Linear stability analysis in a liquid layer with a surface velocity gradient

Jarosław Białecki* and Janusz A. Hołyst†
Faculty of Physics and Center of Excellence for Complex Systems Research, Warsaw University of Technology, Koszykowa 75, PL–00-662 Warsaw, Poland

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A case of combined planar Couette-Poiseuille flow corresponding to vanishing horizontal flux has been generalized by the introduction of a model for the surface velocity gradient. A relation corresponding to the Orr-Sommerfeld equation has been derived for this model. The critical value of the surface velocity gradient has been obtained. At the critical point, the corresponding critical Reynolds number equals infinity. Using an approximated method we estimated the behavior of the critical Reynolds number for a slightly overcritical surface velocity gradient.

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I. INTRODUCTION

This work is inspired by biomedical research for mass transport across the liquid layer with a surfactant that is expanded or compressed by a moving barrier [1]. The horizontal velocity of the surfactant is a linear function of the horizontal coordinate. Our aim is to investigate if the gradient of the velocity can influence the liquid stability. It should be emphasized that we are not interested in the phenomena connected to the inhomogeneous surface tension, for instance the Marangoni effect, but our objective is to study the instability due to a specific velocity profile across the liquid layer.

The starting point is one of the simplest and also the best known hydrodynamical systems, namely the combined planar Couette-Poiseuille flow. Its stability has been intensively researched since the beginning of the twentieth century using different methods and miscellaneous approximations. The linear analysis for the planar Poiseuille flow gives the critical Reynolds number $R_c = 5772$ [2,3] and for the planar Couette flow $R_c = \infty$, so that the latter flow is absolutely stable with respect to infinitesimal amplitude disturbances. For a combination of both flows, when the Couette component increases from zero, the flow becomes more and more stable [4,5] and the critical Reynolds number increases from $R_c = 5772$ to infinity. Nonlinear analysis and numerical simulations reveal for both planar Couette and Poiseuille flows an undercritical instability [6–8]. For the planar Poiseuille flow an instability with respect to finite disturbances occurs when $R > R_G = 2645$ [3]. The linear stability analysis yields only sufficient criteria for the instability occurrence and says nothing definite about the stability. In turn, the energy method predicts only the sufficient condition for the stability and says nothing definite about the instability [9,10]. The energy analysis shows that the planar Poiseuille flow is stable for $R < R_{ES} \approx 81.5$ [2] and the planar Couette flow is stable for $R < R_{ES} \approx 82.6$ [11].

In this work the case of combination of Poiseuille and Couette flows, corresponding to the vanishing total horizontal flux, has been generalized by introducing into the model a constant surface velocity gradient. It means that due to the boundary condition the velocity of the upper surface of liquid is a linear function of the horizontal coordinate. Because our aim is to estimate the upper limit of critical instability parameters, we use the linear analysis. In Sec. II our model is defined. In Sec. III its characteristic equation corresponding to the Orr-Sommerfeld relation and applied approximations as well as numerical methods are described. In Sec. IV results of numerical computation are presented. In Sec. V the validity range for the employed approximation is estimated. Section VI contains a discussion of obtained results.

II. THE MODEL

We consider a two-dimensional incompressible homogeneous viscous fluid [12]. A dimensionless Cartesian coordinate system is chosen with the z axis positive upwards. The layer of fluid is infinite in x direction, $z = -1$ denotes the bottom of the liquid layer, and $z = 1$ is the upper surface. Both the bottom and the upper surface are supposed to be rigid. For convenience, the origin of the coordinate system is chosen in such a way that fluid velocities at the bottom surface vanish. It follows that the characteristic velocity is that of the upper surface.

The starting point of our consideration is a combination of Poiseuille and Couette flows. In the coordinate system defined above, the boundary conditions are

$$v_x(x, z=-1) = 0, \quad v_x(x, z=1) = 1,$$

$$v_z(x, z=-1) = 0, \quad v_z(x, z=1) = 0. \tag{1}$$

The condition for vanishing of the total horizontal flux is

$$\int_{-1}^{1} v_x(z)\, dz = 0. \tag{2}$$

The stationary velocity $\mathbf{U}$ has nonvanishing horizontal component that depends only on the vertical coordinate z;

$$\mathbf{U} = (U_x(z), 0). \tag{3}$$

It follows from the Navier-Stokes and the continuity equations that the stationary velocity profile is given by function

*Electronic address: bialecki@if.pw.edu.pl
†Electronic address: jholyst@if.pw.edu.pl

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$$U_x = 0.75z^2 + 0.5z - 0.25. \tag{4}$$

Changing the coordinate system to a new one (denoted by asterisk) where fluid velocities of the bottom and upper surfaces possess the same absolute values and opposite directions while the characteristic velocities are normalized as $\max[U_x(z)]=1$, solution (4) can be written as

$$U_x^* = 0.9(1 - z^{*2}) + 0.6z^*. \tag{5}$$

Since the ratio of the Couette to Poiseuille component 0.6/0.9 is greater than the critical ratio corresponding to the absolute stability 0.341/0.970 [2], the flow is absolutely stable with respect to linear perturbations.

Now we extend the model by introducing the velocity gradient γ of the upper surface. The boundary condition (1) changes to

$$v_x(x,z=-1)=0, \quad v_x(x,z=1)=1+\gamma x,$$

$$v_z(x,z=-1)=0, \quad v_z(x,z=1)=0. \tag{6}$$

Gradient γ is assumed to be constant for all x. The existence of the nonzero gradient γ ensure that contrary to form (3) the stationary fluid velocity vector $\mathbf{U}$ consists of two components that are functions of both coordinates x and z.

$$\mathbf{U} = (U_x(x,z), U_z(x,z)). \tag{7}$$

In this model relation (2) leads to appropriate boundary conditions for a stream function.

III. LINEAR STABILITY ANALYSIS

The first step in the linear analysis is to find the stationary velocity field. It follows from the Navier-Stokes and the continuity equations that the stream function $\psi(x,z,t)$ fulfills equation

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial z} + \frac{\partial\psi}{\partial z}\frac{\partial\Delta\psi}{\partial x} - R^{-1}\Delta\Delta\psi = 0. \tag{8}$$

For the stationary flow we can separate variables in the stream function ψ_s,

$$\psi_s(x,z) = F(x)G(z). \tag{9}$$

The form of function $F(x)$ must be chosen in accordance with the boundary condition (6);

$$F(x) = 1 + \gamma x. \tag{10}$$

Parameter γ possesses no intuitive interpretation because in nondimensional coordinate system it is scaled by characteristic velocity. To overcome this drawback we introduce another nondimensional parameter γ_r as

$$\gamma_r = \gamma R, \tag{11}$$

so

$$\gamma_r = \frac{\partial V_X}{\partial X}\frac{H^2}{4\nu}, \tag{12}$$

where V is the surface velocity, X is the horizontal coordinate, and H is the layer thickness, all measured in our dimensional coordinate system.

Inserting Eqs. (9)–(11) into Eq. (8) we obtain the equation for the stationary flow as

$$\gamma_r(G'G'' - GG''') - G'''' = 0 \tag{13}$$

with boundary conditions

$$G(-1) = G'(-1) = G(1) = 0, \quad G'(1) = 1. \tag{14}$$

The second step is to examine the stability of the stationary solution against an infinitesimal disturbance. We assume a disturbed stream function ψ_d in the form

$$\psi_d(x,z,t) = (1 + \gamma_r R^{-1}x)G(z) + \phi(z)e^{i\alpha(x-ct)}, \tag{15}$$

where the second term on the right-hand side of Eq. (15) is a small perturbation. Let us point out that in spite of the lack of translation symmetry along the x direction, we have assumed a periodical disturbance. We do not consider a solution for the whole x axis but only for a small x district where the surface velocity is almost constant. In the following, we search for the critical Reynolds number only within the considered x district. The main role of the surface velocity gradient is to change the undisturbed velocity profile. Inserting Eq. (15) into Eq. (8) and collecting all terms of the first order of perturbation $\phi(z)$ we obtain the equation

$$\begin{aligned} 0 = &-(i\alpha R)^{-1}(D^2-\alpha^2)^2\phi + (G'-c)(D^2-\alpha^2)\phi - G'''\phi \\ &+ \gamma_r R^{-1}(-i\alpha GD - i\alpha^{-1}G''D + i\alpha^{-1}GD^3)\phi \\ &+ \gamma_r R^{-1}x(G'D^2 - G''' - G'\alpha^2)\phi, \end{aligned} \tag{16}$$

where $D = \partial/\partial z$. Function ϕ should obey the homogeneous boundary conditions

$$\phi(-1) = \phi'(-1) = 0, \tag{17}$$

$$\phi(1) = \phi'(1) = 0. \tag{18}$$

The first line of Eq. (16) is the Orr-Sommerfeld equation, while the next two lines appear due to the surface velocity gradient. One has to point out that the influence of this gradient on stability properties of the considered model is also hidden in function $G(z)$, following from Eq. (13) that corresponds to the undisturbed flow. In our model we assume that one can neglect the last line of Eq. (16) proportional to $\gamma_r R^{-1}x$. The validity of this approach, i.e., the accuracy range of such an approximation will be *a posteriori* checked in Sec. IV.

Because the disturbed problem is of the fourth order we can choose four particular solutions

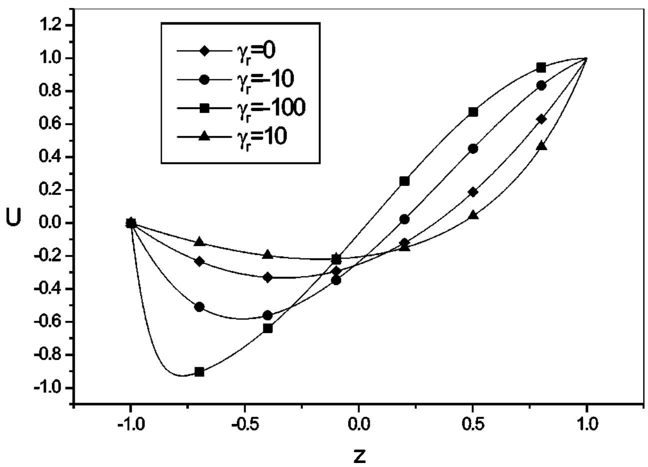

FIG. 1. Undisturbed stationary velocity profile.

$$\phi_0(-1)=1,\quad \phi_1(-1)=0,\quad \phi_2(-1)=0,\quad \phi_3(-1)=0,$$

$$\phi_0'(-1)=0,\quad \phi_1'(-1)=1,\quad \phi_2'(-1)=0,\quad \phi_3'(-1)=0,$$

$$\phi_0''(-1)=0,\quad \phi_1''(-1)=0,\quad \phi_2''(-1)=1,\quad \phi_3''(-1)=0, \tag{19}$$

$$\phi_0'''(-1)=0,\quad \phi_1'''(-1)=0,\quad \phi_2'''(-1)=0,$$

$$\phi_3'''(-1)=1.$$

Functions ϕ_0 and ϕ_1 vanish due to Eq. (17), so nontrivial solution consists only of ϕ_2 and ϕ_3. Fulfilling the boundary condition (18) is equivalent to vanishing of the determinant

$$\begin{vmatrix} \phi_2(1) & \phi_3(1) \\ \phi_2'(1) & \phi_3'(1) \end{vmatrix} = 0. \tag{20}$$

A solution of the undisturbed problem (13) with the boundary condition (14) has been found using the shooting method [14] where the differential equation has been integrated using the fourth order Runge-Kutta method.

Equation (16), representing the disturbed problem, is a stiff equation, so similar to the undisturbed problem (13) that one cannot simply integrate it for the boundary conditions (17) and (18). The problem is solved as follows. We introduce function

$$F(z)=\begin{vmatrix} \phi_2(z) & \phi_3(z) \\ \phi_2'(z) & \phi_3'(z) \end{vmatrix}. \tag{21}$$

Using the method of compound matrix [13] we integrate function $F(z)$ over z, and applying the shooting approach we look for parameter c such that condition (20) is fulfilled and solutions ϕ_2 and ϕ_3 obey Eqs. (16)–(18).

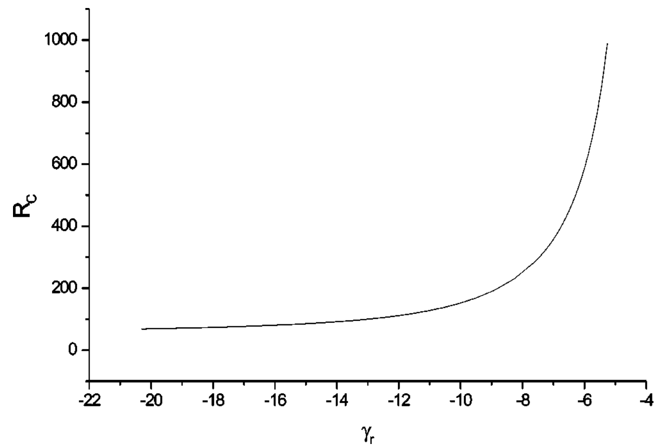

FIG. 2. Critical Reynolds numbers vs the surface velocity gradient.

IV. RESULTS

First let us consider how the surface velocity gradient affects undisturbed, stationary velocity profile (Fig. 1). The problem can be solved exactly. If the gradient vanishes ($\gamma_r=0$) the velocity profile is parabolic. For a negative gradient value (compression of surface) the extreme of the profile becomes deeper and an inflection point occurs. These two features suggest that for a negative gradient the flow becomes unstable. For a positive value of γ_r the extreme is flatter and the inflection point does not appear to be what suggests a stable character of this solution.

Solutions of the disturbed problem confirm this forecast. The instability does appear for a negative value of the surface velocity gradient when $\gamma_r<\gamma_{rC}=-4.21$. Figure 2 shows that the critical Reynolds number is infinite for γ_{rC} and rapidly decreases below $R=100$ when absolute value of γ_r increases above 4.21.

Now let us consider the shape of the disturbance. The spatial distribution of the stream function (only its real part has the physical meaning) can be easily found by putting the computed function $G(x)$ into the last term of Eq. (15). Of course, this term describes only the disturbance possessing an infinitesimally small amplitude. An example of such a disturbance is depicted in Fig. 3. One can see that the disturbance possesses the shape of a chain of skewed vortexes.

The wave number α of the critical disturbance (Fig. 4) starts from zero for $\gamma_r=\gamma_{rC}$; while the gradient γ_r is decreasing, the critical wave number is gowing up. This behavior is similar to the instability of combined planar Poiseuille

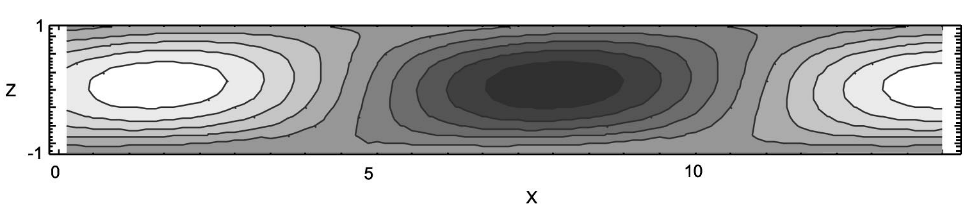

FIG. 3. The real part of the stream function of the critical disturbance for $\gamma_r=6$.

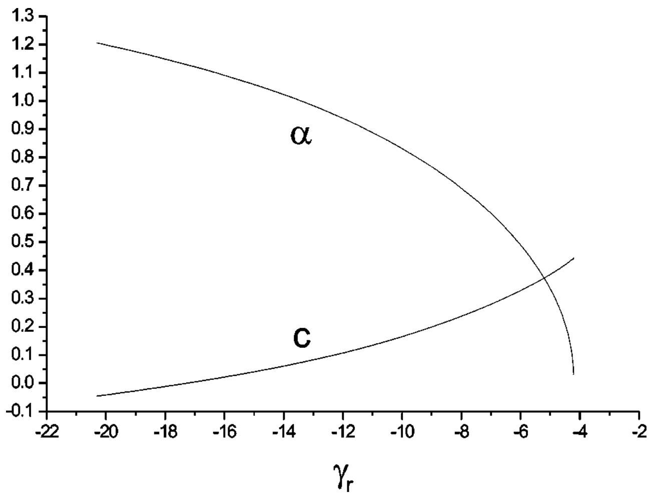

FIG. 4. Phase velocity c and wavelength α of the critical disturbance vs surface velocity gradient.

and Couette flow [2] that starts from the infinite wavelength when Couette component is increased over the critical value.

The phase velocity c of the critical disturbance (Fig. 4) starts from $c=0.44$ for $\gamma_r=\gamma_{rC}$, monotonically decreases with decreasing γ_r, and crosses zero at $\gamma_r=-17.2$. It follows that for $\gamma_{rC}>\gamma_r>-17.2$ the phase velocity of the disturbance possesses the opposite direction to the surface velocity, while for $\gamma_r<-17.2$ the directions of both velocities are the same.

V. VALIDITY OF USED APPROXIMATION

One has to check at least *a posteriori* the validity of our approach since we have neglected terms proportional to $\gamma_r R^{-1}x$ in Eq. (16). One can easily see that the value of critical surface velocity gradient γ_{rC} is *exact* because for the infinite Reynolds number the omitted term and other terms proportional to $\gamma_r R^{-1}$ disappear. It follows that the whole influence of the velocity gradient at $\gamma_r=\gamma_{rC}$ on the system stability is due to changes of the *stationary* velocity profile described by function $G(z)$ given by Eq. (13). The results for $\gamma_r<\gamma_{rC}$ are approximated but it is easy to compare coefficients of appropriate derivatives of ϕ in the neglected part of Eq. (16) to the corresponding coefficients in the nonneglected part. Let us define functions D_i as

$$D_i=\max_z\left|\frac{N_i}{M_i}\right|x^{-1}, \tag{22}$$

where N_i is a coefficient of the ith derivative of ϕ standing by the neglected part of Eq. (16) and M_i is the corresponding coefficient of the non-neglected part of Eq. (16). In the omitted third line of Eq. (16) there are only zeroth and second derivatives of ϕ, so we have

$$D_0=\max_z\left|\frac{\gamma_r R^{-1}(-G'''-G'\alpha^2)}{i\alpha^3R^{-1}-\alpha^2(G-c)-G'''}\right|, \tag{23}$$

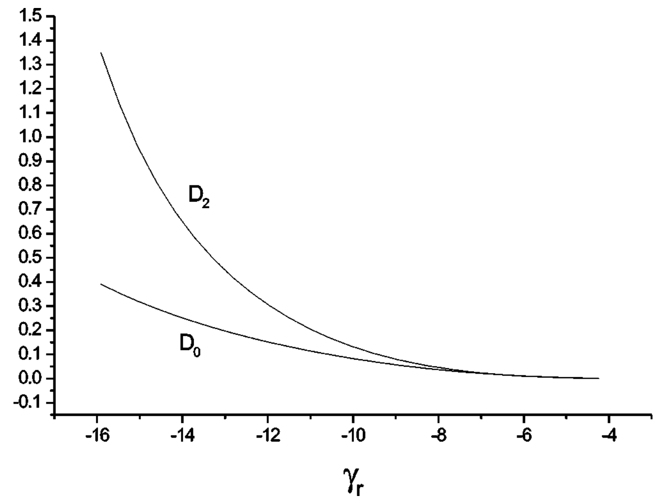

FIG. 5. Functions D_0 and D_2 vs surface velocity gradient.

$$D_2=\max_z\left|\frac{\gamma_r R^{-1}G'}{-2i\alpha R^{-1}+G-c}\right|. \tag{24}$$

Both functions D_0 and D_2 converge to zero when $\gamma_r\to\gamma_{rC}$ (Fig. 5), which indicates that our approximation is self-consistent.

Another parameter that can be used for the estimation of correctness of our approach is a change of surface velocity Δv at the distance of one wavelength of critical disturbance. The approximation is valid when $\Delta v \ll 1$. Figure 6 shows that it occurs in the neighborhood of γ_{rC}. Comparing Figs. (5) and (6) we see that the validity of our approach is limited to $|\gamma_r|<10$.

VI. CONCLUSIONS

Existence of the surface velocity gradient can have a substantial destabilizing effect for the shear flow. In the considered case of combined planar Couette and Poiseuille flows, positive values of gradient (expansion of surface) do not

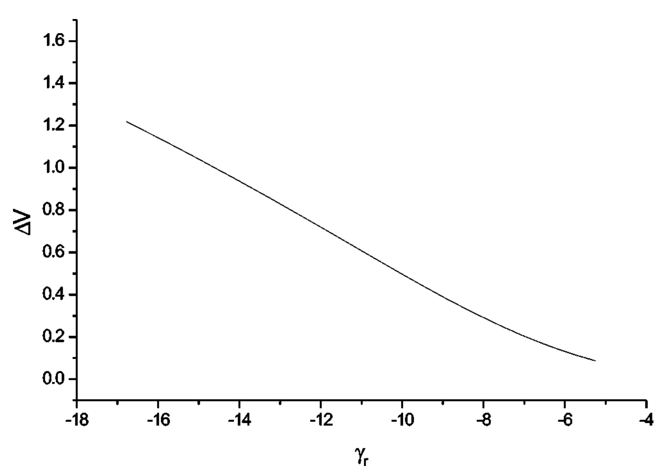

FIG. 6. Changes of surface velocity gradient along one wavelength of critical disturbance vs surface velocity gradient.

cause instability at all. However, if the value of the surface velocity gradient is negative (compression of the surface) the instability is possible. The instability begins when the absolute value of the dimensionless surface velocity gradient is larger than $|\gamma_r| > 4.21$. When the instability starts the corresponding critical Reynolds number R_c is infinite, but for increasing $|\gamma_r|$ it rapidly decreases to values $R \approx 100$.

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