Supremacy distribution in evolving networks

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We study a supremacy distribution in evolving Barabasi-Albert networks. The supremacy $s_i$ of a node $i$ is defined as the total number of all nodes that are not older than $i$ and can be linked to it by a directed path (including the node $i$). The nodes form a basin connected to the node $i$ as its in-component. For a network with a characteristic parameter $m=1, 2, 3, \ldots$, the supremacy of an individual node increases with the network age as $t^{1+m/2}$ in an appropriate scaling region. It follows that there is a relation $s(k) \sim k^{m+1}$ between a node degree $k$ and its supremacy $s$, and the supremacy distribution $P(s)$ scales as $s^{-1-2/(1+m)}$. Analytic calculations basing on a continuum theory of supremacy evolution and on a corresponding rate equation have been confirmed by numerical simulations.

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I. INTRODUCTION

During the past few years, there has been much interest in the modeling of networks [1–5], and several parameters describing the network structure have been considered. The examples are degree distribution $P(k)$ [2,6], mean path length [7–10], betweenness centrality (load) [9,11], or first and higher-order clustering coefficients [12–14]. Universal scaling has been observed for some of these parameters in computer simulations and in real data describing such objects as the Internet, WWW, scientific collaboration networks, or food webs [3–5]. Here we study a parameter that can play an important role in the description of a class of directed networks. We call the parameter a supremacy since it describes the number of nodes that are subordinated to a certain node. The parameters is equal to the size of the basin connected to a certain node $i$ or to the size of its in-component [15–17]. In the next section, we define our parameter and show its relevance for different problems of complex networks. Section III includes a continuum theory for the supremacy time evolution $s_i(t)$ and the supremacy probability distribution $P(s)$ in the Barabasi-Albert (BA) model with $m=1$, in Sec. IV we find and solve a corresponding rate equation, while in Sec. V a generalization of our problem for the BA model with $m>1$ is presented.

II. THE MODEL

Let us consider the BA network with the characteristic parameter $m=1$ [1,2]. At the moment $t$, a node $i$ is created and it attaches to some older node in the network according to the preferential attachment rule (PAR). Then in the next time steps, other nodes are created and are attached to the node $i$ or to other nodes of the network following PAR. As a result, at the moment $t>t_i$ there is a subgraph in a form of a tree $T(i, t)$ beginning in the node $i$ and containing all nodes that are not older than the node $i$ and that are connected to $i$ by directed paths as in Fig. 1. If we assume that the node $i$ represents a scientist who wrote an important paper [18] or a politician who created an influential party [19,20], we can consider all nodes belonging to the tree as his/her successors. If the tree $T(i, t)$ contains $s_i$ nodes, then the number $s_i$ is the measure of the supremacy or the predominance of the node $i$ at time $t$. Since the evolution of the network is governed by PAR and all properties of the network are described by some probability distributions, we are interested in the supremacy distribution $P(s)$ in the network.

The resulting subgraph $T(i, t)$ can be also interpreted as a cluster of connected sites in the directed percolation problem [8,21–25], and the supremacy of a node $i$ is just the size of such a cluster starting from the site $i$. The subgraph $T(i, t)$ has been called in [8,17,25] an in-component of the node $i$ and has been considered in [15] for description of the Internet structure and in [16] for scaling relations in food webs. It has been proved [15] that the supremacy distribution in the

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The solution of simulations presented in Fig. 2 while for larger
where we took into account the initial condition
write the following equation for changes of
the node degree linearly in time comparing to the square-root dependence of
results, we get a simple relation between the node supremacy
and the node degree,

\[ s_i = 1 + \sum_{j=1}^{s_i} (k_j^{(i)} - 1) = 1 + K(i) - s_i. \]

Thus \( K(i) = 2s_i - 1 \) and we have a simple equation

\[ \frac{\partial s_i(t)}{\partial t} = \frac{2s_i - 1}{2t}, \]

with the solution

\[ s_i(t) = \frac{1}{2} \left( \frac{t}{t_i} + 1 \right) \]

where we took into account the initial condition \( s_i(t=t_i) = 1. \)
The solution (4) means that the node supremacy increases linearly in time comparing to the square-root dependence of
the node degree \([2], i.e., k_i(t) = \sqrt{t_i} t_i. \)
Combining the last two results, we get a simple relation between the node supremacy and the node degree,

\[ s(k) = \frac{1}{2} \left[ k^2 + 1 \right]. \]

In the region \( k \leq 100, \) this formula fits well with numerical
simulations presented in Fig. 2 while for larger \( k, \) differences
between the analytic theory and the numerical simulations
are observed.

The probability density \( P(s) \) for the supremacy distribution
in the network follows from the relation

\[ P(s) ds_i = \tilde{P}(t_i) dt_i, \]

where \( \tilde{P}(t_i) = 1/t \) is the distribution of node attachment times
\( t_i \) for a network of age \( t. \) After a simple algebra, we get

\[ P(s_i) = \frac{1}{t} \left| \frac{\partial s_i}{\partial t_i} \right|^{-1} = \frac{2}{(2s_i - 1)^2}. \]

One can see that the supremacy distribution is a time-independent function. Figure 3 show a comparison of the last equation to numerical data. Let us stress that for \( s \gg 1, \) the supremacy distribution scales as \( P(s) \sim s^{-2} \) while the degree distribution for the BA model \([1,2] \) scales as \( P(k) \sim k^{-3}. \)

### IV. Rate Equation for Supremacy Distribution for \( m=1 \)

Now we show how to get the supremacy distribution using
the rate-equation approach that was introduced by
Krapivsky, Redner, and Leyvraz \([6] \) to study network degree
distribution \( P(k). \) Let \( N(s,t) \) be the number of nodes possessing
the supremacy \( s \) at time \( t. \) The rate equation for \( N(s,t) \) is

\[ \frac{\partial N(s,t)}{\partial t} = \frac{1}{t} \left| \frac{\partial s_i}{\partial t_i} \right|^{-1} = \frac{2}{(2s_i - 1)^2}. \]

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FIG. 4. Schematic illustration of supremacy effects in the BA network with \( m = 2 \). Solid arrows represent connections within the supremacy area/cluster of the top vertex, whereas dashed arrows express connections pointing outside the cluster. Note that there is a single loop in the cluster.

\[
\frac{\partial N(s,t)}{\partial t} = \frac{[2(s-1)-1]N(s-1,t)-(2s-1)N(s,t)}{2t} + \delta_{s,1}. \tag{8}
\]

The first term on the right-hand side of Eq. (8) corresponds to the creation of a new node with the supremacy \( s \). The process is proportional to the number of nodes with the supremacy \( s-1 \) and the corresponding transition probability that follows from the PAR and Eq. (2). The second term corresponds to the creation of a node with a supremacy \( s + 1 \), i.e., to the destruction of a node with a supremacy \( s \), while the last term describes the creation of a node with a supremacy \( s = 1 \). Writing \( N(s,t) = P(s)N_0 \), where \( N_0 = t \) corresponds to the total number of nodes at time \( t \) and \( P(s) \) is the probability of a node with the supremacy value \( s \), we get the recursive equation

\[
P(s) = \frac{2s-3}{2s+1} P(s-1) \quad \text{for } s \geq 2, \tag{9}
\]

where \( P(1) = 2/3 \). The solution of Eq. (9) is

\[
P(s) = \frac{2}{(2s-1)(2s+1)}. \tag{10}
\]

Note that \( s \gg 1 \), the solution (10) coincides with the solution (7) that has been received in the limit of the continuum theory. In fact, the solution (10) was obtained for the first time in [17] with a slightly different approach.

V. SCALING OF SUPREMACY DISTRIBUTION FOR \( m \gg 1 \)

The peculiar feature of the BA model is the independence of the scaling exponent characterizing the degree distribution \( P(k) \sim k^{-\alpha} \) from the model parameter \( m \) describing the number of links that are created by every new node. Below, we show that the scaling exponent of supremacy distribution depends on the parameter \( m \). If we neglect all loops existing in the BA network with the characteristic parameter \( m \gg 1 \) (see Fig. 4), then we can easily repeat our considerations from Secs. III and IV. Instead of Eq. (2), we get

\[
s_i = 1 + \sum_{j=1}^{s_i} (k_i^{(j)} - m) = K(i) + 1 - ms_i, \tag{11}
\]

and the time evolution of the supremacy is described by

\[
s_i(t) = \frac{m}{m + 1} \left( \frac{t}{t_i} \right)^{(m+1)/2} + \frac{1}{m + 1}. \tag{12}
\]

thus the relation between the degree and the supremacy is

\[
s(k) = \frac{m}{m + 1} \left( \frac{k}{m} \right)^{m+1} + \frac{1}{m + 1}. \tag{13}
\]

It follows that for dense networks with \( m \gg 1 \), the supremacy \( s_i(t) \) increases in time much faster than the node degree \( k_i(t) \). Figure 2 shows a comparison of the result (13) to numerical data for \( m = 2 \). One can see that the predicted scaling of \( s(k) \) breaks down completely for large values of \( k \) where the plot \( s(k) \) saturates. The reason is the presence of loops for \( m \gg 1 \) appear in the network and that have been neglected in our approach. If \( m \gg 1 \), the result (13) is valid mainly for vertices with a small degree \( k_i \) (and a small supremacy \( s_i \)) since loops are sparse in small clusters starting from such nodes. The saturation effect does not appear for the BA model with the parameter \( m = 1 \) where loops are absent.

Taking into account Eq. (12), we get the supremacy distribution in the form

\[
P(s_i) = \frac{1}{t} \left| \frac{\partial s_i}{\partial t_i} \right|^{-1} = \frac{2}{m} \left( \frac{m+1}{m} s_i - 1 \right)^{-(m+3)/(m+1)}. \tag{14}
\]

We see that the scaling exponent for the supremacy distribution is equal to \( \delta = -1 - 2l/(1 + m) \), and in contrast to the scaling exponent of degree distribution it depends on the parameter \( m \). The result (14) is in good agreement with numerical simulation for BA networks; see Fig. 3. The rate equation for \( m > 1 \) is similar to Eq. (8), i.e.,

\[
\frac{\partial N(s,t)}{\partial t} = \frac{[(1+m)(s-1)-1]N(s-1,t)}{2t} - \frac{[(1+m)s-1]N(s,t)}{2t} + \delta_{s,1}. \tag{15}
\]

The resulting solution for the probability \( P(s) \) can be written as the following product:

\[
P(s) = \frac{2}{m+2} \prod_{i=2}^{s} \frac{[(i-1)(m+1) - 1]}{[i(m+1) + 1]} \tag{16}
\]

for \( s > 1 \), where \( P(1) = 2/(m+2) \). For dense networks \( m \gg 1 \), the solution (16) can be approximately written as

\[
P(s) \approx \frac{2}{ms}, \tag{17}
\]

which coincides with Eq. (14).

VI. CONCLUSIONS

In conclusion, we introduced a universal parameter (a supremacy) that describes vertices in directed networks. The
parameter is equal to the size of a cluster starting from the site in a directed percolation model. We have shown that for the Barabasi-Albert model there is a relationship between the supremacy and the vertex degree. It follows that there are universal scaling laws describing the time evolution of the supremacy and corresponding supremacy distributions in BA models. Contrary to the scaling results for the node degree, the corresponding scaling exponents of supremacy depend on the characteristic model parameter \( m \). Numerical simulations are in good agreement with analytical estimations for nodes with a small and medium supremacy, especially for the case \( m=1 \), where no loops are present in the system. The results for \( m>1 \) show that the influence of loops on the supremacy distribution is negligible for the evolving BA networks considered.

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