, where P_1 is the first Fourier component of the output signal. In the adiabatic approximation P_1 can be derived [7] from the modulated TC probability P

$$P_1=T_0^{-1}\sum_{n=0}^{T_0-1} P[q_0+q_1\cos(\omega_0 n)]\exp(-i\omega_0 n), \quad (1)$$

where the integer $T_0=2\pi/\omega_0\gg 1$ is the modulation period. The SPA at a point q_0 can be thus considered as a functional of the TC probability on the interval $[q_0-q_1,q_0+q_1]$. If $P(q)$ is smooth and differentiable, then for q_1 small enough, in the linear approximation $P_1=(q_1/2)(dP/dq)|_{q_0}$. However, as will be shown below, $P(q)$ can be a complicated and nondifferentiable function. Then it is better to replace the derivative at a single point q_0 by its average, i.e., the difference quotient on the interval $[q_0-q_1,q_0+q_1]$. Thus we get

$$P_1=P'q_1/2, \quad (2)$$

where $P'=[P(q_0+q_1)-P(q_0-q_1)]/(2q_1)$. This approximation, although simple, provides a useful intuition about the shape of the SPA and yields a good quantitative assessment as well. Typically the probability $P(q)=0$ for sufficiently small q (no noise) and grows monotonically with q (with monotonic derivative) that can lead to only one maximum of P' and the corresponding single maximum of σ. However, if the derivative fluctuates "at the scale of q_1" multiple maxima of the SPA (i.e., multiresonance) can be expected.

Here we focus on noisefree SR in dynamical systems in the vicinity of chaotic crises [8]. In such systems below the critical value q_c of a control parameter q there exists a chaotic attractor, above it the attractor converts to a chaotic saddle as a consequence of a collision with the border of its basin of attraction (in other words, with the basin of escape), and chaotic transients can be observed. The system bounces around the saddle for some time, and then rapidly escapes to some other part of the phase space. The escape probability, i.e., the inverse of the average transient time obeys the power scaling law

$$P(q)=C(q-q_c)^{\gamma}, \quad (3)$$

where C is a constant and $\gamma\geqslant 1/2$ is the scaling exponent (henceforth, without loss of generality, we assume $q_c=0$). It is known, however, that the scaling law (3) gives only a rough trend of the function $P(q)$, and sometimes quite large oscillations around it can appear [8,9]. They are caused by fractal structures of the attractor and, possibly, basin of escape colliding in crisis. In the following we investigate SR in such systems with periodically modulated control parameter. The escape events can be treated as TC events [10] that produce one-step long pulses in the, otherwise zero, output signal. It is shown that the oscillations of $P(q)$ lead to the SMR.

Let us begin with an example of the Hénon map: $x_{n+1}=p-x_n^2-Jy_n$, $y_{n+1}=x_n$ with $J=-0.3$ that shows a bound-

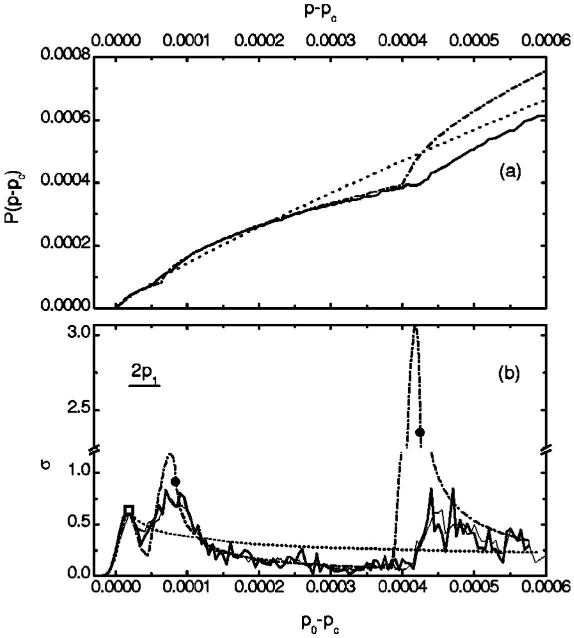

FIG. 1. The escape probability (a) and the SPA (b) for the crisis in Hénon map: thick solid lines — direct numerical simulations, thin solid line in (b) — the SPA obtained from the numerical escape probability using Eq. (2), dotted lines — the plain power scaling law (3) and the corresponding SPA, dash-dotted lines — model (5) curves. The parameters are $p_1=2\times10^{-5}$, $T_0=1024$, $\gamma=0.858$, $\alpha=0.158$, $\eta=0.517$, $C=0.386$, $\zeta=0.554$, and $a=0.65$. The dots denote the maxima $\sigma_{max}^{(k)}$ estimated from Eq. (7) with (from left) $k=5$ and $k=4$, and the square — the first maximum from Eq. (4).

ary crisis at $p_c=1.426\,921\,11\ldots$: there is a strange attractor for $p<p_c$ and divergence to infinity for $p>p_c$. After every escape event the trajectory is reinjected on the precritical attractor. Figure 1 shows the TC probability $P(p)$ and the corresponding curves $\sigma(p_0)$ obtained by replacing p with $p(n)=p_0+p_1\cos(\omega_0 n)$ [$p-p_c$, p_0-p_c, and p_1 correspond to q, q_0 and q_1 in Eqs. (3) and (1), respectively]. Roughly regular oscillations of $P(q)$ due to the fractal structure of the attractor [9] can be seen and the SPA shows multiple maxima. It is apparent that the approximation (2) reproduces the actual SPA from the escape probability very well.

To make a comparison, the plain power law (3) and the corresponding SPA have been plotted in Fig. 1 (dotted lines). In this case only the first maximum is recovered. It occurs since for $q<0$ there is $P(q)=0$ so the difference quotient P' increases monotonically for $-q_1<q_0<q_1$, and then for $q_0>q_1$ decreases if $\gamma<1$ or further increases if $\gamma>1$. So when $\gamma<1$ a maximum of the curve σ vs q_0 appears at $q_0=q_1$, while when $\gamma>1$ there is no maximum at all, at least within the region where Eq. (3) holds true. In our example $\gamma=0.858$ and the first maximum in Fig. 1(b) results from the power law alone. From Eqs. (2) and (3) its height is

$$\sigma_{max}=(2^{\gamma}Cq_1^{\gamma-1}/4)^2. \tag{4}$$

Using the value of C fitted from the log-log plot of $P(p)$ we obtain σ_{max} whose value agrees well with the height of the first maximum in Fig. 1(b). From Eq. (4) it follows that the height of the first maximum decreases with q_1 as $\sigma_{max}\propto q_1^{2\gamma-2}$. However, similarly as Eq. (3), this scaling law gives only a general trend for a sufficiently large range of q_1 because of the fractal-induced oscillations.

All the other maxima in the SPA in Fig. 1(b) are a consequence of seemingly tiny oscillations of $P(p)$. In order to assess $P(p)$ and $\sigma(p_0)$ we use a simple geometric model valid for two-dimensional maps [9,11]. Assume that at the crisis point $p-p_c\equiv q=0$ the attractor is locally a set $\mathcal{A}^{(k)}$ of weighted parabolas accumulating at the point of collision with the basin of escape that is modeled by the half-plane $y>0$:

$$\mathcal{A}^{(k)}=\bigcup_{i=0}^{k}\mathcal{A}_i\cup\mathcal{A}_{k+1}, \tag{5}$$

where

$$\mathcal{A}_i=\{(x,y):y=-x^2-a\alpha^i+q\},$$

$$\mathcal{A}_{k+1}=\{(x,y):y=-x^2+q\},$$

the relative measure distribution $\mu(\mathcal{A}_i)=(1-\eta)\eta^i$ and $\mu(\mathcal{A}_{k+1})=\eta^{k+1}$, and the parameters $\alpha,\eta\in(0,1)$, $a>0$. Taking a larger k means to take a finer approximation of the infinite fractal set. The probability of the escape event is proportional to the measure of the attractor overlapping the basin of escape, $P(q)=\zeta\mu(q)$, $\zeta=$const. The function $P(q)$ reproduces the power law trend (3) with regular oscillations superimposed on it. The parameters α, η, and γ can be derived from the eigenvalues of the periodic orbit mediating in the crisis and the relation $\gamma=\ln\eta/\ln\alpha+1/2$ [9]. Similarly, ζ can be obtained from C and the other model parameters. The model curve can be fitted to numerical data [Fig. 1(a)] by choosing properly the parameter a.

Using the model (5) with the modulated $q(n)=q_0+q_1\cos(\omega_0 n)$, where $q_0\equiv p_0-p_c$, $q_1\equiv p_1$, the curve $\sigma(q_0)$ can be evaluated analytically. The theoretical curve in Fig. 1(b) deviates from the numerical one for larger $p-p_c$; this is connected with the corresponding deviation of the curves $P(p)$ in Fig. 1(a). The derivative of the model curve $P(q)$ fluctuates: it is infinite at the beginning of every oscillation $q=a\alpha^i$ and decreases for $q\to a\alpha^{i-1}$. It follows that for $q_0>q_1$ we can, in general, observe a series of maxima in the corresponding SPA to the right of the first maximum originating from the power scaling law (3). All the smaller scale oscillations of $P(q)$ yield only some modulation of the rising slope of the first maximum.

Let us estimate the height $\sigma_{max}^{(k)}$ of the maxima of the SPA appearing approximately at $q_{0,max}^{(k)}=a\alpha^k+q_1$ when k fulfils the condition $a\alpha^{k-1}>2q_1$. The top of the parabolic segment $\mathcal{A}_k$ touches then the basin boundary exactly once per modulation period. Peaks of this kind dominate in Fig. 1(b). The TC probability at the interval $q\in[a\alpha^k,a\alpha^{k-1}]$ can be approximated as [9]

$$P(q)=\zeta[\eta^{k+1}\sqrt{q}+\eta^k(1-\eta)\sqrt{q-a\alpha^k}]. \tag{6}$$

Replacing q with $q(n)$ in Eq. (6), expanding the first root in the Taylor series around $q_{0,max}^{(k)}$ up to the first-order term in q_1, and evaluating the Fourier transform of Eq. (6) in the continuous time approximation we get

$$\sigma_{max}^{(k)}=\left(\frac{\zeta\eta^k}{4}\right)^2\left[\frac{\eta}{\sqrt{q_{0,max}^{(k)}}}+\frac{8(1-\eta)}{3\pi}\sqrt{\frac{2}{q_1}}\right]^2. \tag{7}$$

A similar formula follows from inserting Eq. (6) into Eq. (2).

Let us now consider the case when not only the attractor but also the basin of escape is a fractal set. This happens, e.g., for crises in the kicked spin map [12]. The model describes the motion of a classical magnetic moment (spin) $\boldsymbol{S}$, $|\boldsymbol{S}|=S$, in the field of uniaxial anisotropy and impulse transversal magnetic field $\tilde{B}(t)=B\Sigma_{n=1}^{\infty}\delta(t-n\tau)$ given by the Hamiltonian $H=-A(S_z)^2-\tilde{B}(t)S_x$, where $A>0$ is the anisotropy constant. The time evolution is determined by the Landau-Lifschitz equation with damping, $\dot{\boldsymbol{S}}=\boldsymbol{S}\times\boldsymbol{B}_{eff}-(\lambda/S)\boldsymbol{S}\times(\boldsymbol{S}\times\boldsymbol{B}_{eff})$, where $\boldsymbol{B}_{eff}=-dH/d\boldsymbol{S}$ and $\lambda>0$ is the damping parameter. The equation can be integrated and denoting by $\boldsymbol{S}_n$ the spin vector just after the nth field pulse one finds a two-dimensional map $\boldsymbol{S}_{n+1}=T[\boldsymbol{S}_n]$ whose explicit form is given in Ref. [12].

At $S=1$, $A=1$, $\tau=2\pi$, and $\lambda=0.143\,700\,2\cdots$ the spin map exhibits attractor merging crisis at $B_c=1.2$: for $B<B_c$ two separate symmetric chaotic attractors corresponding to the spin "up" ($S_z>0$) and "down" ($S_z<0$) states coexist, whereas for $B>B_c$ the attractors merge and the spin jumps between these two states. The borders of the basins of escape are fractal [9] and the TC probability $P(B)$, i.e., the probability of jump between the two spin orientations, exhibits complicated oscillations originating from the overlap of the fractal structures of precritical attractors and their basins [Fig. 2(a)]. In order to study noisefree SR the pulse amplitude is modulated, $B\rightarrow B(n)=B_0+B_1\cos(\omega_0 n)$, where $B(n)$ denotes the nth pulse amplitude. The resulting curves $\sigma(B_0)$ exhibiting multiple maxima are shown in Fig. 2(b). Note that although the curve $P(B)$ is very complicated the SPA is still predicted very well using the approximation (2). As in the former case the first maximum follows from the trend (3) while all the others are the consequence of the complicated shape of the TC probability.

In order to study this case analytically [11], apart from the model of the fractal attractor (5) the model of the fractal basin is assumed [9] as a set $\mathcal{B}^{(l)}$ of stripes accumulating at the point of collision $y=0$:

$$\mathcal{B}^{(l)}=\bigcup_{j=0}^{l}\mathcal{B}_j\cup\mathcal{B}_{l+1}, \tag{8}$$

where $\mathcal{B}_j=\{(x,y):\beta^j(b-b_E)\leqslant y\leqslant\beta^j b\}$, $\mathcal{B}_{l+1}=\{(x,y)0\leqslant y\leqslant\beta^{l+1}b\}$, and $0<\beta<1$, $b>b_E>0$. The model curve $P(q)$, where $q\equiv B-B_c$, can be fitted to the numerical data [Fig. 2(a)] by choosing properly the parameters a and b. The other parameters can be evaluated either as in the case of the Hénon map, or from the magnified plots of the collision region of the fractal attractor and basin of escape [9].

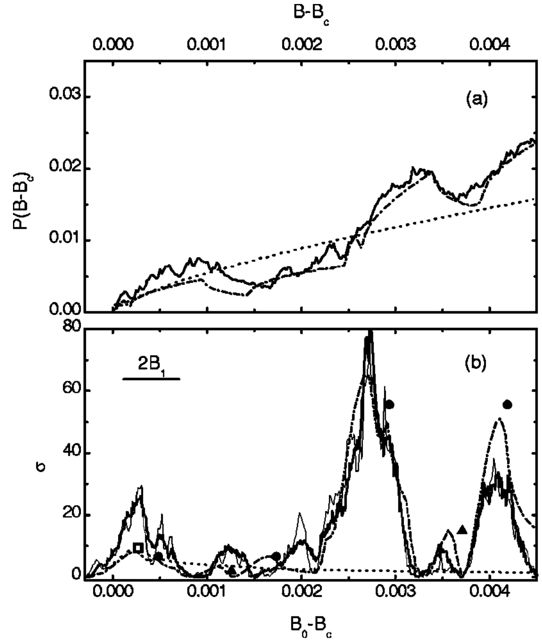

FIG. 2. As in Fig. 1 for the crisis in the spin map; the dash-dotted lines result now from combined models (5) and (8). The parameters are $B_1=3\times10^{-4}$, $T_0=1024$, $\gamma=0.707$, $\alpha=0.002\,34$, $\eta=0.285$, $\beta=0.124$, $b_E=3.27$, $C=0.715$, $\zeta=1.90$, $a=1.05$, and $b=4.022$. The dots denote the maxima $\sigma_{max}^{(k,l)}$ estimated from Eq. (7) with (from left) $(k,l)=(2,4)$, (2,3), (1,4), (1,3), the triangles denote the maxima $\tilde{\sigma}_{max}^{(k,l)}$ estimated from Eq. (9) with (from left) $(k,l)=(2,4)$ and (1,4), and the square — the first maximum from Eq. (4).

Strong maxima of the SPA resulting from the overlap of the two fractal structures again appear to the right of the first maximum in Fig. 2(b). There are maxima connected either with entering the stripe $\mathcal{B}_l$, [positive slope of $P(B)$] or the "hole" between stripes $\mathcal{B}_l$ and $\mathcal{B}_{l-1}$ [negative slope of $P(q)$] by the attractor branch $\mathcal{A}_k$. These maxima appear approximately at $q_{0,max}^{(k,l)}=a\alpha^k+b'\beta^l+q_1$, where $b'=b$ or $b'=b-b_E$ in the two above-mentioned cases, respectively; the top of the parabolic segment $\mathcal{A}_k$ touches the lower border of the stripe or hole again once per modulation period. The height of the maxima can be estimated as in the previous example, using the approximation (6) for the attractor and considering only one stripe $\mathcal{B}_l$ of the basin. In the first case, under the assumptions $a\alpha^{k-1}>2q_1$, $\beta^l b_E>2q_1$, the height of the maximum $\sigma_{max}^{(k,l)}$ can be obtained from Eq. (7) after substituting $q_{0,max}^{(k,l)}$ for $q_{0,max}^{(k)}$. In the second case, provided that $a\alpha^{k-1}>2q_1$, $\beta^{l-1}(b-b_E)-\beta^l b>2q_1$, the height can be estimated as

$$\tilde{\sigma}_{max}^{(k,l)}=\left(\frac{\zeta\eta^k}{4}\right)^2\left[\frac{\eta}{\sqrt{q_{0,max}^{(k,l)}-b'\beta^l}}-\frac{\eta}{\sqrt{q_{0,max}^{(k,l)}-b\beta^l}}+\frac{1-\eta}{\sqrt{q_{0,max}^{(k,l)}-b'\beta^l-a\alpha^k}}-\frac{8(1-\eta)}{3\pi}\sqrt{\frac{2}{q_1}}\right]^2. \tag{9}$$

Comparison between the theoretical and numerical results in

Fig. 2(b) shows that the agreement is not as good as in Fig. 1(a). Nevertheless, the shape of the curve $\sigma(B_0)$ and the order of magnitude of the maxima are qualitatively predicted.

Note that in both above-mentioned cases the curve $P(q)$ has a self-similar structure. Maxima of the SPA can be described by the introduced models for q_0 of order q_1. The effect of the smaller scale structure of $P(q)$ is typically hidden in the rising slope of the first maximum. For $q \gg q_1$ our simple models do not recover the smaller scale oscillations of $P(q)$ that are, however, present and lead to a series of virtually random sharp maxima of $\sigma(q_0)$.

To conclude, in this paper we showed that TC systems with a nonmonotonous derivative of the TC probability as a function of the control parameter are a generic class of models in which SMR appears. In particular, we considered the neighborhood of chaotic crises where noisefree SMR occurs as a consequence of the fractal structure of precritical attractors and, possibly, of their basins. This fractal structure is well reflected in the SPA; thus SR seems to be a subtle tool for the investigation of such structures. The SPA, in turn, can be obtained easily and with high accuracy from the TC probability using the modified linear approximation (2). It should be stressed that SMR near crises appear naturally as a consequence of the dynamical properties of the system, and not of an arbitrarily introduced potential with some invariance properties [6]. We believe that similar kind of mechanism leads to SMR also in other systems where the probability of the event determining SR (e.g., escape rate from potential well in bistable systems) has a nonmonotonous derivative (cf. [4]).

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