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How to control a chaotic economy?*

Janusz A. Hołyst[1,2], Tilo Hagel[2], Günter Haag[3], Wolfgang Weidlich[2]

[1] Institute of Physics, Warsaw Technical University, Koszykowa 75, PL-00-662 Warsaw, Poland
[2] Institut für Theoretische Physik II, Universität Stuttgart, Pfaffenwaldring 57/III, D-70550 Stuttgart, Germany
[3] Institut für Angewandte Wirtschaftsforschung, IAW, Ob dem Himmelreich 1, D-72074 Tübingen, Germany

Abstract. An economic system which exhibits chaotic behaviour has been stabilized on various periodic orbits by use of the Ott-Grebogi-Yorke method. This procedure has been recently applied to controlling chaotic phenomena in physical, chemical and biological systems. We adopt this method successfully for Feichtinger's generic model of two competing firms with asymmetrical investment strategies. We show that the application of this control method to the particular economic process considered brings a substantial advantage: one can easily switch from a chaotic trajectory to a regular periodic orbit and simultaneously improve the system's economic properties. Numerical simulations are presented in order to illustrate the effectiveness of the whole procedure.

Key words: Chaotic economy – Control of chaos – Deterministic chaos

1 Introduction

It is now generally accepted that many evolutionary processes of economic systems should be described by nonlinear equations (Haag 1990, Weidlich 1991, Weidlich and Braun 1992, Mosekilde and Thomsen 1992, Lorenz 1993, Haag, Hilliges and Teichmann 1993). However, one characteristic feature of nonlinear models is the possible appearance of deterministic chaos. This means that, although the constitutive equations of motion are deterministic, their solutions may exhibit a chaotic (non-periodic) structure (Chen 1988, Moseklide and Larsen 1988, Haag 1992, Puu 1992, Feichtinger 1992).

The observation of time series in economics always suggests the presence of stochastic and chaotic elements in the interactions and decisions of market agents.

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Correspondence to: J.A. Hołyst; FAX: 48(2)6282 171; e-mail: jholyst@if.pw.edu.pl

Hence, entrepreneurs or other decision makers in the economic system may face the difficult task of dealing with an economic system which behaves in an unpredictable way. However, there exist various causes for the unpredictability.

Firstly, it may occur that an otherwise regularly behaving economic system is disturbed by **exogenous** stochastic perturbations and random shocks. These influences are of course not predictable since they are caused by a variety of external factors.

Secondly, the chaotic dynamics of an economic system can be generated by the **endogenous** nonlinear dynamics without any external influence of the interacting relevant variables. This is a case of deterministic chaos that will be treated in this paper.

In the case of a chaotic economy instruments on the firm level (e.g. changes in the stock of strategic investments) which enable the management to influence and to control the market dynamics in a predictable way are highly welcome. Here we shall discuss a method for transforming the irregular dynamics into a regular one in the case of deterministic chaos. This ingenious method was recently proposed by Ott, Grebogi and Yorke (OGY) (Ott et al. 1990). It enables one to force the chaotic trajectory onto a periodic orbit by a correction mechanism. This particular mechanism has the form of a small, time-dependent perturbation of a certain control parameter. The OGY method has already been successfully applied to controlling chaos in physical (Ditto, Rauseo and Spano 1990), chemical (Parmananda, Sherard, Rollins and Dawald 1993) and biological (Garfinkel, Spano, Ditto and Weiss 1992) systems. The aim of this work is to point out that the OGY method can also be of great importance for the control in case of a chaotic economy.

In Section II of our paper we give a brief review of the OGY control method. In Section III we remind of the main properties of a chaotic model consisting of two competing firms which was recently developed by Feichtinger (Feichtinger 1992). This model will be treated as a generic model of a chaotic economy. We want to stress that the aim of this work is not to give a detailed foundation of the model. In Section IV we present results of our analytical and numerical calculations that demonstrate the stabilization of the chaotic trajectory in Feichtinger's model by use of the OGY method.

2 The method

In order to control the chaotic system we introduce an algorithm developed by Ott, Grebogi and Yorke (1990) in the form that was discussed in Dressler and Nitsche (1992). The main points of the method are based on the following observations:

(a) A chaotic solution of a nonlinear dynamic system (i.e. a chaotic attractor) can have even an infinite number of **unstable periodic orbits** in its immediate neighbourhood.
(b) In the neighbourhood of a periodic solution the evolution of the system can be approximated by an appropriate **local linearization** of the equations of motion.
(c) Small perturbations of any control parameter p of the equations of motion can shift the chaotic trajectory towards the so-called **stable manifold** of the chosen periodic orbit.
(d) The points belonging to the stable manifold approach the periodic solution in the course of time.

The application of the method becomes very straightforward for two-dimensional dissipative systems with a discrete time evolution. In Section III the method will be applied to the Feichtinger model (Feichtinger 1992). Let us assume that the evolution of the system is described by a two-dimensional map:

$$\mathbf{r}_{n+1} = \mathbf{F}(\mathbf{r}_n; p) \tag{1}$$

where $n = 1,2,3\ldots$ are subsequent discrete points in time, $\mathbf{r}_n = [x_n, y_n]$ pairs of real numbers describing our system and p a control parameter. Let us assume that we are able to temporally change the parameter p around its value p^0. Furthermore we assume that there exists an unstable periodic solution of (1). In the simplest case it is a so-called fixed point $P(\mathbf{r}^f)$, i.e. a point $\mathbf{r}^f$ which fulfills the equation

$$\mathbf{r}^f = \mathbf{F}(\mathbf{r}^f; p^0). \tag{2}$$

Let us now consider the points of the chaotic (strange) attractor with the same control parameter p^0. According to (a) some of these points lie in the neighbourhood of the fixed point $P(\mathbf{r}^f)$. According to (b) we approximate the evolutionary equation (1) using a linearization of the function $\mathbf{F}(\mathbf{r}_n; p^0)$ in $\mathbf{r}_n - \mathbf{r}^f = \delta\mathbf{r}_n$. Besides small deviations of the parameter p from p^0 shall also be allowed. In this way we obtain

$$\delta\mathbf{r}_{n+1} \cong \hat{J}^{\mathbf{F}}_{\mathbf{r}^f}\delta\mathbf{r}_n + \mathbf{w}\delta p_n \tag{3}$$

with $\delta p_n = p_n - p^0$. $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$ is the Jacobian matrix of the function $\mathbf{F}(\mathbf{r}_n; p)$ at the point $(\mathbf{r}^f; p^0)$

$$\hat{J}^{\mathbf{F}}_{\mathbf{r}^f} = \begin{bmatrix} \dfrac{\partial F^x(x,y;p)}{\partial x}, & \dfrac{\partial F^x(x,y;p)}{\partial y} \\ \dfrac{\partial F^y(x,y;p)}{\partial x}, & \dfrac{\partial F^y(x,y;p)}{\partial y} \end{bmatrix}_{(\mathbf{r}=\mathbf{r}^f; p=p^0)} \tag{4}$$

$\mathbf{w}$ is a vector defined by

$$\mathbf{w} = \left(\frac{\partial \mathbf{F}}{\partial p}\right)_{(\mathbf{r}=\mathbf{r}^f; p=p^0)} \tag{5}$$

The Jacobian $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$ possesses the eigenvalues λ_1, λ_2 and the eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ defined by the equations

$$\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}\mathbf{e}_{1(2)} = \lambda_{1(2)}\mathbf{e}_{1(2)}. \tag{6}$$

The unstable fixed point $\mathbf{r}^f$ in the neighbourhood of the chaotic attractor must be a so-called saddle point (Schuster 1988). This is equivalent to the condition $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or vice versa. The corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ define the stable and unstable directions of the fixed point $\mathbf{r}^f$ in the domain of linearity (see Fig. 1). Assuming that the eigenvectors $\mathbf{e}_{1(2)}$ are normalized, i.e. $|\mathbf{e}_{1(2)}| = 1$, one can now find a pair of vectors $\mathbf{f}_1$, $\mathbf{f}_2$ that are perpendicular to the unstable and stable axis, respectively:

$$\mathbf{f}_1 \cdot \mathbf{e}_2 = \mathbf{f}_2 \cdot \mathbf{e}_1 = 0, \tag{7}$$

$$\mathbf{f}_1 \cdot \mathbf{e}_1 = \mathbf{f}_2 \cdot \mathbf{e}_2 = 1. \tag{8}$$

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ together form a so-called contravariant basis.

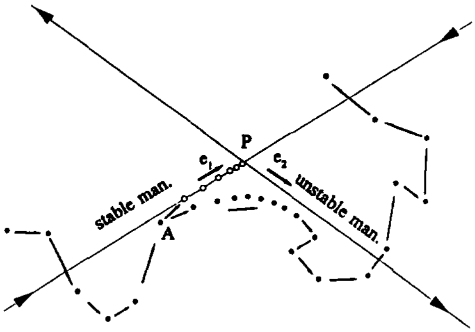

Fig. 1. Schematic diagram of the stabilization procedure. Black circles and blackarrows represent the unperturbed chaotic trajectory that in point A approaches the neighbourhood of the stable manifold of the unstable fixed point P. By means of a perturbation of a control parameter the trajectory can be shifted to this manifold. Further evolution towards the fixed point is represented by white circles. Along the stable manifold the fixed point is attractive

Our key conception is to vary the parameter p in an appropriate way (see (3)) in order to shift the position of the trajectory $\mathbf{r}_n$ of the system to the stable manifold of the fixed point $\mathbf{r}^f$. Because of the attractivity of the fixed point along the stable direction the trajectory will approach the fixed point $\mathbf{r}^f$. If $\mathbf{r}_{n+1}$ is not on the stable manifold, the trajectory will fail to reach the fixed point $\mathbf{r}^f$ and within a few time steps it will disappear somewhere in the state space. To cause the trajectory of the system to lie on the stable manifold, the vector $\delta\mathbf{r}_{n+1}$ should possess no component perpendicular to the stable axis. The condition for such a shift can easily be expressed using the contravariant eigenvector $\mathbf{f}_2$:

$$\mathbf{f}_2 \cdot \delta\mathbf{r}_{n+1} = 0. \tag{9}$$

Here the usefulness of the contravariant basis becomes clear. The contravariant basis is introduced in order to obtain a vector $\mathbf{f}_2$ which is perpendicular to the stable direction (in high-dimensional systems to all stable directions). $\delta\mathbf{r}_{n+1}$ to lie on the stable manifold is only guaranteed if $\delta\mathbf{r}_{n+1}$ is perpendicular to $\mathbf{f}_2$. Taking into account that $\delta\mathbf{r}_n$ can be expanded as follows:

$$\delta\mathbf{r}_n = (\mathbf{f}_1 \cdot \delta\mathbf{r}_n)\mathbf{e}_1 + (\mathbf{f}_2 \cdot \delta\mathbf{r}_n)\mathbf{e}_2, \tag{10}$$

condition (9) together with (3),(6), and (10) leads to the following shift of the control parameter p (Dressler 1992):

$$\delta p_n = -\frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w}} \mathbf{f}_2 \cdot \delta\mathbf{r}_n. \tag{11}$$

Fig.1 illustrates our procedure. For higher periodic orbits that consist of finite series $[\mathbf{r}_1^f, \mathbf{r}_2^f, \ldots \mathbf{r}_k^f]$ with $\mathbf{r}_{n+1}^f = \mathbf{F}(\mathbf{r}_n^f; p)$ for $n = 1, 2 \ldots (k-1)$ and $\mathbf{r}_1^f = \mathbf{F}(\mathbf{r}_k^f; p)$ the stabilization can also be easily performed if in the equations (4–5) one uses the k-th iteration of the function $\mathbf{F}$, $\mathbf{F}^k(\mathbf{r}; p) = \underbrace{\mathbf{F} \circ \mathbf{F} \circ \mathbf{F} \circ \ldots \circ \mathbf{F}}_{k\text{-}times}(\mathbf{r}; p)$ instead of the function $\mathbf{F}(\mathbf{r}; p)$. Then each of the points $\mathbf{r}_1^f, \mathbf{r}_2^f, \ldots \mathbf{r}_k^f$ is a fixed point of the function $\mathbf{F}^k(\mathbf{r}; p)$ and one can evaluate the corresponding values of parameter perturbations (11) in k-steps (Ott, Grebogi, York 1990, Dressler, Nitsche 1992).

3 The model

Recently a very illustrative model of two competing firms X and Y was introduced by Feichtinger (Feichtinger 1992). This model is a nonlinear extension of Richardson-type models of arms race. Richardson-type models are motivated and described in detail by Behrens (Behrens, 1992).

It is not the purpose of the present article to give further justifications of this model. Instead we take it as a proxy for a class of nonlinear economic models to which the OGY method can be applied.

In its economic version the model decribes two competing firms X and Yacting on the same market of goods where x_n and y_n are the sales of the firms X and Y, respectively measured on a discrete time scale $n = 0,\ 1, 2 \ldots$ Both firms observe carefully one another because, although they have the opportunity to invest, their investment strategies depend on each other and are assumed to be asymmetric: firm X will invest more if it has an advantage over firm Y while firm Y will invest more if it is in a disadvantageous position compared to firm X. These assumptions mean that both firms invest willingly provided that $x_n > y_n$ and they both diminish their investments in the opposite situation. In case that neither of the firms invests we shall assume an exponential decay of their sales. Taking this into account we can write the following system of two coupled difference equations for the evolution of sales of both firms:

$$x_{n+1} = F^x(x_n, y_n; p) = (1 - \alpha)x_n + \Phi^x(x_n, y_n; a, c) \tag{12}$$

$$y_{n+1} = F^y(x_n, y_n; p) = (1 - \beta)y_n + \Phi^y(x_n, y_n; b, c) \tag{13}$$

where

- constants α and β (with $0 < \alpha, \beta < 1$) are time rates of sales decay of both firms under zero investments,
- functions $\Phi^x(x_n, y_n; a, c)$ and $\Phi^y(x_n, y_n; b, c)$ describe the influence of investments at time n on the sales at time $n+1$, with the parameters a, b, c describing the investment behaviour.

The presence of the symbol $p \in \{a, b, c, \alpha, \beta\}$ in (12) and (13) represents the dependence of functions $F^x(x, y; p)$ and $F^y(x, y; p)$ on the parameters of the system. To be in agreement with the firms' investment strategies we need to choose some step-like functions for $\Phi^x(x_n, y_n; a, c)$ and $\Phi^y(x_n, y_n; b, c)$. According to Feichtinger (1992) we assume

$$\Phi^x(x_n, y_n; a, c) = \frac{a}{1 + \exp(-c(x_n - y_n))} \tag{14}$$

and

$$\Phi^y(x_n, y_n; b, c) = \frac{b}{1 + \exp(-c(x_n - y_n))}. \tag{15}$$

Parameters a and b describe the efficiencies of investments of both firms or scales of their investments. Parameter c is a measure of the "elasticity" of the investment strategies. Equations (12–13) together with (14–15) form a two-dimensional map $\mathbf{r}_{n+1} = \mathbf{F}(\mathbf{r}_n; p)$ with $\mathbf{r}_n = [x_n, y_n]$ and $\mathbf{F} = [F^x, F^y]$ that fully defines the evolution of our discrete dynamical system. Depending on the specific values of parameters α, β,

a, b, and c the solutions of (12–13) can be regular or chaotic (Feichtinger, 1992). It is easy to see that for chaotic behaviour the conditions

$$\alpha < \beta, a < b \tag{16}$$

are necessary since in all other cases the time paths become monotonic (e. g. $\lim_{n\to\infty} x_n, y_n = 0$ in case $\alpha > \beta, a < b$).

4 Results

Choosing as a generic example the following set of parameter values $a^0 = 0.16$, $b^0 = 0.9$, $c^0 = 105$, $\alpha^0 = 0.46$, $\beta^0 = 0.7$ one can easily observe chaotic solutions of x_n and y_n (Fig. 2). The mean values of x_n and y_n averaged over the chaotic attractor are $\bar{x}_{chaotic} \approx 0.01813$ and $\bar{y}_{chaotic} \approx 0.06702$. Let us now assume that firm X is not satisfied with the chaotic character of its sales (or country X would like to avoid rapid changes of its armament potential) and that it will try to stabilize its market position (or country X would like to stabilize the global armaments level). One of the possibilities to achieve this aim is to settle down the chaotic trajectory on the fixed point $\mathbf{r}^f = [x^f, y^f]$ of the map (12–13) by appropriately changing the scale a of its investments. The values of x^f and y^f can easily be found numerically as solutions of the system of two coupled transcendental algebraic equations

$$x^f = (1-\alpha^0)x^f + \frac{a^0}{1+\exp(-c^0(x^f - y^f))}, \tag{17}$$

$$y^f = (1-\beta^0)y^f + \frac{b^0}{1+\exp(-c^0(x^f - y^f))}. \tag{18}$$

For the chosen set of parameter values we obtain $x^f \approx 0.01182216$, $y^f \approx 0.04369976$ (see point P in Fig. 2). It is worth stressing that $\mathbf{r}^f$ is an **unstable** fixed point, i.e. the system will stay in the neighbourhood of $\mathbf{r}^f$ only for a limited time and then escape into other domains of the "phase space" (x_n, y_n). The Jacobian $\hat{J}^{\mathrm{F}}_{\mathbf{r}_0}$ of the map (12–13) at the fixed point $\mathbf{r}^f$ reads

$$\hat{J}^{\mathrm{F}}_{\mathbf{r}_0} \approx \begin{bmatrix} 1.091602, & -0.5516017 \\ 3.102760, & -2.802760 \end{bmatrix}. \tag{19}$$

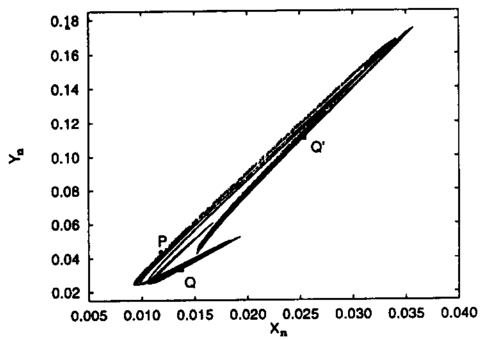

Fig. 2. Chaotic evolution of sales x_n and y_n in the Feichtinger model (40000 points are depicted). Unstable fixed point P and unstable period two (Q,Q') are marked

Its eigenvalues are $\lambda_1 \approx 0.5866502$ and $\lambda_2 \approx -2.297808$. The contravariant vector $\mathbf{f}_2$ corresponding to the unstable direction is $\mathbf{f}_2 \approx [-1.089834, 1.190519]$.

Furthermore we can write the vector $\mathbf{w}$ (5) as $\mathbf{w} = [0.03398870, 0.0]$. So the final form of the control equation (11) reads:

$$\delta a_n = -62.03245[-1.089834(x_n - 0.01182216) + 1.190519(y_n - 0.04369976)] \quad (20)$$

where -62.03245 is the numerical value of the factor $\dfrac{-\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w}}$.

Suppose now that firm X is able to change its control parameter a. (This is reasonable because a describes market investments of X and is its only accessible parameter. The second parameter in the investment function, c, is a physical parameter which cannot be varied by the firm.) We can interprete a as the investment budget of firm X. Normally such a budget is fixed or is allowed to vary only within the small interval $(a^0 - \Delta a; a^0 + \Delta a)$ where Δa is some fixed value. Looking at Fig. 5 it becomes obvious why firm X has to choose the OGY method to control its market sales: It seems reasonable to assume that firm X can change its investment budget somewhat. However, no non-chaotic domain can be reached in this way since there is $a^0 < 0.16$ whereas the first non-chaotic interval begins at $a^0 = 0.207$. Thus firm X would have to increase its investment budget by 29.375% to shift it into the domain of regular behaviour. This seems impracticable in most cases. On the other hand firm X can regulate the dynamic market behaviour by changing a only slightly if the OGY method is used.

Then the process of controlling can be performed as follows:

- If at time n the specific values of sales x_n and y_n are such that $|\delta a_n| > \Delta a$ (where δa_n is calculated from (20)), the control **will not be activated** and firm X will invest by using its "standard" value of the parameter $a_n = a^0$.
- If at time n the specific values of sales x_n and y_n are such that $|\delta a_n| < \Delta a$, then the control **will be activated** and firm X will change its market investment parameter to $a_n = a^0 + \delta a_n$ according to (20).

The numerical simulation of this process of investment strategy is presented in Figs. 3a–3c. During the first 100 steps the system is not controlled at all and the chaotic evolution of the sales x_n and y_n can easily be seen. At time $n = 100$ (depicted by arrows A in Figs. 3a–3c) firm X decides to control the market by changing its investment parameter $a^0 = 0.16$ by the maximum shift of $\Delta a = 0.008$, i.e. 5% of the "standard value" of a^0. Firm X has to wait for the occasion to activate this control strategy until time $n = 199$ (arrows B) because up to this moment sales x_n, y_n do not fulfill the condition $|\delta a_n| < 0.008$ and so the control procedure cannot be activated. At $n = 199$ the control is switched on for the first time and one can see from Fig. 3c that the value of the parameter a_n changes from $a_{198} = 0.16$ to $a_{199} \approx 0.164$. From this moment on the control is permanently applied until time $n = 250$ (arrows C) when firm X decides to switch off the control. It is worth mentioning that, except for the first few steps, the values of the perturbations of the control parameter a were extremely small and only the last digit was oscillating, $\delta a_n \approx a^\epsilon \approx 16 \cdot 10^{-7}$ for $n > 212$. The value of the constant a^ϵ depends on the numerical precision of our calculations. For "absolutely exact" calculations one must take $a^\epsilon = 0$. In Fig. 3a we can see the stabilization of the sales of firm X at a level $x^f \approx 0.0118$ for $n > 200$. At the same time the sales of firm Y **are also stabilized** (Fig. 3b) at a level $y^f \approx 0.0437$.

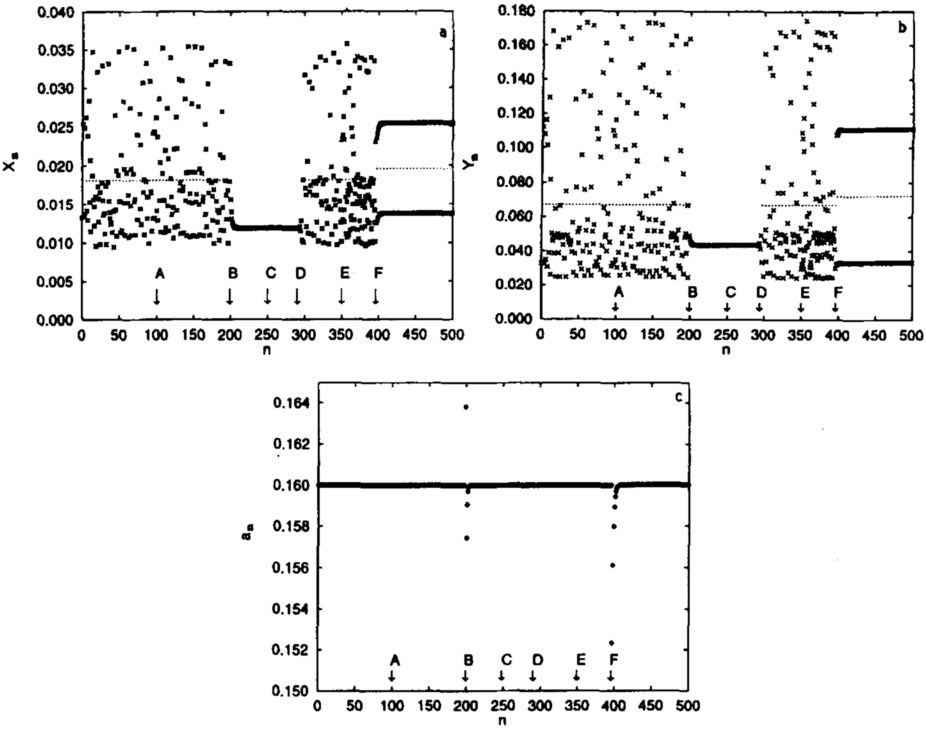

Fig. 3. Time dependence of sales x_n, sales y_n, and values of the investment parameter a_n. Results of the stabilization on the fixed point and on period two can be easily seen. Dashed lines depict the **mean values** of the sales in various time intervals (chaotic or periodic ones). The meaning of the points A, B, C, D, E, and F is explained in the text

Unfortunately the values of x^f (as well as of y^f) are smaller than the mean values of the sales of firm X (and Y) in the chaotic region $n = 0,1,2...199$

$$\frac{x^f}{\bar{x}_{chaotic}} \approx \frac{y^f}{\bar{y}_{chaotic}} \approx 0.65. \tag{21}$$

Thus, for this kind of investment behaviour an uncontrolled, chaotic evolution of the market will be preferred if firm X wants to maximize its sales. However, the switching from chaotic to non-chaotic behaviour **always** results in the serious advantage that the firm can perform market forecasting, in other words, it will gain more certainty about the volume of its sales x_n during the next time steps. In a chaotic system the trajectories are extremely sensitive to the initial conditions and so there would be only little knowledge of this kind.

On the other hand, regarding equations (12–13) as a model of arms races (Behrens 1992), the consequence of the control (20) is undoubtedly a **stabilization and decrease** (by around 35%) of the military potentials of both countries.

At time $n = 250$ (arrow C in Figs. 3) we let firm X switch off its market control (from now on the control parameter remains constant, i.e. $a_n = a_{250}$ for $n > 250$). Despite this the market stays stable for about 45 steps as one can see from

Figs. 3a–3b (arrow D). It is not until now that the previous stabilization is "forgotten" and the system starts again to behave in a chaotic way. At time $t = 350$ (arrow E) firm X decides to switch on the market control again. However, at this particular time the control is used to stabilize the market on period two. In fact, one can easily find in a numerical way that there is

$$F(\mathbf{r}_1^f; p^0) = \mathbf{r}_2^f, \tag{22}$$

$$F(\mathbf{r}_2^f; p^0) = \mathbf{r}_1^f, \tag{23}$$

with $\mathbf{r}_1^f \approx [0.02535584,\ 0.1110008]$, $\mathbf{r}_2^f \approx [0.01371228,\ 0.03341212]$ (see points Q and Q′ in Fig. 2). Using now the point $\mathbf{r}_1^f$ as the fixed point of the transformation $\mathbf{F}(\mathbf{F}(\mathbf{r}; p^0))$ the calculations are analogous to those performed for the fixed point $\mathbf{r}^f$. For example, the Jacobian $\hat{J}_{\mathbf{r}_1}^{\mathrm{FF}}$ of the function $\mathbf{F}(\mathbf{F}(\mathbf{r}; p^0))$ at the point $\mathbf{r}_1^f$ is the product of the Jacobians $\hat{J}^{\mathrm{F}}$ of the function $\mathbf{F}(\mathbf{r}; p^0)$ at the points $\mathbf{r} = \mathbf{r}_2^f$ and $\mathbf{r} = \mathbf{r}_1^f$, i.e.

$$\hat{J}_{\mathbf{r}_1}^{\mathrm{FF}} = \hat{J}_{\mathbf{r}_2}^{\mathrm{F}} \cdot \hat{J}_{\mathbf{r}_1}^{\mathrm{F}} \tag{24}$$

and in our case it can be written explicitly as

$$\hat{J}_{\mathbf{r}_1}^{\mathrm{FF}} \approx \begin{bmatrix} 1.180227, & -0.4870004 \\ 4.995705, & -2.646558 \end{bmatrix}. \tag{25}$$

The Jacobian (25) possesses the eigenvalues $\lambda_1 \approx 0.3750580$ and $\lambda_2 \approx -1.841390$. The contravariant vector $\mathbf{f}_2$ corresponding to the unstable direction is $\mathbf{f}_2 \approx [-2.283011, 1.380863]$. The control procedure for period two was first activated at time $n = 397$ (arrow F) and after few further iterations the system was stabilized. Now the sales of the firms X and Y oscillate between x_1^f and x_2^f and between y_1^f and y_2^f, respectively. Comparing the mean values of the solutions of period two the chaotic mean values

$$\frac{x_1^f + x_2^f}{2\bar{x}_{chaotic}} \approx \frac{y_1^f + y_2^f}{2\bar{y}_{chaotic}} \approx 1.077 \tag{26}$$

one can see that the stabilization of the chaotic dynamics on period two brings a 7.7% increase of sales for **both competing firms** although the control procedure was performed by firm X only. This effect is of course at the cost of all other firms acting on the market and influencing the dynamics of our model.

It is worth mentioning what would happen if both firms tried to control the system simultaneously but with an incompatible choice of fixed points, e.g. firm X seeking to adjust a so as to settle onto x^f, y^f whilst firm Y trying to adjust b so as to settle onto the orbit of period two. It is clear that if the control mechanism is strictly performed as mentioned above (control should be activated for $|\delta a_n| < \Delta a$ and $|\delta b_n| < \Delta b$, respectively) the firm whose control is activated first will succeed. Once the system is stabilized onto the fixed point or the orbit of period two, the other firm will have no possibility to get the system out of the dynamics imposed by the controlling firm without drastically changing its investment behaviour.

The time τ required to reach a stabilized orbit (the length of a chaotic transient) depends on the system parameters, the value of the maximal perturbation of the control parameter Δa, and on initial conditions. The mean value $\langle \tau \rangle$ calculated for an ensemble of initial conditions behaves as (Ott, Grebogi and Yorke 1990)

$$\langle \tau \rangle \sim (\Delta a)^{-\gamma} \tag{27}$$

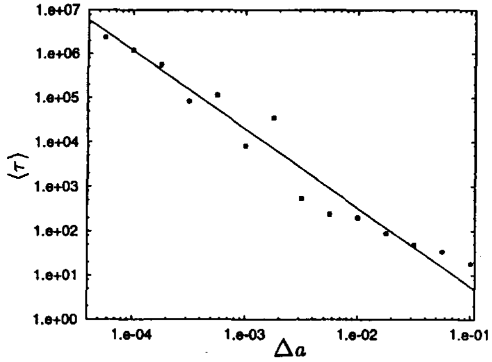

Fig. 4. Mean values $\langle \tau \rangle$ of the time needed to create the stabilized orbit as a function of the values Δa of the maximal allowed perturbation of the control parameter. Symbols represent numerical data, a line shows the result of theoretical predictions (Eqs. 27, 28)

with the exponent γ depending on the eigenvalues of the Jacobian calculated at the fixed point:

$$\gamma = 1 - \frac{\ln|\lambda_2|}{2\ln|\lambda_1|}. \tag{28}$$

Because there is $|\lambda_1| < 1$ and $|\lambda_2| > 1$, the exponent γ is always positive. Therefore the mean time $\langle \tau \rangle$ decreases with an increase of the parameter Δa. The numerical results are displayed in Fig. 4. They are in quite a good agreement with (27–28).

We assumed here that the control was performed by firm X only. However the whole procedure could be used by firm Y as well for which the parameter b should be perturbed.

There are several problems that must be solved before the idea of controlling a chaotic economic system presented here can be practically applied:

- The influence of additional **non-deterministic noise** can substantially limit the "life time" of the stabilized orbit. This non-deterministic noise can arise from an uncertain $b = b_n = \bar{b} + \tilde{\varepsilon}_n$ where $\bar{b}$ is a long run average value of b and $\tilde{\varepsilon}_n$ a small stochastic perturbation. This is reasonable because a complete knowledge of the value b in firm Y's investment function Φ^y cannot be assumed. In order to avoid this destabilizing effect of external noise the amplitude Δa should be large enough (Ott, Grebogi and Yorke 1990).
- In case of market models with continuous time-dependence an appropriate Poincaré surface should be performed (Schuster 1988).
- Even in the case that no exact theoretical model describing the economic dynamics is known the stabilization problem can be solved by use of the Grassberger-Procaccia method (Schuster 1988, Roy, Murphy, Maier and Gills 1992).

5 Conclusions and outlook

We have shown in this paper that for a simple model of an economic system consisting of two competing firms on a goods market an effective control of chaos is

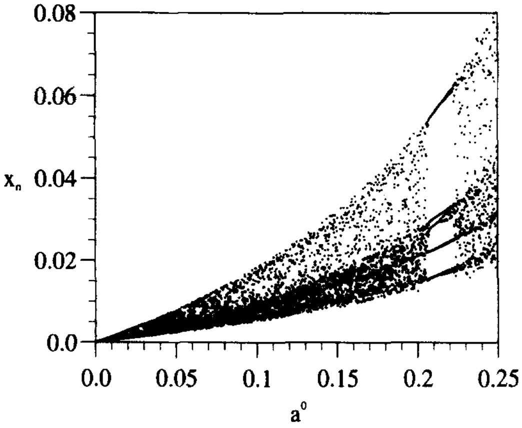

Fig. 5. Bifurcation diagram of the variable x for the investment parameter a^0. Chaos occurs in the whole range $0 < a^0 < 0.207$ ($a^0 = 0.16$ in the model). Therefore it is reasonable for firm X to use the OGY method

possible. Here the variables behaving chaotically are the market shares of the firms. The decision makers of one firm take influence on the economy by controlling the stock of its market investments in dependence on the current sales of both firms. This control was performed using a small time-dependent feedback calculated from the Ott-Grebogi-Yorke algorithm. It is one of our main results that a moderate variation of few percent of the corresponding parameter is sufficient in order to prevent chaos and to lock-in onto a predictable dynamical mode. The effect of this control is a periodic evolution of the system. As there are several periodic orbits available, the system can easily be switched by appropriate market strategies between different types of periodic motion (period one, period two, etc.) and the optimal solution can be chosen. As a consequence we conclude that, at least for particular cases, the control of chaos is not costly. It may even save a lot of management power and improper investments if the control starts at the "right" time when the trajectory of the system is close to an unstable periodic orbit. So far our results are based on discrete-time dynamic equations. The control of chaos in continuum models of economic processes will be investigated in further work. The influence of external noise (exogenous fluctuations) on the effectivity of the chaos-control mechanism is of practical importance and will also be investigated. The method can also be used for empirical systems where no equations of motion are known provided that the dynamics of such systems is "sufficiently deterministic", i.e. the level of "indeterministic noise" is low enough. A further point of current work is the application of the Grassberger-Procaccia analysis to real economic time series which exhibit a chaotic structure in order to construct their dynamics and to investigate control mechanisms of chaos.

References

Behrens D (1992) Two-and three-dimensional models on the arms race. Diplomarbeit, Institut für Ökonometrie, Operations Research und Systemtheorie, Technische Universität Wien
Chen P (1988) Empirical and theoretical evidence of economic chaos. Syst Dynam Rev 4: 81–108

Ditto WL, Rauseo SN, Spano ML (1990) Experimental control of chaos. Phys Rev Lett 65: 3211–3214
Dressler U, Nitsche G (1992) Controlling chaos using time delay coordinates. Phys Rev Lett 68: 1–4
Feichtinger G (1992) Nonlinear threshold dynamics: further examples for chaos in social sciences. In: Haag G, Mueller U, Troitzsch KG (eds) Economic evolution and demographic change. Springer, Berlin
Garfinkel A, Spano ML, Ditto WL, Weiss JN (1992) Controlling cardiac chaos. Science 257: 1230–1235
Haag G (1990) A master equation formulation of aggregate and disaggregate economic decision-making. Sistemi Urbani 1: 65–81
Haag G (1992) Chaotic behaviour in spatial systems and forecasting. Rev d'Econ Reg Urb 5: 735–751
Haag G, Hilliges M, Teichmann K (1993) Towards a dynamic disequilibrium theory of economy. In: Nijkamp P, Regiani A (eds) Nonlinear evolution of spatial economic systems. Springer, Berlin
Lorenz HW (1993) Nonlinear dynamical economics and chaotic motion. Springer, Berlin
Moseklide E, Larsen ER (1988) Deterministic chaos in the beer production-distribution model. Syst Dynam Rev 4: 131–147
Mosekilde E, Thomsen JS (1992) Nonlinear interactions in the economy. In: Haag G, Mueller U, Troitzsch KG (eds) Economic evolution and demographic change. Springer, Berlin
Ott E, Grebogi C, Yorke JA (1990) Controlling chaos. Phys Rev Lett 64: 1196–1199
Parmananda P, Sherard P, Rollins RW, Dewald HD (1993) Control of chaos in an electrochemical cell. Phys Rev A 47: R3003–3006
Puu T (1992) A chaotic process with slow feed back: the case of business cycles. In: Haag G, Mueller U, Troitzsch KG (eds) Economic evolution and demographic change. Springer, Berlin
Roy R, Murphy TW, Maier TD, Gills Z (1992) Dynamical control of a chaotic laser: experimental stabilization of a globally coupled system. Phys Rev Lett 68: 1259–1262
Schuster HG (1988) Deterministic chaos. VCH Verlagsgesellschaft, Weinheim
Weidlich W (1991) Physics and social science–the approach of synergetics. Phys Rep 204: 1–163
Weidlich W, Braun M (1992) The master equation approach to nonlinear economics. J Evol Econ 2: 233–265