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# Destructive Role of Competition and Noise for Control of Microeconomical Chaos†

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**Abstract**—The problem of control of chaos in a microeconomical model describing two competing firms with asymmetrical investment strategies is studied. Cases when both firms try to perform the control simultaneously or when noise is present are considered. For the first case the resulting control efficiency depends on the system parameters and on the maximal values of perturbations of investment parameters for each firm. Analytic calculations and numerical simulations show that competition in the control leads to 'parasitic' oscillations around the periodic orbit that can destroy the expected stabilization effect. The form of these oscillations is dependent on non-linear terms describing the motion around periodic orbits. An analytic condition for stable behaviour of the oscillation (i.e. the condition for control stability) is found. The values of the mean period of these oscillations is a decreasing function of the amplitude of investment perturbation of the less effective firm. On the other hand, amplitudes of market oscillations are increasing functions of this parameter. In the presence of noise the control can be also successful provided the amplitude of allowed investment changes is larger than some critical threshold which is proportional to the maximal possible noise value. In the case of an unbounded noise, the time of laminar epochs is always finite but their mean length increases with the amplitude of investment changes. Computer simulations are in very good agreement with analytical results obtained for this model. © 1997 Elsevier Science Ltd

## 1. INTRODUCTION

It is generally accepted that dynamics of many economical systems should be described by non-linear equations [7, 19, 20, 13, 11, 9] and it is well known that such systems can exhibit a phenomenon of *deterministic chaos* [2, 12, 8, 16, 1, 5]. Recently a method of *chaos control* has been proposed [14] and this method has been successfully applied to control deterministic chaos in numerous physical [3, 17], chemical [15] and biological [6] experiments. In the paper [10] we adopted this method for a generic *economic model* of two

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competing firms. In the present paper we extend our former results by assuming that the control procedure is simultanously applied by both firms or that the noise is present in the system. We will show that in such a situation a *competition in control* can again lead to *chaotic market behaviour*, although both partners try to stabilize the system. Similarly the presence of *noise* destabilizes the system if the amplitude of such a noise is above some critical value.

In Section 2 we give a brief overview of the OGY [14] algorithm of controlling chaos as well as of our previous results on the control of a chaotic economical model [10]. In Sections 3–7 the problems of control simultaneously performed by both firms in a fully deterministic model are discussed while in the Section 8 the influence of noise is studied.

## 2. SINGLE-SIDE CONTROL OF CHAOS IN AN ECONOMICAL MODEL

Let us assume that there are two firms $X,Y$ competing in the same market of goods and that, due to their active investment strategies, their sales $x_n$, $y_n$ at the time moments $n = 1,2,3...$ evolve according to equations [1, 5]

$$x_{n+1} = F^x(x_n,y_n;p) = (1-\alpha)x_n + \frac{a}{1+\exp[-c(x_n-y_n)]} \tag{1}$$

$$y_{n+1} = F^y(x_n,y_n;p) = (1-\beta)y_n + \frac{b}{1+\exp[-c(x_n-y_n)]} \tag{2}$$

where the constants $\alpha$ and $\beta$ (with $0<\alpha,\beta<1$) are the time rates of the decay of sales of both firms under zero investment while the second parts of equations (1) and (2) describe the influence of investments at time $n$ on the sales at time $n+1$. Parameters $a$ and $b$ describe the efficiencies of investments of both firms or scales of their investments while parameter $c$ is a measure of the 'elasticity' of the investment strategies. The symbol $p$ on the right-hand side of equations (1) and (2) stands for the dependence of the all parameters $a$, $b$, $c$, $\alpha$, $\beta$. Equations (1) and (2) together form a two-dimensional map $\mathbf{r}_{n+1} = \mathbf{F}(\mathbf{r}_n;p)$, where $\mathbf{r}_n = [x_n,y_n]$ and $\mathbf{F} = [F^x,F^y]$, which fully defines the evolution of our discrete dynamical system. Depending on the specific values of the parameters $\alpha$, $\beta$, $a$, $b$ and $c$ the solutions of equations (1) and (2) can be regular or chaotic [1, 5]. It is interesting that equations (1) and (2) can also be connected with a model of army races between two countries with asymmetric armament policies [1, 5].

The chaotic dynamics of any model that can be described by a map of the form $\mathbf{r}_{n+1} = \mathbf{F}(\mathbf{r}_n;p)$, can be easy controlled [14] if one makes use of the existence of *unstable periodic* solutions that can occur in the close neighbourhood of any chaotic trajectory. The simplest of such solutions is an unstable fixed point $\mathbf{r}^f$ that fulfills the equation

$$\mathbf{r}^f = \mathbf{F}(\mathbf{r}^f;p^0) \tag{3}$$

for some value of the parameter $p = p^0$. Linearizing the map $\mathbf{F}(\mathbf{r}_n;p)$ around the solution $\mathbf{r} = \mathbf{r}^f$ and around the parameter value $p = p^0$ one can easy calculate a value $\delta p_n$ of the time-dependent change of the parameter $p$ that is needed to shift the chaotic trajectory to a so-called *stable manifold* of the fixed point $\mathbf{r}^f$ [14]. In the case of a two-dimensional map, i.e. in the case when $\mathbf{r} = [x,y]$, this change of parameter can be written as [4]

$$\delta p_n = \frac{-\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w}_p} \mathbf{f}_2 \cdot \delta \mathbf{r}_n \tag{4}$$

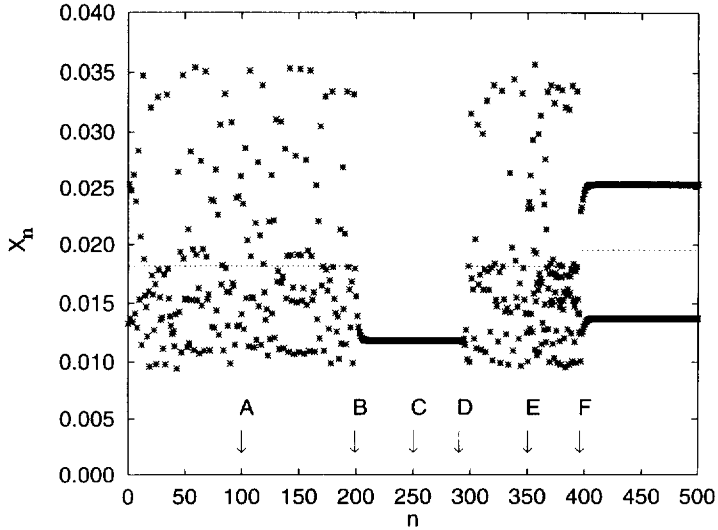


Fig. 1. Time dependence of sales $x_n$. Results of the stabilization on the fixed point and on period two can be easily seen. Dashed lines show the *mean values* of the sales at various time intervals (chaotic or period ones). The meanings of the points A, B, C, D, E and F are explained in the text.

where the constant $\lambda_2$ is the larger (in modulus) eigenvalue of the Jacobian $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$ of the map $\mathbf{F}(\mathbf{r}_n;p)$ calculated at the unstable fixed point

$$\hat{J}^{\mathbf{F}}_{\mathbf{r}^f} = \begin{bmatrix} \dfrac{\partial F^x(x,y;p)}{\partial x}, & \dfrac{\partial F^x(x,y;p)}{\partial y} \\ \dfrac{\partial F^y(x,y;p)}{\partial x}, & \dfrac{\partial F^y(x,y;p)}{\partial y} \end{bmatrix}_{(\mathbf{r}=\mathbf{r}^f;p=p^{(0)})}, \tag{5}$$

the vector $\mathbf{f}_2$ is a contravariant vector connected to the Jacobian $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$ (see Appendix A) while the vector $\mathbf{w}_p$ is defined as

$$\mathbf{w}_p = \left(\frac{\partial \mathbf{F}}{\partial p}\right)_{(\mathbf{r}=\mathbf{r}^f;p=p^{(0)})}. \tag{6}$$

It is important to stress that when the system is placed on the stable manifold, it is attracted towards the fixed point in the course of time. In such a way one can switch from the chaotic trajectory to the period one if only one changes appropriately the model parameter $p$. Below we present how such a control of chaos has been performed for the Behrens–Feichtinger model [equations (1) and (2)] [10]. In Fig. 1 the sales $x_n$ of firm $X$ are depicted (the behaviour of sales of firm $Y$ looks very similar). During the first 100 steps the system was not controlled at all and the chaotic evolution of the sales $x_n$ can easily be seen. At time $n = 100$ (depicted by arrow A in Fig. 1) firm $X$ decided to control the market, assuming the maximum value of change of its investment parameter $\Delta a = 0.008$, which is 5% of the 'standard value' of its investment parameter $a^0 = 0.16$. Firm $X$ had to wait for the occasion to activate this control strategy up to time $n = 199$ (arrow B) because up to this moment the sales $x_n$, $y_n$ did not fulfill the condition $|\delta a_n| < 0.008$ and the control procedure could not be activated. At $n = 199$ (arrow B) the control was switched on for the first time and, beginning from this moment, control was permanently used up to the time $n = 250$ (arrow C) when 'firm $X$ decided' to switch the control off. It is worthwhile pointing out that if one excludes the first few steps, then the next values of the perturbations of the control parameter $a$ were extremely small. Unfortunately the value of $x^f$ is smaller than the mean values of the sales of firm $X$ in the chaotic region $n = 0,1,2...199$

$$\frac{x^f}{\bar{x}_{\text{chaotic}}} \approx 0.65 \tag{7}$$

This means that for this kind of investment behaviour, an uncontrolled, chaotic evolution of the market would be preferred (for this particular case) if firm $X$ wants to maximize its sales. However, switching from chaotic to non-chaotic behavior *always* brings the strong advantage that the firm can perform market forecasting, in other words, it can be more sure of the volume of its sales $x_n$ during the next time steps. Such knowledge is very limited for chaotic trajectories because of its sensitivity to initial conditions.

On the other hand, if we consider equations (1) and (2) as a model of army races [1, 5], then the control brings no doubt about the positive effects of both the *stabilization and decreasing* (by around 35%) of the military potentials of both countries.

At time $n = 250$ (arrow C at Fig. 1) it is assumed that firm $X$ had switched off its market control. One can see, however, that the market remained stable for about 45 steps (up to the arrow D) when it 'forgot' about the previous stabilization and again started to behave in a chaotic fashion. Then at time $t = 350$ (arrow E) firm $X$ decided to switch the market control on again. However, at this particular time, control was used to stabilize the market on period two $(\mathbf{r}_1^f, \mathbf{r}_2^f)$ such that $F(\mathbf{r}_1^f;p^0) = \mathbf{r}_2^f$ and $F(\mathbf{r}_2^f;p^0) = \mathbf{r}_1^f$. The control procedure for period two was first activated at time $n = 397$ (arrow F) and after a few further iterations the system was stabilized. Now the sales of firm $X$ oscillate between $x_1^f$ and $x_2^f$. If one compares the mean values of the 'period two' solutions with the 'chaotic' mean values

$$\frac{x_1^f + x_2^f}{2\bar{x}_{\text{chaotic}}} \approx 1.077 \tag{8}$$

one can see that the stabilization of the chaotic dynamics on 'period two' brought a 7.7% increase in sales for firm $X$. It is interesting that although the control procedure is only performed by firm $X$, one can also observe the stabilization of sales $y_n$ of firm $Y$ and similarly, the fixed point $y_f$ brings a decrease of sales $y_n$ while orbit two brings an increase [10].

## 3. SIMULTANEOUS CONTROL OF CHAOTIC MARKET BY BOTH FIRMS

Let us assume now that both firms simultaneously try to change market behaviour from chaotic to periodic and they attempt to shift a chaotic trajectory towards an unstable fixed point $\mathbf{r}^f$. Corresponding values of changes of the investment parameters $\delta a_n$ and $\delta b_n$ needed for such a control follow from equation (4) and can be written as

$$\delta a_n = -\frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w}_\mathbf{a}} \mathbf{f}_2 \cdot \delta \mathbf{r}_n \equiv A \mathbf{f}_2 \cdot \delta \mathbf{r}_n, \tag{9}$$

$$\delta b_n = -\frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w}_\mathbf{b}} \mathbf{f}_2 \cdot \delta \mathbf{r}_n \equiv B \mathbf{f}_2 \cdot \delta \mathbf{r}_n, \tag{10}$$

where the vectors $\mathbf{w}_\mathbf{a}$, $\mathbf{w}_\mathbf{b}$ are

$$\mathbf{w}_\mathbf{a} = \left[\frac{1}{1 + \exp[-c^0(x^f - y^f)]}, 0\right] \tag{11}$$

$$\mathbf{w}_\mathbf{b} = \left[0, \frac{1}{1 + \exp[-c^0(x^f - y^f)]}\right]. \tag{12}$$

We will assume that the firms can start the control provided that $|\delta a_n| \leq \Delta a$ or $|\delta b_n| \leq \Delta b$. Thus it follows that there are parameters of *control efficiency* for each firm $s_x = \Delta a|\mathbf{f}_2 \cdot \mathbf{w_a}|$, $s_y = \Delta b|\mathbf{f}_2 \cdot \mathbf{w_b}|$ and the larger value of these parameters decides which of the firms $X,Y$ is more effective in its control strategy and can start the control first. Let us assume that $s_x < s_y$. Then firm $Y$ starts to control the market first in the time point $t = n_0$, i.e. there is $|\delta b_{n_0}| \leq \Delta b$. We assume that $|\delta a_{n_0}| > \Delta a$, otherwise the discussion can be reduced to the situation occurring at a certain moment $t = n_0 + \tau_1$ (see below). In the case where the control policy is successful, i.e. non-linear terms represented by corrections of the type $\kappa_{u,ss}\xi^2{}_{s,n_0}$ (see Appendix B) do not destroy the control, the distance between the market trajectory $\mathbf{r}_n$ and the fixed point $\mathbf{r}^f$ decreases with time. After one or several steps of time, this distance becomes so small that firm $X$ can also start its control procedure at a time moment $n_0 + \tau_1$ ($\tau_1 > 0$) when $|\delta a_{n_0+\tau_1}| \leq \Delta a$. Because the firms act *independently* there is no coordination between their control strategies and we shall show that, in the case when they *both* try to stabilize the market, their combined influence on the market does not always lead to system stabilization. The dynamics of the market is then described by the following equation

$$\delta \mathbf{r}_{n+1} \approx \hat{J}^{\mathbf{F}}_{\mathbf{r}^f}\, \delta \mathbf{r}_n + \mathbf{w}_a\, \delta a_n + \mathbf{w}_b\, \delta b_n. \tag{13}$$

for $n \geq n_0 + \tau_1$. Putting the values of $\delta a_n$ and $\delta b_n$ given by equations (9) and (10) into equation (13) and projecting the resulting equation onto the stable and unstable directions of the Jacobian matrix $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$ one gets, after some algebra,

$$\xi_{s,n+1} = \xi_{s,n}\lambda_1 - \xi_{u,n}\lambda_2 d, \tag{14}$$

$$\xi_{u,n+1} = -\lambda_2 \xi_{u,n}, \tag{15}$$

$$\delta a_{n+1} = -\lambda_2 a_n, \tag{16}$$

$$\delta b_{n+1} = -\lambda_2 b_n. \tag{17}$$

where

$$\xi_{s,n} = \delta \mathbf{r}_n \cdot \mathbf{f}_1, \tag{18}$$

$$\xi_{u,n} = \delta \mathbf{r}_n \cdot \mathbf{f}_2$$

and

$$d = \frac{\mathbf{f}_1 \cdot \mathbf{w_a}}{\mathbf{f}_2 \cdot \mathbf{w_a}} + \frac{\mathbf{f}_1 \cdot \mathbf{w_b}}{\mathbf{f}_2 \cdot \mathbf{w_b}}. \tag{19}$$

However because there is $|\lambda_2| > 1$, the above results mean that in the case of the simultaneous control of both firms, the absolute value of the unstable component $\xi_{u,n}$ of the vector $\delta \mathbf{r}_n$ and the absolute values of the perturbations of the investment parameters $\delta a_n$ and $\delta b_n$ *increase in time*. In other words, the system is farther and farther from the fixed point $\mathbf{r}^f$ (more precisely from the stable manifold leading to this point) and larger and larger values of perturbations of the investment parameters are needed to fulfill the control strategies. This is the opposite situation to the case when only a single firm controls the market and the distance $\mathbf{r}_n$ decreases in time. It follows that there must be a moment $n_0 + \tau_1 + \tau_2$ when there is $|\delta a_{n_0+\tau_1+\tau_2}| > \Delta a$ and firm $X$, which is less effective in its control action than firm $Y$ (because $s_x < s_y$), has to break its stabilization efforts and put $\delta a_{n_0+\tau_1+\tau_2} = 0$.

## 4. THE CONDITION FOR THE PERMANENT CONTROL

Following the last section we need to distinguish two cases:
(a) If $|\delta b_{n_0+\tau_1+\tau_2}| \leq \Delta b$, firm $Y$ will continue its stabilization policy for several time steps *alone* until firm $X$ switches on its control again.
(b) If $|\delta b_{n_0+\tau_1+\tau_2}| > \Delta b$, firm $Y$ must switch the control off *at the same moment* as firm $X$. In such a case, the market will evolve for some time along an *uncontrolled* chaotic trajectory until it returns to the neighbourhood of the fixed point $\mathbf{r}^f$ and at least firm $Y$ will be able to start its control procedure again.

Obviously if *only* case (a) appears, then the market oscillates in the neighbourhood of the fixed point $\mathbf{r}^f$ whereas if case (b) is possible, then such oscillations are interrupted by chaotic intervals. The condition for the absence of case (b), i.e. the condition for the *stability* of oscillations around the fixed point $\mathbf{r}^f$, can easily be deduced if one takes into account that from equations (9) and (10) we have

$$\frac{\delta b_n}{\delta a_n} = \frac{\mathbf{f}_2 \cdot \mathbf{w_a}}{\mathbf{f}_2 \cdot \mathbf{w_b}} \tag{20}$$

However, the maximum possible value of $|\delta a_{n_0+\tau_1+\tau_2}|$ is equal to $|\lambda_2 \Delta a|$ (because $|\delta a_{n_0+\tau_1+\tau_2-1}| \leq \Delta a$) and calculating the maximum possible value of $|\delta b_{n_0+\tau_1+\tau_2}|$ we get the following condition for *permanent control*

$$\Delta b \geq \left| \lambda_2 \Delta a \frac{\mathbf{f}_2 \cdot \mathbf{w_a}}{\mathbf{f}_2 \cdot \mathbf{w_b}} \right| = \Delta b_c. \tag{21}$$

The above relation can be rewritten in the form

$$\frac{s_y}{s_x} \geq |\lambda_2| \tag{22}$$

which can be interpreted in such a way that the oscillations are stable, i.e the control is successful provided the control efficiency parameter of the more efficient firm $Y$ is larger than the control efficiency parameter of the less efficient firm $X$ by at least a factor equal to the absolute value of the larger eigenvalue of the Jacobian matrix $\hat{J}^{\mathbf{F}}_{\mathbf{r}^f}$. The validity of the condition (21) was checked numerically and a very good agreement between the analytical and numerical values of the critical parameter $\Delta b_c$ has been observed. In Fig. 2(a) and (b) one can see that if the maximal value of the changes of the investment parameter $\Delta b$ for the more efficient firm $Y$ is less than the corresponding critical value $\Delta b_c$, then the market is controlled only during some time intervals. On the other hand, we have used $\Delta b = \Delta b_c$ for Fig. 2(c) and one can see that in this case the control is permanent. Corresponding results describing the evolution of the sales $y_n$ of the second firm $Y$ are quite similar to the results presented in Fig. 2(a)–(c) for the sales $x_n$ of firm $X$.

## 5. FORM OF OSCILLATIONS

To get the analytic form of the stable oscillations presented at Fig. 2(c) we followed our analysis from the previous section and divided the whole period of oscillation into two time intervals. During the first time interval, $n_0 \leq n < n_0 + \tau_1$, only firm $Y$ is active with its control efforts. However, during the second interval, $n_0 + \tau_1 \leq n < n_0\tau_1 + \tau_2$, both firms are active. Without losing the generality we will assume that $n_0 = 0$.

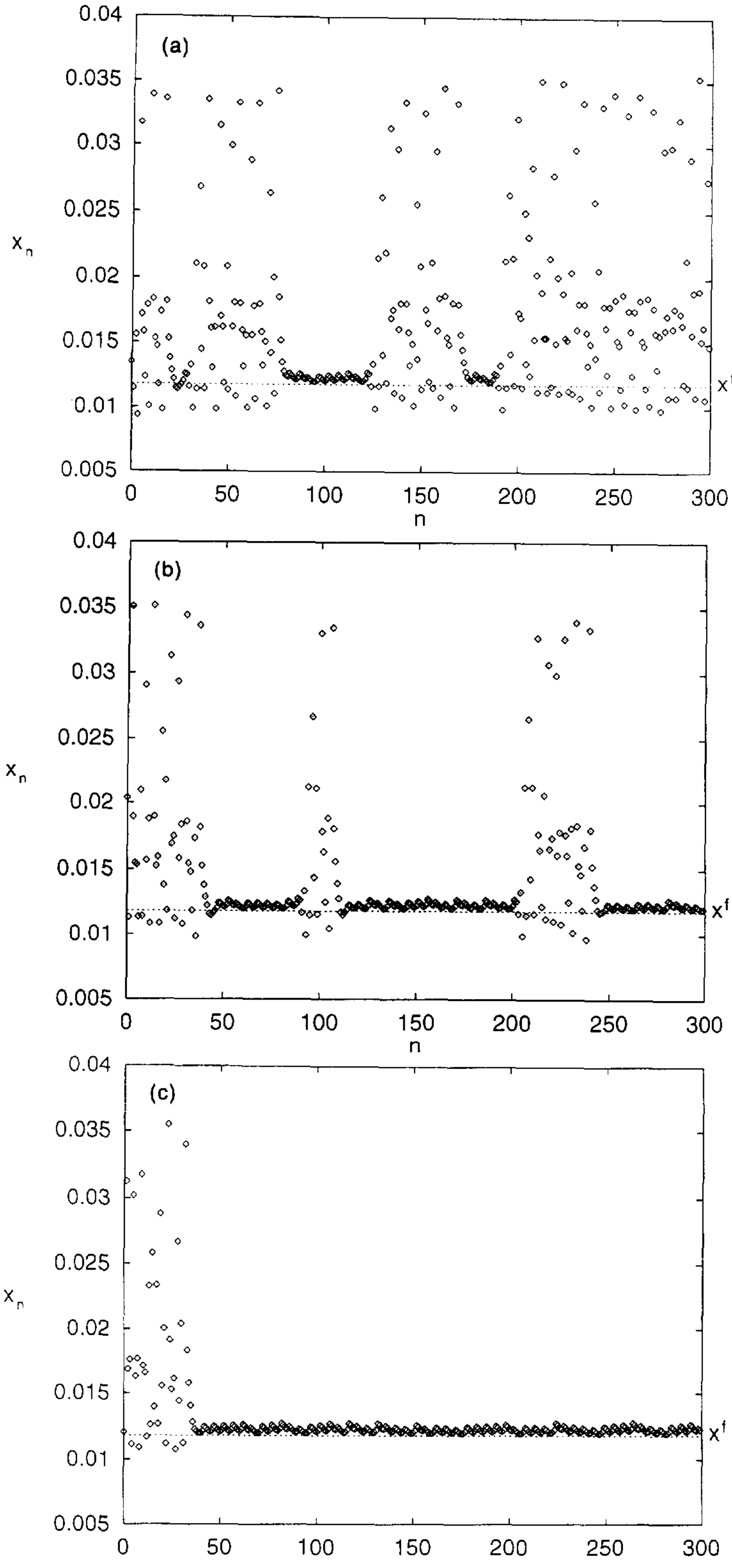


Fig. 2. Evolution of the sales of firm $X$ for various values of maximal perturbations $\Delta b$ of the investment parameters of firm $Y$ and the constant value $\Delta a = 0.016$ of the maximal perturbations of the investment parameter of firm $X$. (a) $\Delta b = 0.97\Delta b_c$, (b) $\Delta b = 0.995\Delta b_c$ and (c) $\Delta b = \Delta b_c$.

The market evolution during the first period can be obtained by combining equations (13) and (10) and putting $\delta a_n = 0$ instead of equation (9). One gets, as a result

$$\delta \mathbf{r}_{n+1} \approx \hat{J}^{\mathbf{F}}_{\mathbf{r}'} \delta \mathbf{r}_n - \mathbf{w_b} \frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w_b}} \mathbf{f}_2 \cdot \delta \mathbf{r}_n \tag{23}$$

From equation (23) we determine, after some algebra, that stable and unstable components of the vector $\mathbf{r}_n$ evolve as follows

$$\xi_{s,n+1} \approx \lambda_1 \xi_{s,n} - \lambda_2 d_b \xi_{u,n} \tag{24}$$

$$\xi_{u,n+1} \approx 0. \tag{25}$$

where the constant $d_b$ is equal to

$$d_b = \frac{\mathbf{f}_1 \cdot \mathbf{w_b}}{\mathbf{f}_2 \cdot \mathbf{w_b}} \tag{26}$$

Equation (25) indicates that if the standard OGY control is used [14], then the unstable component $\xi_u$ vanishes. To obtain a more precise value of this component we need to extend the standard OGY analysis by investigations of *non-linear contributions* to system equations. Taking only the quadratic terms in the corresponding Taylor series (see Appendix B for details), instead of equation (25) we obtained the following result

$$\begin{aligned} \xi_{u,n+1} &\approx \kappa_{u,ss} \xi^2_{s,n} + \kappa_{u,uu} \xi^2_{u,n} + \kappa_{u,bb} (\delta b_n)^2 + \kappa_{u,su} \xi_{s,n} \xi_{u,n} + \kappa_{u,sb} \xi_{s,n} \delta b_n + \kappa_{u,ub} \xi_{u,n} \delta b_n \\ &\equiv g(\xi_{u,n}, \xi_{s,n}, \delta b_n) \end{aligned} \tag{27}$$

where the coefficients $\kappa_{u,ss}$, $\kappa_{u,uu}$, $\kappa_{u,su}$, $\kappa_{u,sb}$, $\kappa_{u,ub}$, $\kappa_{u,bb}$ are defined in Appendix B.

To get a description of the market evolution during the second time period we iterated equations (14) and (15) and as a result we obtained the following solutions for the stable and unstable components of the market vector during the time interval corresponding to control of both firms

$$\xi_{s,\tau_1+k} \approx \xi_{s,\tau_1} \lambda_1^k + \xi_{u,\tau_1} \lambda_2 \frac{(-\lambda_2)^k - \lambda_1^k}{\lambda_2 + \lambda_1} d \tag{28}$$

$$\xi_{u,\tau_1+k} \approx \xi_{u,\tau_1} (-\lambda_2)^k \tag{29}$$

where $0 < k \leq \tau_2$. The combined results representing irregular oscillations of sales $x_n$ and $y_n$ of both firms $X$ and $Y$ as well as the perturbations of their investment parameters $a_n$ and $b_n$ are presented in Fig. 3. One can see that the agreement between the analytical and numerical results is quite good.

In Fig. 4 the time evolution of the sales $x_n$ of firm $X$ is presented for various values of parameter $\Delta a$. Although the evolution is not completely periodic, one can observe that an increase of parameter $\Delta a$ leads to a decrease of the mean period of these oscillations and to an increase of their mean amplitudes. The quantitative analysis of these phenomena will be the aim of the next sections.

## 6. MEAN PERIOD OF MARKET OSCILLATIONS

We will try to estimate the mean value of *the period* $\tau$ of market oscillations seen in Figs 2–4. According to the discussion from the previous section, this period can be expressed as $\tau = \tau_1 + \tau_2$ where the period $\tau_1$ corresponds to the time length when the control is performed by firm $Y$ alone while the period $\tau_2$ describes the time length when the control is performed by firms $X$ and $Y$ simultaneously.

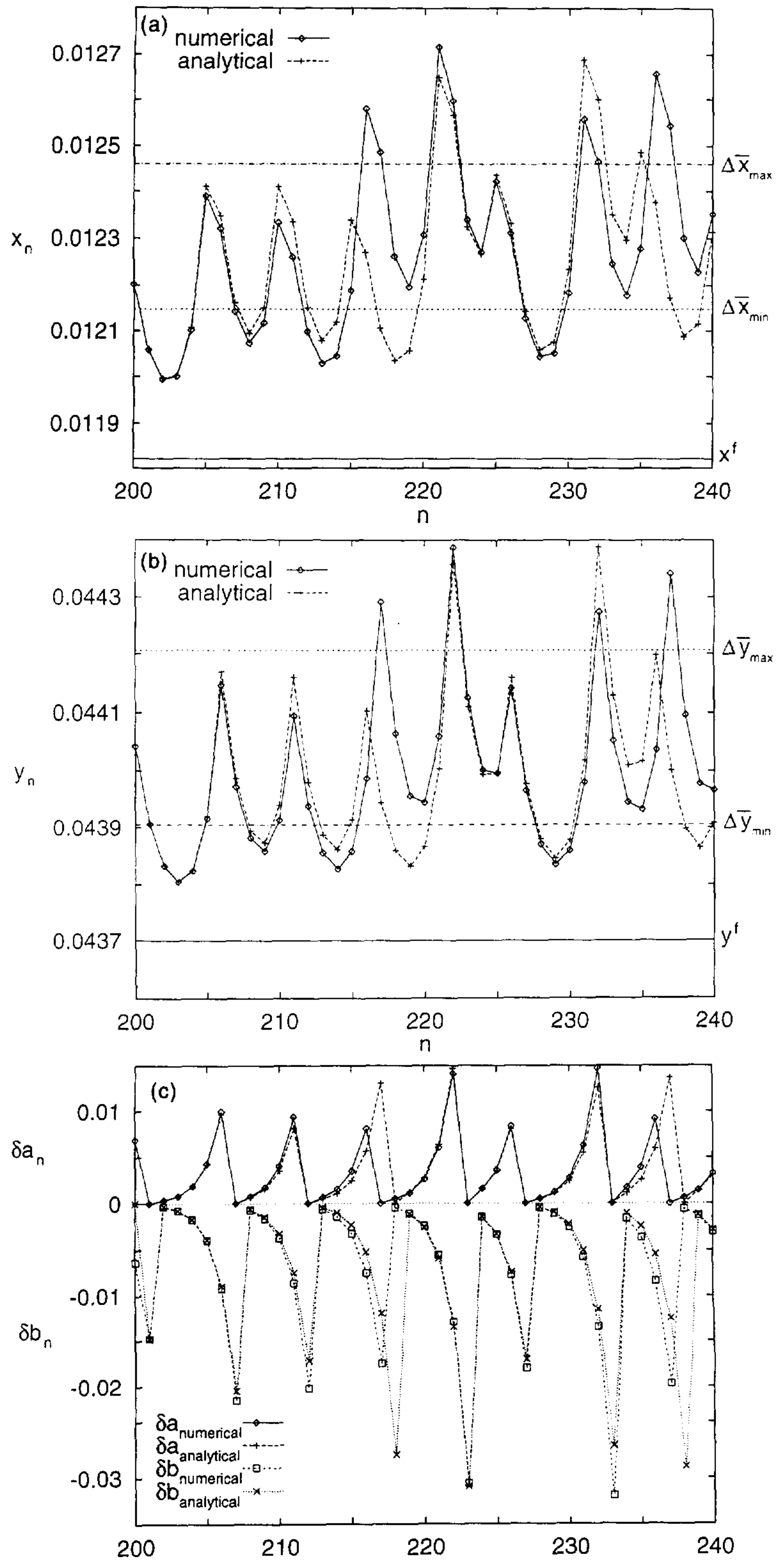


Fig. 3. Analytical and numerical results representing an irregular oscillatory evolution of firms sales $x_n$, $y_n$ and perturbations of investment parameters $\delta a_n$, $\delta b_n$ as described by equations (27)–(29).

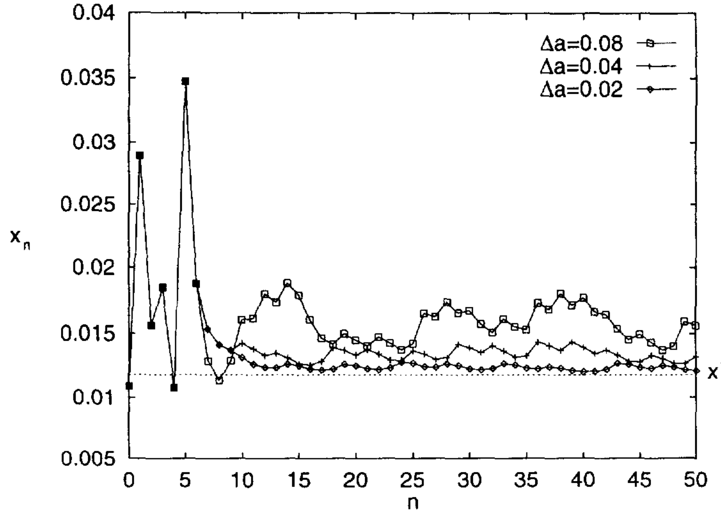


Fig. 4. Oscillations of the sales $x_n$ of firm $X$ for $\Delta b = 0.2$ and for various values of the parameter $\Delta a$.

For simplicity we will restrict our investigations to the case $\tau_1 = 1$ when after one step of the control performed by firm $Y$ the market is so close to the unstable fixed point $\mathbf{r}^f$ that firm $X$ can also start its control procedure. The condition for such a situation can be obtained as follows. If firm $X$ can start the control at the moment $n = \tau_1 + \tau_2 + 1$ then it must be

$$|\xi_{u,\tau_1+\tau_2+1}| \leq \xi_u^{\Delta a} \equiv \left|\frac{\mathbf{f}_2 \cdot \mathbf{w}_a \Delta a}{\lambda_2}\right| \tag{30}$$

However

$$\xi_{u,\tau_1+\tau_2+1} = g(\xi_{u,\tau_1+\tau_2}, \xi_{s,\tau_1+\tau_2}, \delta b_{\tau_1+\tau_2}) \tag{31}$$

where the function $g$ on the right-hand side of the above equation is defined by equation (27). The values of arguments of this function can be expressed with the help of equation (29) and the asymptotic form of equation (28) that is valid for $\tau_2 > > 1$

$$\xi_{s,\tau_1+\tau_2} \approx -(-\lambda_2)^{\tau_2+1} \xi_{u,\tau_1} \frac{d}{\lambda_1 + \lambda_2} \tag{32}$$

and from equation (10) we have

$$\delta b_{\tau_1+\tau_2} = B \xi_{u,\tau_1+\tau_2} \tag{33}$$

Putting equations (29), (32) and (33) into equation (31) we get, after some algebra,

$$\xi_{u,\tau_1+\tau_2+1} \approx E \lambda_2^{2\tau_2} \xi_{u,\tau_1}^2 \approx E \xi_{u,\tau_1+\tau_2}^2 \tag{34}$$

where the coefficient $E$ is defined as

$$E = \kappa_{u,uu} + \kappa_{u,su} \frac{\lambda_2 d}{\lambda_1 + \lambda_2} + \kappa_{u,ss}\left(\frac{\lambda_2 d}{\lambda_1 + \lambda_2}\right)^2 + \kappa_{u,sb} \frac{\lambda_2 B d}{\lambda_1 + \lambda_2} + \kappa_{u,ub} B + \kappa_{u,bb} B^2 \tag{35}$$

Taking into account that

$$\xi_u^{\Delta a} |\lambda_2| \geq |\xi_{u,\tau_1+\tau_2}| > \xi_u^{\Delta a} \tag{36}$$

we finally get from equations (30) and (34) the following condition for the case $\tau_1 = 1$

$$|E \lambda_2 \mathbf{f}_2 \cdot \mathbf{w}_a \Delta a| \leq 1 \tag{37}$$

Now we can estimate the value of the period $\tau_2$. Putting in equation (34) $\xi_{u,\tau_1+\tau_2+1} \approx \xi_{u,\tau_1}$

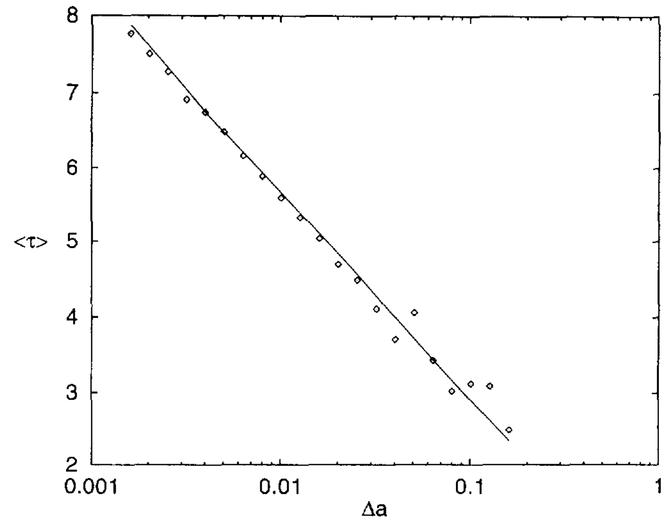


Fig. 5. Mean values of the market oscillation periods as a function of the parameter $\Delta a$ and the parameter $\Delta b = 4\Delta a >> \Delta b_c$. The line represents the theoretical dependence equation (40).

we get $1 \approx E\lambda_2^{2\tau_2}\xi_{u,\tau_1}$. Combining it with equation (29) we can write the inequalities equation (36) as

$$|\xi_u^{\Delta a}\lambda_2| \geq |E\lambda_2^{\tau_2}|^{-1} > \xi_u^{\Delta a} \tag{38}$$

Taking from equation (30) the definition of $\xi_u^{\Delta a}$, from the inequalities equation (38) we get the following inequalities for the period $\tau_2$

$$-\frac{\log|E\mathbf{f}_2\cdot\mathbf{w_a}\Delta a|}{\log|\lambda_2|} \leq \tau_2 < -\frac{\log|E\mathbf{f}_2\cdot\mathbf{w_a}\Delta a|}{\log|\lambda_2|} + 1 \tag{39}$$

Assuming for the mean value of the period $\tau_2$ the mean value of its lower and upper limits and putting $\tau_1 = 1$ we finally get the mean value of the total oscillation period as

$$\langle\tau\rangle \approx -\frac{\log|E\mathbf{f}_2\cdot\mathbf{w_a}\Delta a|}{\log|\lambda_2|} + \frac{3}{2} \tag{40}$$

The last relation is in a very good agreement with the numerical simulations which can be seen in Fig. 5 and which show that this period is a decreasing function of the parameter $\Delta a$, $\Delta b$. The small differences between analytical and numerical results can only be seen for low values of $\langle\tau\rangle$ where the approximation equation (32) is no longer valid.

## 7. AMPLITUDE OF MARKET OSCILLATIONS

To obtain the values of the amplitudes of market oscillations we use relations between $x_n, y_n$ and $\xi_{s,n}$, $\xi_{u,n}$

$$\delta x_n = \xi_{s,n}e_s^x + \xi_{u,n}e_u^x \tag{41}$$

$$\delta y_n = \xi_{s,n}e_s^y + \xi_{u,n}e_u^y \tag{42}$$

After a tedious algebra, similar to that presented in Section 6, we found that a mean value of one extreme of $\delta x_n$ can be written as

$$\delta\bar{x}_{extr1} = A_1^x\lambda^\gamma{}_1 + A_2^x(-\lambda_2)^\gamma \tag{43}$$

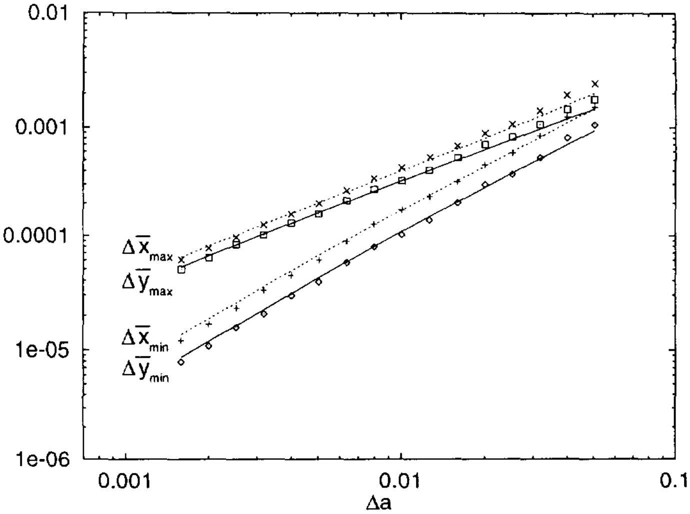


Fig. 6. Mean values of the market oscillation amplitudes as a function of the parameter $\Delta a$ and the parameter $\Delta b = 4\Delta a >> \Delta b_c$. The lines represent the theoretical values equations (43), (47) and (48).

where

$$A_1^x = e_s^x \operatorname{sign}(E)\xi_u^{\Delta a}\left(\frac{1-\lambda_2}{2}\right)\lambda_2\left(\frac{\lambda_1 d}{\lambda_1+\lambda_2} - d_b\right) - e_s^x\frac{\lambda_2 E d}{2(\lambda_1+\lambda_2)}(1+\lambda_2^2)(\xi_u^{\Delta a})^2 \tag{44}$$

$$A_2^x = \frac{E}{2}(\xi_u^{\Delta a})^2(1+\lambda_2^2)\left(e_s^x\frac{\lambda_2 d}{\lambda_1+\lambda_2} + e_u^x\right) \tag{45}$$

$$\gamma = \frac{\log\left(\dfrac{-\Gamma_2 A_2^x}{\Gamma_1 A_1^x}\right)}{\Gamma_1 - \Gamma_2} \tag{46}$$

and $\Gamma_1 = \log\lambda_1$, $\Gamma_2 = \log(-\lambda_2)$. The second extreme of $x_n$ is either

$$\delta\bar{x}_{extr2a} = A_1^x + A_2^x \tag{47}$$

or

$$\delta\bar{x}_{extr2b} = \operatorname{sign}(E)\xi_u^{\Delta a}\left(\frac{1+|\lambda_2|}{2}\right)\left(e_s^x\frac{\lambda_2 d}{\lambda_1+\lambda_2} + e_u^x\right) \tag{48}$$

One can get the corresponding extreme values of $y_n$ by changing $e_s^x \to e_s^y$ and $e_u^x \to e_u^y$ in the above equations. To get above results we used relations $A_1^x A_2^x > 0$, $1 > \lambda_1 > 0$, $\lambda_2 < -1$ and we assumed $\tau_1 = 1$, $\tau_2 >> 1$.

The comparison of equations (43), (47) and (48) with numerical simulations is presented in Fig. 6. We see that the agreement is very good. The fact that the amplitudes of (stable) oscillations of $x_n$ and $y_n$ increase with the values of the amplitudes of investment changes $\Delta a$, $\Delta b$, can be intuitively understood: if $\Delta a$, $\Delta b$ increase then the market can go further from the fixed point $\mathbf{r}^f$ and the oscillations should be larger.

## 8. INFLUENCE OF NOISE

It is clear that no economic model includes all features of mechanisms governing the economic processes nor is the information on the market situation that is available to decision makers complete. Such an approximation or a lack of knowledge can be modelled by additional degrees of freedom that influence the system as *a noise*. To do this, to the right-hand side of equations (1) and (2) we add an additional term $\mathbf{g}_n = [g^x_n, g^y_n]$ representing a pair of random variables

$$\mathbf{r}_{n+1} = \mathbf{F}(x_n, y_n; p) + \mathbf{g}_n \tag{49}$$

We consider two kinds of noisy perturbations: white noise (WN) and gaussian noise (GN). In the first case we assume that the variables $g^x$ and $g^y$ can take, with a constant probability $\mathrm{Pr}_{WN}(g^x, g^y) = (4\sigma^x_{WN}\sigma^y_{WN})^{-1}$, any value from intervals $[-\sigma^x_{WN}; \sigma^x_{WN}]$ and $[-\sigma^y_{WN}; \sigma^y_{WN}]$, respectively, where the constants $\sigma^x_{WN}, \sigma^y_{WN} \geq 0$. In the second case there are no limits for possible noise values but the noise with a small modulus is more likely than the noise with a large modulus, i.e.

$$\mathrm{Pr}_{GN}(\mathbf{g}) = \frac{1}{2\pi\sqrt{\sigma^x_{GN}\sigma^y_{GN}}} \exp\left[-\frac{(g^x)^2}{2(\sigma^x_{GN})^2} - \frac{(g^y)^2}{2(\sigma^y_{GN})^2}\right], \tag{50}$$

where $\sigma^x_{GN}, \sigma^y_{GN} \geq 0$.

Let us assume that the noise is present during the single-side control performed by the firm $x$. The first question we can ask is: when does the presence of noise destroy the control algorithm? The answer to this question follows directly from the combination of equations for the noisy dynamics equation (49) with the expression equation (4) for the control parameter $\delta a_n$. In fact, if the system is stabilized on the periodic orbit then $\delta\mathbf{r}_n \approx \mathbf{g}_{n-1}$ and the value of the control parameter is

$$\delta a_n = -\frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w_a}} \mathbf{f}_2 \cdot \mathbf{g}_{n-1} \tag{51}$$

However we have assumed that $|\delta a_n| < \Delta a$ and it follows that in the presence of the white noise the control is *permanent* if

$$\Delta a > \left|\frac{\lambda_2}{\mathbf{f}_2 \cdot \mathbf{w_a}}\right| (|\mathbf{f}^x_2|\, \sigma^x_{WN} + |\mathbf{f}^y_2|\, \sigma^y_{WN}) = \Delta a_{crit} \tag{52}$$

while for the gaussian noise the permanent control is possible only in the limit $\Delta a \to \infty$. The validity of the condition (52) has been checked numerically. In Fig. 7(a) and (b) the values of the maximal control parameter $\Delta a$ are *slightly below* the critical value $\Delta a_{crit}$ and one can see that 'laminar' time periods when the system is under control and the values of sales $x_n$ oscillate around the fixed point value $x_f$ are interwoven with periods of chaotic behaviour. On the other hand, for Fig. 7(c) the maximal control parameter $\Delta a$ is *slightly above* the critical value $\Delta a_{crit}$ and one observes that the system stays permanently under control. For all these figures we have used $\sigma^y_{GN} \gg \sigma^x_{GN}$ because the control is performed by firm $X$ which is well informed about its own market situation but has a limited information about the competing firm $Y$.

The second question we can ask is: what is the mean length $\langle n \rangle_L$ of observed 'laminar' intervals? To calculate this value we first need to calculate the probability $p_{\Delta a}$ that for a given value of the maximum control parameter $\Delta a$ the system will stay under control for one time step. Taking into account equation (51), this probability can be expressed as the following integral

$$p_{\Delta a} = \int_{-\infty}^{+\infty} \mathrm{d}g^x \int_{Sg^x - R}^{Sg^x + R} \mathrm{Pr}\left(g^x, g^y\right) \mathrm{d}g^y \tag{53}$$

where $S = -f^x_2/f^y_2$, $R = \Delta a|\mathbf{f}_2 \cdot \mathbf{w_a} \,\|\, \lambda_2 f^y_2|^{-1}$ and $\mathrm{Pr}(g^x, g^y)$ represents either the white noise distribution $\mathrm{Pr}_{WN}(\mathbf{g})$ or the gaussian noise distribution $\mathrm{Pr}_{GN}(\mathbf{g})$. In the first case, the integral can be performed analytically and the result is a piecemeal linear function of the parameter $\Delta a$. In the second case, the integration can be done numerically. Now the probability that

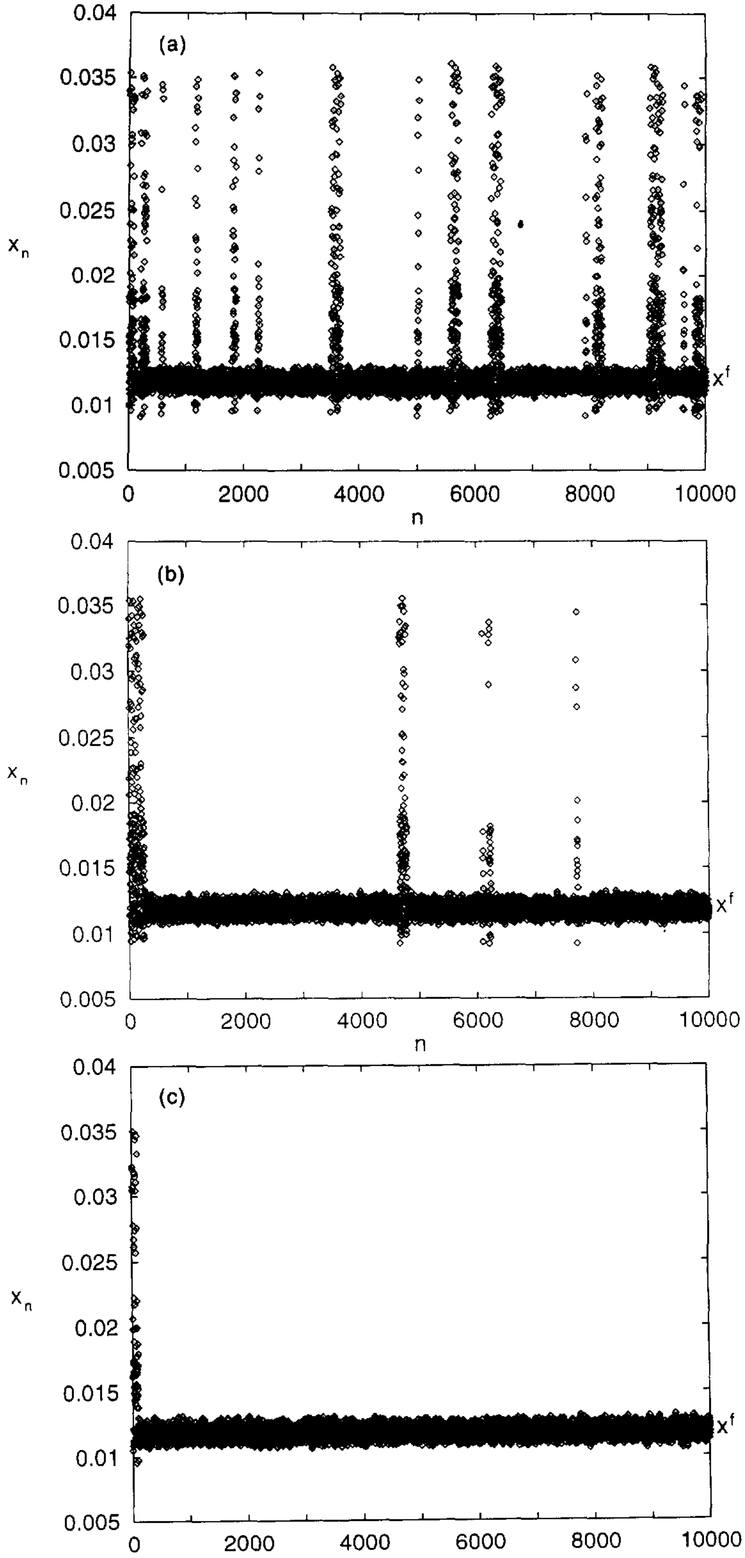


Fig. 7. Influence of white noise on the sales of firm $X$. Maximal amplitudes of the noise are $\sigma_x = 4 \times 10^{-5}$, $\delta_y = 20 \times 10^{-5}$ thus $\Delta a_{crit} = 0.0174744$. For all figures there is $\Delta b = 0$. $\Delta a = 0.017$ for (a), $\Delta a = 0.0172$ for (b) and $\Delta a = 0.0175$ for (c).

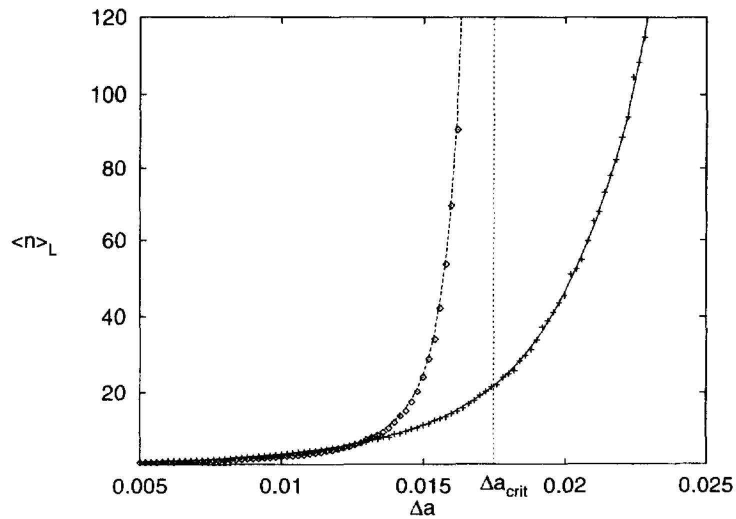


Fig. 8. Influence of the control amplitude $\Delta a$ on the mean length of the laminar market periods for white noise and for gaussian noise.

the system is controlled during $n$ steps but the $(n+1)$ step breaks the control can be written as $P_n = (p_{\Delta a})^n(1-p_{\Delta a})$. The mean value of controlled periods can then be written as

$$\langle n \rangle_L = \sum_{n=0}^{\infty} n P_n = \frac{p_{\Delta a}}{1-p_{\Delta a}} \tag{54}$$

The result of equation (54) is presented in Fig. 8 as the function of the parameter $\Delta a$ for the white and gaussian noise together with corresponding statistics of numerical simulations. We used the same values of the parameters $\sigma^x_{GN}, \sigma^y_{GN}$ as for Fig. 7 and we put $\sigma^x_{WN} = \sqrt{3}\sigma^x_{GN}, \sigma^y_{WN} = \sqrt{3}\sigma^y_{GN}$ to insure that mean square values of the noise components $g_x$, $g_y$ have the same values for the cases of white and gaussian noise. The fact that for white noise in the limit $\Delta a \to \Delta a_{crit}$ we have $\langle n \rangle_L \to \infty$ follows from the fact that for $\Delta a \geq \Delta a_{crit}$ the control is permanent [cf. Fig. 7(c)].

## 9. CONCLUSIONS

We observed that a competition of control between two firms that have the same aim to stabilize the chaotic market *again* makes the system *chaotic*. This phenomenon occurs especially when the 'control efficiencies' of both firms are comparable. On the other hand, when there is a large difference in the control potentials of both firms then the market can be easily controlled by the 'more efficient firm'. The result of competitive control is the occurrence of 'parasitic' oscillations around the stabilized fixed point of the market evolution. The amplitude of these oscillations increases with the increase of the amplitude of the 'less efficient firm' and above some critical value these oscillations destabilize the *temporary state of control*. In order to get a qualitative description of this behaviour one has to extend the standard linear analysis of the controlled system by higher order contributions. Such an approach has proved to be successful and we obtained very good agreement between numerical simulations and analytical estimations of the mean values of market oscillation periods and oscillation amplitudes. The noise can also destroy the control, for the unbounded noise the market evolution is always divided into epochs of laminar (controlled) and chaotic behaviour. For the bounded noise, the system can stay permanently under control provided that the control efficiency is enough large compared to the maximal noise value.

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## APPENDIX A

The Jacobian $\hat{J}^{\mathbf{F}}_{\mathbf{r}_f}$ possesses the eigenvalues $\lambda_1$, $\lambda_2$ and eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ defined by the equations

$$\hat{J}^{\mathbf{F}}_{\mathbf{r}_f}\mathbf{e}_{1(2)} = \lambda_{1(2)}\mathbf{e}_{1(2)} \tag{A1}$$

The unstable fixed point $\mathbf{r}^f$ in the neighborhood of the chaotic attractor must have a character of a so-called saddle fixed point [18]. This is equivalent to the fact that one eigenvalue fulfills $|\lambda_1| < 1$ while for the other eigenvalue there is $|\lambda_2| > 1$. The corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ define the stable and unstable directions of the fixed point $\mathbf{r}^f$, respectively. Assuming that the eigenvectors $\mathbf{e}_{1(2)}$ are normalized, i.e. $|\mathbf{e}_{1(2)}| = 1$ one can now find a pair of vectors $\mathbf{f}_1$, $\mathbf{f}_2$ that are perpendicular to the unstable and stable axis, respectively

$$\mathbf{f}_1\cdot\mathbf{e}_2 = \mathbf{f}_2\cdot\mathbf{e}_1 = 0, \tag{A2}$$

$$\mathbf{f}_1\cdot\mathbf{e}_1 = \mathbf{f}_2\cdot\mathbf{e}_2 = 1. \tag{A3}$$

The vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ together form a so-called contravariant basis.

## APPENDIX B

Taking into account equations (1), (2) and (18) we can expand the unstable component of the vector $\mathbf{r}_{n+1}$ as follows

$$\xi_{u,n+1} \approx \left[\frac{\partial \mathbf{F}}{\partial \xi_s}\xi_{s,n} + \frac{\partial \mathbf{F}}{\partial \xi_u}\xi_{u,n} + \frac{\partial \mathbf{F}}{\partial b}\delta b_n + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial \xi^2_s}\xi^2{}_{s,n} + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial \xi^2_u}\xi^2{}_{u,n} + \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial b^2}(\delta b_n)^2 + \frac{\partial^2 \mathbf{F}}{\partial \xi_s \partial \xi_u}\xi_{s,n}\xi_{u,n} \right.$$
$$\left. + \frac{\partial^2 \mathbf{F}}{\partial \xi_s \partial b}\xi_{s,n}\delta b_n + \frac{\partial^2 \mathbf{F}}{\partial \xi_u \partial b}\xi_{u,n}\delta b_n\right]\cdot \mathbf{f}_2 \tag{B1}$$

where we have assumed that there are no changes of the parameter $a_n$. However, due to relation (4), the linear terms in above equation disappear. The remaining quadratic coefficients can be expressed with the help of the following relations

$$\frac{\partial}{\partial \xi_s} = \sum_k \frac{\partial q^k}{\partial \xi_s}\frac{\partial}{\partial q^k}, \qquad \frac{\partial}{\partial \xi_u} = \sum_k \frac{\partial q^k}{\partial \xi_u}\frac{\partial}{\partial q^k} \tag{B2}$$

(where $q^k = x,y$) and taking into account that

$$\frac{\partial q^k}{\partial \xi_s} = e_1^{q^k} \equiv e_1^k, \qquad \frac{\partial q^k}{\partial \xi_u} = e^{q^k}{}_2 \equiv e_2^k. \tag{B3}$$

As result we get

$$\xi_{u,n+1} \approx \kappa_{u,ss}\xi^2_{s,n} + \kappa_{u,uu}\xi^2_{u,n} + \kappa_{u,bb}(\delta b_n)^2 + \kappa_{u,su}\xi_s\xi_u + \kappa_{u,sb}\xi_{s,n}\delta b_n + \kappa_{u,ub}\xi_{u,n}\delta b_n \tag{B4}$$

where

$$\begin{aligned}
\kappa_{u,ss} &= \tfrac{1}{2}\sum_{k,l,m} e_1^k e^l{}_1 f_2^m \frac{\partial^2 F^m}{\partial q^k \partial q^l} \\
\kappa_{u,uu} &= \tfrac{1}{2}\sum_{k,l,m} e_2^k e^l{}_2 f_2^m \frac{\partial^2 F^m}{\partial q^k \partial q^l} \\
\kappa_{u,us} &= \sum_{k,l,m} e_2^k e^l{}_1 f_2^m \frac{\partial^2 F^m}{\partial q^k \partial q^l} \\
\kappa_{u,ub} &= \sum_{k,m} e_2^k f_2^m \frac{\partial^2 F^m}{\partial q^k \partial b} \\
\kappa_{u,sb} &= \sum_{k,m} e_1^k f_2^m \frac{\partial^2 F^m}{\partial q^k \partial b} \\
\kappa_{u,bb} &= \tfrac{1}{2}\sum_{m} f_2^m \frac{\partial^2 F^m}{\partial b \partial b}
\end{aligned} \tag{B5}$$

A further simplification of results follows from the fact that due to the relation equation (4), the 'variable' $\delta b_n$ is proportional to the 'variable' $\xi_{u,n}$ and due to equations (1) and (2), the coefficient $\kappa_{u,bb}$ vanishes for the considered model.