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Higher order clustering coefficients in Barabási–Albert networks

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Abstract

Higher order clustering coefficients $C(x)$ are introduced for random networks. The coefficients express probabilities that the shortest distance between any two nearest neighbours of a certain vertex i equals x , when one neglects all paths crossing the node i . Using $C(x)$ we found that in the Barabási–Albert (BA) model the average shortest path length in a node's neighbourhood is smaller than the equivalent quantity of the whole network and the remainder depends only on the network parameter m . Our results show that small values of the standard clustering coefficient in large BA networks are due to random character of the nearest neighbourhood of vertices in such networks.

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1. Introduction

During the last few years studies of random, evolving networks (such as the Internet, WWW, social networks, metabolic networks, food webs, etc.—for review see Refs. [1,2, and references therein]) have become a very popular research domain among physicists. A lot of efforts were put into investigation of such systems in order to recognise their structure and to analyse emerging complex dynamics. We learned that the networks are far from being random as Erdős and Rényi assumed in random graph

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theory [3,4], but surely they are not so ordered as crystals. Despite network diversity, most of the real web-like systems share three prominent features [1,2]:

- The average shortest path length l is small. In order to connect two nodes in a network typically only a few edges need to be passed.
- The average clustering coefficient C is large. Two nodes having a common neighbour are also likely to be neighbours.
- The probability that a randomly selected node has exactly k nearest neighbours follows a power-law (scale-free) distribution $P(k) \sim k^{-\gamma}$ with $2 < \gamma < 3$ in most of real systems.

A considerable number of network models has been studied in order to capture the above characteristics. Most of these are based on two ingredients originally introduced by Barabási and Albert [5,6]: continuous network growth and preferential attachment. In Barabási–Albert (BA) model, network starts to grow from an initial cluster of m fully connected sites. Each new node that is added to the network creates m links that connect it to previously added nodes. The preferential attachment means that the probability of a new link to end up in a vertex i is proportional to the connectivity k_i of this vertex. The validity of the preferential attachment was confirmed within real networks analyses [7–9]. The BA algorithm generates networks with the desirable scale-free distribution $P(k) \sim k^{-3}$ and small values of the average shortest path. One can also observe a phase transition for spins located at BA network vertices with a critical temperature increasing as a logarithm of the system size [10–13]. The only striking discrepancy between the BA model and real networks is that the value of the clustering coefficient predicted by the theoretical model decays very fast with network size and for large systems is typically several orders of magnitude lower than found empirically.

In this paper we extend the standard definition of the clustering coefficient by introducing higher order clustering coefficients that describe interrelations between vertices belonging to the nearest neighbourhood of a certain vertex in complex network. Global characteristics like the standard clustering coefficient and the average shortest path do not provide a useful insight into complex network structure and dynamics. We hope that the higher order clustering coefficient analyses in real systems [14] may give some guidelines as to how to model clustering mechanisms. Here, we study higher order clustering coefficients in BA model. Our results provide a vivid evidence that the BA networks are blind to clustering mechanisms.

2. Model description

The standard clustering coefficient C is one of global parameters used to characterise the topology of complex networks. For the first time it was introduced by Watts and Strogatz [15] to characterise local transitivity in social networks. Clustering coefficient gives the probability that two nearest neighbours of the same node are also mutual neighbours. Let us focus on a selected node i in a network, having k_i edges which

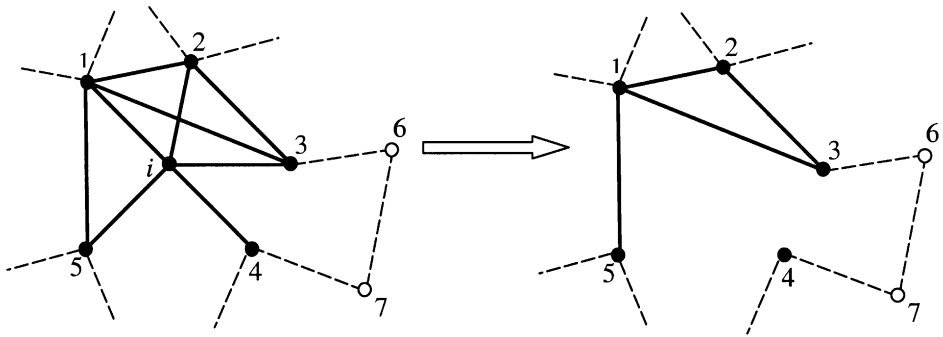


Fig. 1. Stage 1: Network in the vicinity of a node i . Stage 2: After removing the node i and its adjacent links.

connect it to k_i other nodes. The value of the clustering coefficient of the node i is given by the ratio between the number of edges E_i that actually exist between these k_i nodes and the total number $k_i(k_i - 1)/2$ of such edges that could exist in the neighbourhood of i

$$C_i = \frac{2E_i}{k_i(k_i - 1)}. \tag{1}$$

The clustering coefficient of the whole network is the average of all individual C_i 's

We define a *clustering coefficient of order x* for a node i as the probability that there is a distance of length x between two neighbours of a node i . Putting the number of such x -distances equal to $E_i(x)$, the higher order clustering coefficients follow:

$$C_i(x) = \frac{2E_i(x)}{k_i(k_i - 1)}. \tag{2}$$

$C(x)$ is the mean value of $C_i(x)$ over the whole network. Note that $C(x)$ reduces to the standard clustering coefficient for $x=1$ and $\sum_x C(x)=1$ for the BA networks with $m \geq 2$.

The above definition becomes comprehensible after examining Fig. 1. Let us assume a node i with $k_i = 5$. The node determines its nearest neighbourhood that in this case consists of vertices $\{1, 2, 3, 4, 5\}$. The aim is to find higher order clustering coefficients $C_i(x)$ of the node i . The table below gives the shortest distances between vertices adjacent to the node i . The distances were taken from the Fig. 1 (Stage 2)

	1	2	3	4	5
1	—	1	1	4	1
2	1	—	1	4	2
3	1	1	—	3	2
4	4	4	3	—	5
5	1	2	2	5	—

Summing over all pairs of vertices one obtains

$$\begin{matrix} x \\ E_i(x) \\ C_i(x) \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 2 & 1 \\ 0.4 & 0.2 & 0.1 & 0.2 & 0.1 \end{pmatrix}.$$

3. Numerical results

Fig. 2 shows the higher order clustering coefficients dependence on the network size N . Nodes determined as belonging to the nearest neighbourhood of any vertex in BA

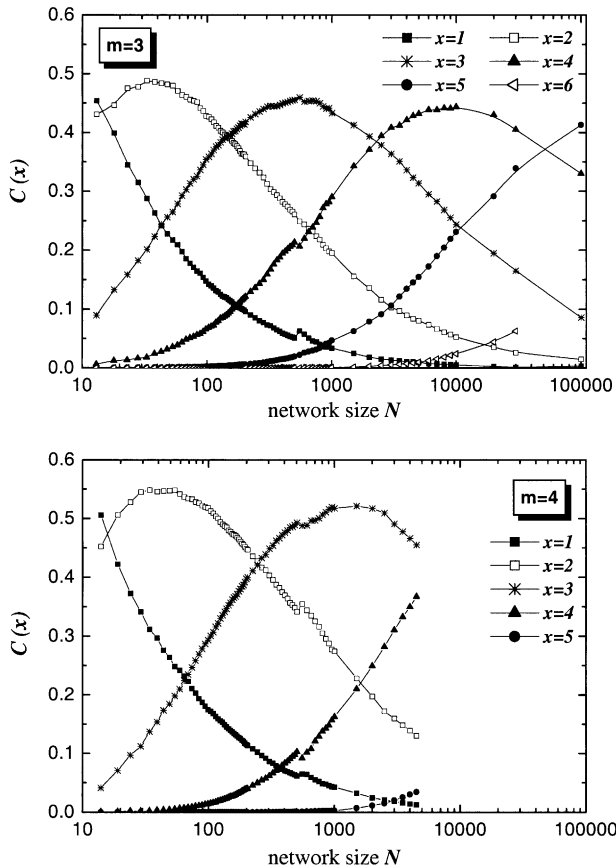


Fig. 2. Higher order clustering coefficients versus network size N for $m = 3$ and 4.

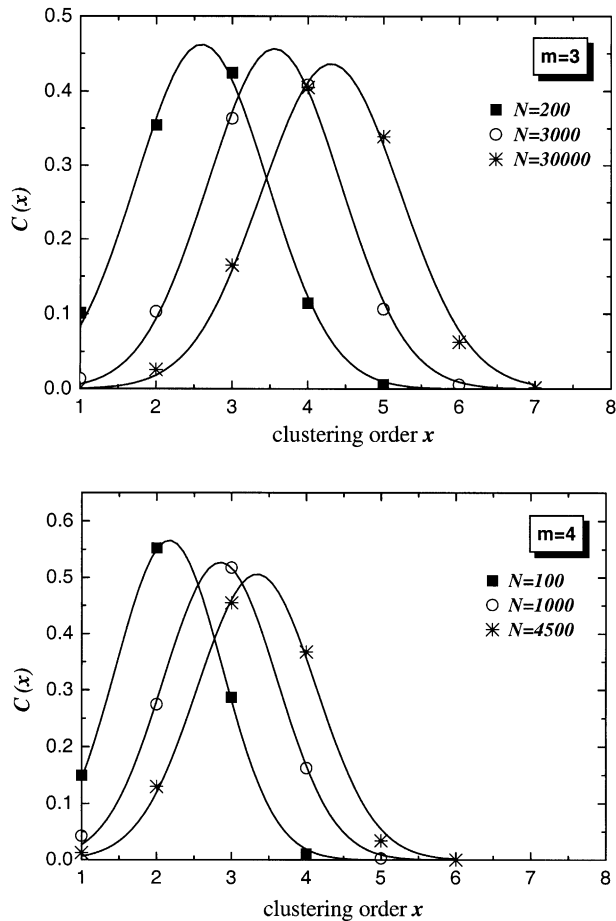


Fig. 3. Higher order clustering coefficient distributions in BA networks of given sizes N for $m = 3$ and 4.

network are more likely to be second ($x = 2$), third ($x = 3$) and further neighbours when N increases.

We investigated distributions of the clustering coefficients $C(x)$ at a given system size N in BA networks with $m = 3$ and 4. We found that regardless of N the distributions of $C(x)$ (when extending x to real values) fairly good fit the normalised Gaussian curve (Fig. 3). Moreover, we observed that the standard deviation of these distributions depends only on the parameter m of the network. The Gaussian-like patterns of $C(x)$ in BA model express *random* character of clustering relationships. In fact, it is known that the distribution of distances between randomly chosen sites in BA network is Gaussian. It follows that interrelations in the nearest neighbourhood of any vertex in BA model are similar to interrelations between randomly chosen vertices in the BA network. Distinctly, there are no mechanisms responsible for clustering in BA algorithm.

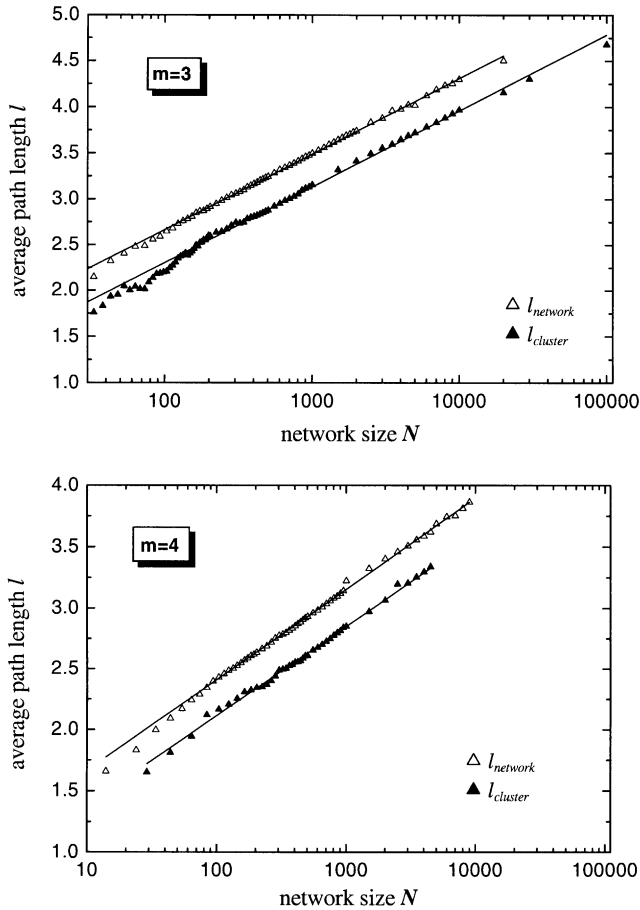


Fig. 4. Characteristic path lengths $l_{cluster}$ and $l_{network}$ versus network size N for $m = 3$ and 4 . For $m = 3$: $l_{network}$ is fitted to $0.82 \ln(N) + 1.02$ and $l_{cluster}$ is fitted to $0.83 \ln(N) + 0.65$. For $m = 4$: $l_{network}$ is fitted to $0.74 \ln(N) + 0.92$ and $l_{cluster}$ is fitted to $0.73 \ln(N) + 0.65$.

The above-described observation can be verified by a functional dependence of centres x_c of the distribution $C(x)$ on the network size N . We realised that the value x_c (Fig. 3) expresses both: the order x when $C(x)$ obtains maximum and the average shortest path length $l_{cluster}$ between vertices belonging to the neighbourhood of any vertex within a network. Fig. 4 shows that in BA networks the average shortest path $l_{cluster}$ scales with the system size N in the same way as the average shortest path length for the whole network $l_{network}$ [1], i.e., in the first approximation as

$$l_{cluster} \sim \ln(N). \tag{3}$$

We found that the mean distance between two nearest neighbours of the same node $l_{cluster}$ equals to the mean distance between any two nodes in the network $l_{network}$

minus a constant $A(m)$

$$l_{cluster} = l_{network} - A(m), \quad (4)$$

where $A(m)$ is approximately independent on the network size N and equals 0.37 and 0.27 for $m = 3$ and 4, respectively.

4. Conclusions

In summary, we quantified the structural properties of BA networks by the higher order clustering coefficients $C(x)$ defined as probabilities that the shortest distance between any two nearest neighbours of a certain vertex i equals x , when neglecting all paths crossing the node i . We estimated that the average shortest path length in the node's neighbourhood is smaller than the equivalent whole network quantity and the remainder depends only on the network parameter m . Our results show in a vivid way that the absence of the clustering phenomenon in BA networks is due to the random character of the nearest neighbourhood in these networks.

Recently, some alternative algorithms have been suggested to account for the high clustering found in real web-like systems. Holme and Kim [16] have extended the standard BA model adding the trial formation rule. Networks built according to their guidelines exhibit both the high clustering and the scale-free nature. Barabási et al. [17,18] and independently Dorogovtshv et al. [19] have found that scale-free random networks can be modelled in a deterministic manner by so-called pseudofractal scale-free networks. They argued that the high clustering in real networks might result from their hierarchical topology. It would be interesting to analyse the higher order clustering coefficients in these networks and compare it with real data [14].

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