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Phase Transitions and Hysteresis in a Cellular Automata-Based Model of Opinion Formation

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A particular case of a cellular automata-based model of two-state opinion formation in social groups with a strong leader is studied. We consider a 2D Euclidian geometry of "social space" and mutual interactions $\propto 1/r^n$. The model shows an interesting dynamics which can be analytically calculated. There are two stable states of the system: a cluster around the leader and unification. Unstable clusters may also appear. A variation in parameters such as the leader's strength or the "social temperature" can change the size of a cluster or, when they reach some critical values, make the system jump into another state. For a certain range of parameters the system exhibits bistability and hysteresis phenomena. We obtained explicit formulas for the cluster size, critical leader's strength, and critical "social temperature." These analytical results are verified by computer simulations.

KEY WORDS: Cellular automata; opinion formation; bistability; phase transitions.

1. INTRODUCTION

During the last 20 years there has been a great deal of interest in the application of paradigms of theoretical physics to the description of social and economic phenomena. The usefulness of methods based on the concepts of synergetics, the master equation,[1] cellular automata,[2,3] neural networks,[4] dynamical systems, and deterministic chaos[5] has been proven in several papers. The main advantage of these methods when applied to

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social or economical models[6–22] is the *quantitative* character of the results they provide.

Cellular automata[2] (CA) are one of these interdisciplinary concepts, first introduced by von Neumann.[23] CA are dynamical systems in which space and time are discrete. They consist of a lattice of cells which can have a finite (usually small) number of values. The cells are updated synchronously according to a definite local rule which is the same for each cell. Cellular automata have been extensively applied in physics, chemistry, biology, and the theory of computation,[3] and they are considered to be an alternative or complementary way of describing nature as compared to differential equations. Despite their simplicity, they often exhibit a very complex dynamics. They have been also used for modeling the behavior of groups of individuals.[8]

The theory of *social impact* (SI) formulated by Latané[24] has led to a new class of CA models.[22, 25] It claims that the impact imposed on an individual by a group of people is a certain function of their strength, immediacy, and number. The concept of "social space" is important here. Each pair of individuals is characterized by a distance in this space and the absolute value of social impact is some decreasing function of this distance. This impact is in fact a combined effect of several types of social influences that an individual can experience from others and that are known in the sociological literature[24] as *behavioral contagion*, *conformity*, *compliance*, *group pressure*, *imitation*, *normative influence*, *observational learning*, *social faciliation*, *suggestion*, and *vicarious conditioning*. The SI theory is able to describe such various social phenomena[24] as *stage fright and embarrassment*, *news interest*, *tipping in restaurants*, or *bystander intervention*. Although very simplified, the SI models show complex processes like *clusterization* and *polarization* of opinion which are also observed in real social groups.[26]

Opinions are often represented in SI models by binary states. This is certainly relevant for yes/no questions, but it was also proved by social surveys that in the case of *important* questions with *many* possible answers the distribution of attitudes is *bimodal*, i.e., most people share one out of two opposite opinions.[18] The analysis of several SI models based on statistical mechanics (mean-field theory) can be found in ref. 20. One of the main results is a "staircase" cluster dynamics. Starting from a random initial distribution of attitudes, the social system tends to form *clusters* of individuals sharing the same opinion, though we assume that they cannot move. Such a structure enables the minority opinion to survive.

Clusters are basic stable structures appearing in the course of evolution in the SI models. In this paper we examine a particular case of a cluster around a *strong leader*.

In Section 2 we introduce some general assumptions and properties of the SI model. Section 3 is devoted to the problem of size of the followers' cluster appearing around a leader in social space and in Section 4 we study the influence of "social temperature" (i.e., noise) on such a cluster.

2. THE MODEL

Following the paper by Lewenstein *et al.*,[20] we consider a system consisting of N individuals (members of a social group) and we assume that each of them can share one of two opposite opinions on a given subject, denoted as $\sigma_i = \pm 1, i = 1, 2,..., N$. Individuals can influence each other's opinion, and each individual is characterized by two parameters which describe his/her "social strength": persuasiveness $p_i > 0$, which denotes the ability to induce others to adopt his/her opinion, and supportiveness $s_i > 0$, which is the ability to support those who already share it. Every pair of individuals (i, j) is ascribed a distance d_{ij} in social space. Similarly to cellular automata models,[2] the dynamics of the system has a discrete form: the opinion of the ith individual at a moment $t+1$ depends on the *social impact* I_i that he/she experiences at the moment t. The impact is defined as

$$I_i = I_p \left[\sum_{j=1}^{N} \frac{f(p_j)}{g(d_{ij})} (1 - \sigma_i \sigma_j) \right] - I_s \left[\sum_{j=1}^{N} \frac{f(s_j)}{g(d_{ij})} (1 + \sigma_i \sigma_j) \right] \tag{1}$$

Here, $I_p(x)$ and $I_s(x)$ are increasing functions describing the relative "persuading" and "supporting" impact (usually one assumes[20, 24] that $I_{p,s}(x) \propto x^{\alpha}$, where $0.5 \leqslant \alpha \leqslant 1$), $f(x)$ is an increasing function of s_i and p_i, and $g(x)$ is also an increasing function of the social distance d_{ij}. In our model we put $I_p(x) = I_s(x) = \frac{1}{2}x$, $f(x) = x$ and

$$g(d_{ij}) = \begin{cases} d_{ij}^n, & n \geqslant 1 \quad \text{for} \quad i \neq j \\ 1/\beta & \quad\quad\quad\ \text{for} \quad i = j \end{cases} \tag{2}$$

where β is the so-called self-support parameter. We also assume $s_i = p_i \equiv s_i$, i.e., everyone is ascribed only one parameter, which we call "strength." Considering all the above assumptions, the social impact on the ith individual [cf. Eq. (1)] can be rewritten in the form

$$I_i = \sum_{j=1, j \neq i}^{N} \frac{-s_j \sigma_i \sigma_j}{d_{ij}^n} - s_i \beta \tag{3}$$

Changes of opinion in the course of time, i.e., the dynamics of the system, are determined by

$$\sigma_i(t+1) = -\text{sign}[\sigma_i(t)I_i(t)] \tag{4}$$

i.e.,

$$\sigma_i(t+1) = \begin{cases} \sigma_i(t) & \text{for} \quad I_i(t) \leqslant 0 \\ -\sigma_i(t) & \text{for} \quad I_i(t) > 0 \end{cases} \tag{5}$$

The dynamics may be synchronous (all individuals are updated at the same time) or asynchronous (they are updated in consecutive steps one after another in a given order or at random). In our simulations we use the synchronous type of dynamics unless stated otherwise.

The dynamics (4) can be extended in the following way:

$$\sigma_i(t+1) = -\text{sign}[\sigma_i(t)I_i(t) + h_i(t)] \tag{6}$$

i.e.,

$$\sigma_i(t+1) = \begin{cases} \sigma_i(t) & \text{for} \quad I_i(t) + \sigma_i(t)\, h_i(t) \leqslant 0 \\ -\sigma_i(t) & \text{for} \quad I_i(t) + \sigma_i(t)\, h_i(t) > 0 \end{cases} \tag{7}$$

The additional term h_i may express the influence of external factors (e.g., mass media, government policy, special events) supporting one of the opinions. It can also be a random variable modeling all the complexity of human behavior. In the latter case we obtain a noisy dynamics.

The properties of the model for different geometries of social space with s_i and p_i being random variables and exponents in (2) $n \in \{0, 1, 2, 4, 8\}$ were extensively studied both analytically and numerically.[20, 22] It was found[21] that the value $n \approx 2$ gives a good fit to correlation data collected during social surveys on the influence of *physical distance* on social interactions for populations such as students/adults in Boca Raton (Florida), students in Shanghai, or the group of social psychologists attending a Nags Heads Conference.

3. A GROUP WITH ONE LEADER

Consider a circular space of radius R. The individuals are located in the nodes of a quadratic grid, and the distance between nearest neighbors equals 1. This geometric distance models social immediacy.[4] Each of the

[4] The assumptions that individuals are uniformly placed at sites of a regular lattice and that "the social distance" is calculated as the geometric distance can certainly look artificial in a model describing society. However, we stress that just such approximations have been used by social scientists who have successfully simulated various cooperative phenomena occurring in social groups.[22, 25]

individuals but the one in the middle of the circle has equal strength $s_i = 1$. The strength s_L of the individual in the center, whom we will call the "leader," is far greater than that of the others ($s_L \gg 1$). According to the symmetry one can expect that a circular cluster of individuals sharing the opinion of the leader will form around him or her. Our first aim is to calculate the size of the cluster, i.e., its radius a as a function of s_L, R, and β. The state of the system is stable if the impact on every individual is negative. This certainly is the case for those who are distant from the cluster border. Positive (persuasive) impact may appear only close to the border, where individuals of different opinion meet. Therefore we shall consider the impact at the border. It is convenient to replace the sum over individuals in (3) by an integral. This is justified when $a \gg 1$. The surface density of individuals is 1 (one per unit square).

First we shall investigate long-distance interactions, $1/g(d_{ij}) = 1/d_{ij}$. The total impact experienced by an individual at the border of a cluster with radius a can be then derived as

$$I = \frac{s_L}{a} + \iint_{K'((a,0),a)} \frac{1}{r}\, dS - \left(\iint_{K'((a,0),R)} \frac{1}{r}\, dS - \iint_{K'((a,0),a)} \frac{1}{r}\, dS \right) - \beta \tag{8}$$

where r (replacing the notation d_{ij} for the discrete net) is the distance from the origin $(0, 0)$ which is defined by the location of the individual considered (see Fig. 1). The symbol $K((x, y), r)$ stands for a circle with the center at point (x, y) and radius r. The first term in (8) represents the influence of the leader, the second represents the influence of the members of the cluster. The expression in parentheses is the impact exerted by the majority, i.e., the people outside the cluster (we call these members the "majority," although for some values of the parameters s_L, β, and R this group can be smaller than the group of followers around the leader) and β is the self-supportive impact. K' denotes the respective circles excluding the intersection with the circle $K((0, 0), 1/\sqrt{\pi})$, which represents the individual under consideration. We assumed here that he/she shares the opinion of the majority. If the opposite case held, all terms in (8) would change their signs except for the self-support β, which is always negative. The first assumption means that the cluster grows tending to a stable state; the seconds assumption, that it shrinks. The limiting condition for the stability of the cluster is $I = 0$. Calculating the integrals in polar coordinates, one obtains, according to the assumptions mentioned above, two equations differing in the sign of β,

$$I = \frac{s_L}{a} + 8a - 4RE\left(\frac{a}{R}, \frac{\pi}{2}\right) + \sqrt{\pi} \pm \beta = 0 \tag{9}$$

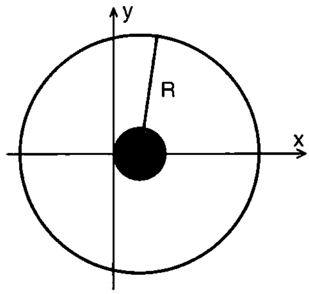

Fig. 1. Scheme of a cluster around a leader.

where $E(k, \varphi) = \int_0^\varphi (1 - k^2 \sin^2 \alpha)^{1/2} \, d\alpha$ is an elliptic integral of the second kind.

In the zeroth-order approximation (valid for $a \ll R$) we have $E(a/R, \pi/2) \approx \pi/2$ and (9) simplifies to a quadratic equation which has two solutions for both $\pm\beta$ (i.e., four solutions altogether):

$$a \approx \frac{1}{16} \{(2\pi R \pm \beta) \pm [(2\pi R \pm \beta)^2 - 32 s_L]^{1/2}\} \tag{10}$$

It follows from (10) that for $s_L > s_{Lc} \approx (2\pi R + \beta)^2/32$ no real solution exists. In this case the general acceptance of the leader's opinion is the only stable state. At the point $s_L = s_{Lc}$ a *discontinuous phase transition* appears. The stable cluster radius changes from $a_c \approx (2\pi R + \beta)/16$ to $a = R$ [unless $\beta > R(16 - 2\pi)$ when $a_c > R$]. If the leader is too weak, on the other hand, it may be impossible for him or her not only to form a cluster, but also to maintain his/her own opinion. The limiting condition for the minimal strength of the leader $s_{L\,\min}$ to resist against the persuasive impact of the majority can be written as

$$\int_0^{2\pi} d\varphi \int_{1/\sqrt{\pi}}^{R} \frac{1}{r} \, r \, dr = \beta s_{L\,\min} \tag{11}$$

Hence,

$$s_{L\,\min} = \frac{2\pi}{\beta} \left(R - \frac{1}{\sqrt{\pi}} \right) \tag{12}$$

For $s_L < s_{L\,\min}$ unification ($a = 0$) will be the only stable state again.

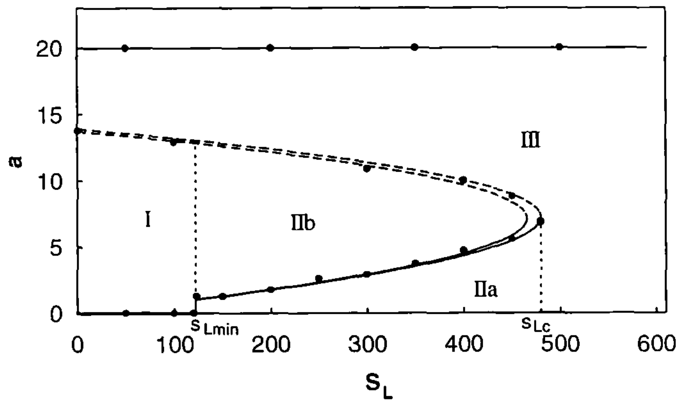

Fig. 2. Cluster radius a vs. leader's strength s_L-phase diagram for a circular social space. Interactions were assumed proportional to the inverse of the mutual distance ($I \propto 1/r$). Lines correspond to analytical results, points to computer simulations.

Figure 2 shows the phase diagram in s_L–a space. All plots are made for a space of radius $R = 20$ (1257 individuals) and $\beta = 1$ unless stated otherwise. The choice $\beta = 1$ means that the individual's own opinion is as important as that of his/her nearest neighbor. We have *two kinds* of stable solutions, i.e., attractors: a homogeneous distribution of opinions (i.e., $a = R$ when the leader's opinion wins, $a = 0$ when it ceases to exist) and a stable cluster corresponding to the stable solution of (9). The curves in Fig. 2 are obtained by solving (9) numerically. Solid lines represent stable fixed points-attractors [they correspond to the solution (10) with the minus sign in front of the square root]; dashed lines represent unstable repellers [corresponding to the plus sign in (10)]. In s_L–a space one can distinguish three basins of attraction. Starting from a state in the area denoted as I, the temporal evolution leads to unification with $a = 0$. The stable cluster attractor divides its basin of attraction into the areas IIa and IIb. All states from III will evolve to unification with $a = 20$. Owing to the two possible signs of the self-support parameter β in (9), the attractor and repeller are split. In the parameter space between the split curves we observed a mixture of stationary and unstationary clusters, where the former correspond to *local equilibria* of the system dynamics. We stress here that, as a result of self-support, even states lying in the space formed by the *split repeller* can be stabilized.

The continuum approximation used in our analytical calculations implies a complete rotational symmetry of the system. In the case of a square lattice the symmetry is reduced to the fourfold axis, and in effect the actually observed clusters are not exactly circular, but rather square

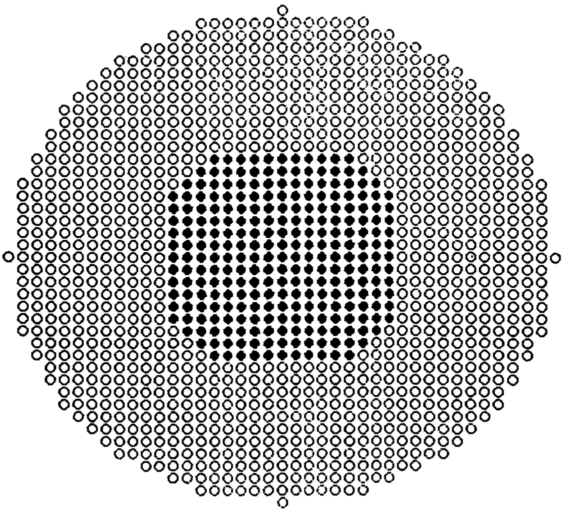

Fig. 3. Octagonal cluster observed for $I \propto 1/r^3$, $s_L = 2000$.

(smaller clusters) or octagonal (larger clusters) (see Fig. 3). we did not observe other polygons, but one can anticipate that they can appear for model parameters allowing still larger clusters. Therefore, the value a corresponding to simulation points in Fig. 2 (as well as in Figs. 5 and 11) denotes an *effective radius* of the cluster, i.e., the radius of a circle with an *area* equal to that of the actually observed octagon (square).

To obtain the effective cluster radius in regions corresponding to the (split) attractor and repeller (Figs. 2, 5, and 11), we used several sets of initial conditions, i.e., circular clusters with various radii, and we waited for an appropriate number of time steps until the clusters reached their equilibrium shapes and sizes. As the clusters evolved to the nearest (locally) stable states we thus observed a broad spectrum of effective cluster radii and we plotted only the cluster parameters corresponding to the states on the border of these regions.

One can also imagine a situation where the opinion of the leader is independent of the influence of the group and his/her strength parameter is varying. In such a case we observe a hysteresis of the (evolving) percentage η of individuals sharing the leader's opinion (Fig. 4). Let us assume that the starting point is the leader holding the "for" opinion ($\sigma_L = +1$) and all the others "against." When we increase s_L, the cluster of "for" followers grows to the critical value s_{Lc}, at which a phase transition to unification occurs. A further increase or decrease of s_L will obviously cause no other changes. But when the leader changes his/her opinion (to "against"), the process of

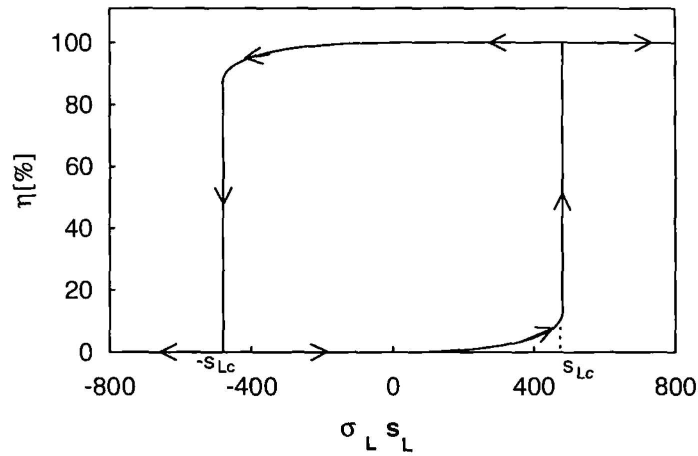

Fig. 4. Hysteresis in the system's behavior. The leader's strength s_L and his/her opinion σ_L change independently of the group.

cluster expansion will be repeated. Thus, for $s_L < s_{Lc}$ the system shows *bistability*.

Similar calculations can be made for other interactions [e.g., for $g(d_{ij}) = (d_{ij})^n$, $n > 1$]. The results are more complicated. We can derive the functions $s_L(a)$ analytically for $n = 2$, 3, and 4, respectively:

$$s_L(a) = a^2 \left\{ -4 \int_0^{\varphi_g} \ln \cos \varphi \, d\varphi - 4\phi_g \ln (2a\sqrt{\pi}) + \pi \ln(\pi a^2) + \int_0^{2\pi} \ln \left[\cos \varphi + \left(\frac{R^2}{a^2} - \sin^2 \varphi \right)^{1/2} \right] d\varphi \pm \beta \right\} \tag{13}$$

$$s_L(a) = a^3 \left\{ \frac{2}{a} \ln \left(\frac{1 + [1 - 1/(4\pi a^2)]^{1/2}}{1 - [1 - 1(4\pi a^2)]^{1/2}} \right) - 4\sqrt{\pi}\phi_g + 2\pi\sqrt{\pi} - \frac{1}{a} \int_0^{2\pi} \ln \left[\cos \varphi + \left(\frac{R^2}{a^2} - \sin^2 \varphi \right)^{1/2} \right]^{-1} d\varphi \pm \beta \right\} \tag{14}$$

$$s_L(a) = a^4 \left\{ \frac{1}{2a^2} (4\pi a^2 - 1)^{1/2} - 2\pi \arccos \frac{1}{2a\sqrt{\pi}} + \pi^2 - \frac{1}{2a^2} \int_0^{2\pi} \ln \left[\cos \varphi + \left(\frac{R^2}{a^2} - \sin^2 \phi \right)^{1/2} \right]^{-2} d\varphi \pm \beta \right\} \tag{15}$$

where $\phi_g = \arccos [1/(2a + \sqrt{\pi})]$.

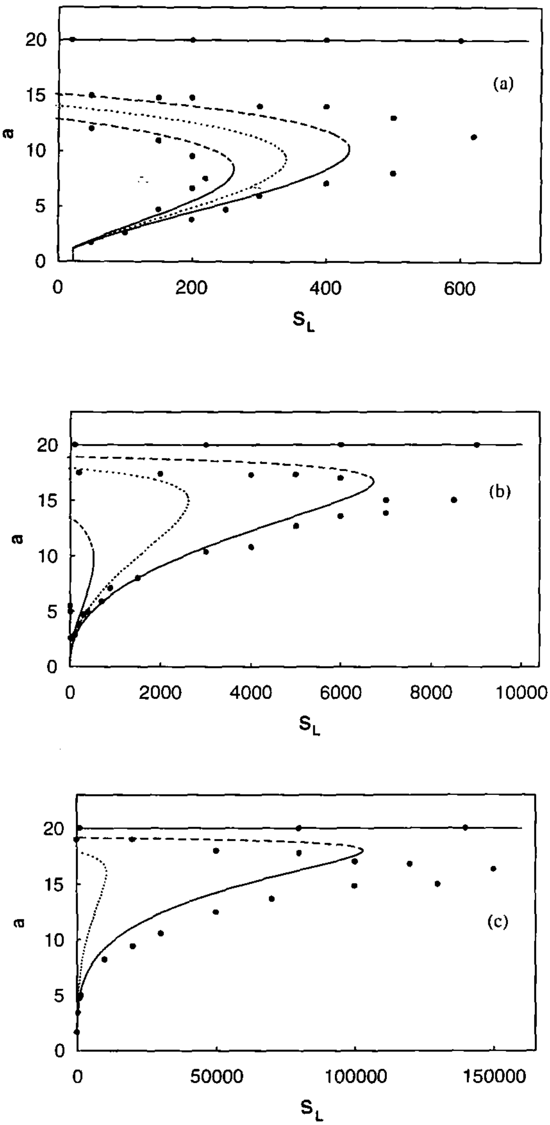

Fig. 5. Cluster radius a vs. leader's strength s_L-phase diagram for a circular social space. Social impact: (a) $I \propto 1/r^2$, (b) $I \propto 1/r^3$, (c) $I \propto 1/r^4$. Lines correspond to analytical results, the dotted line to the case of $\beta = 0$ (no splitting), and the points to computer simulations.

Inverting the above functions (numerically), we obtain the $a(s_L)$ dependences. They are plotted in Fig. 5 with the respective simulation results. Dotted lines are for $\beta = 0$ (no splitting). Again the simulation points denote effective radii of clusters of *extreme size* that are typically octagonal. Qualitatively these diagrams have a structure similar to that of Fig. 2. This means that above a certain critical value of s_L no stable cluster exists and the only possible final state of the system is a total unification $a = R$. Because of the weaker short-range interactions, the same value of the self-support parameter $\beta = 1$ causes a much greater splitting of attractor and repeller; the area of "frozen states" is enlarged. Upon replacing the sum in (3) by an integral we assumed a continuous distribution of individuals, while in our simulation they sit in nodes of a square net. Such an approximation yields good results for long-range interactions. In the case of short-range interactions an individual mostly "feels" his/her nearest neighbors and their exact position becomes important. This is why calculation and simulation results diverge when the range of interactions decreases.

The above results generally hold when the strength parameter s_i of all individuals except the leader is a random variable with a mean value equal to 1, but still $s_i \ll s_L$, instead of being identical for all individuals, which is certainly more realistic. This fact follows simply from the observation that introducing such a random variable will lead to the appearance of additional integrals (or sums) over all possible values of s_i in (8). Assuming that

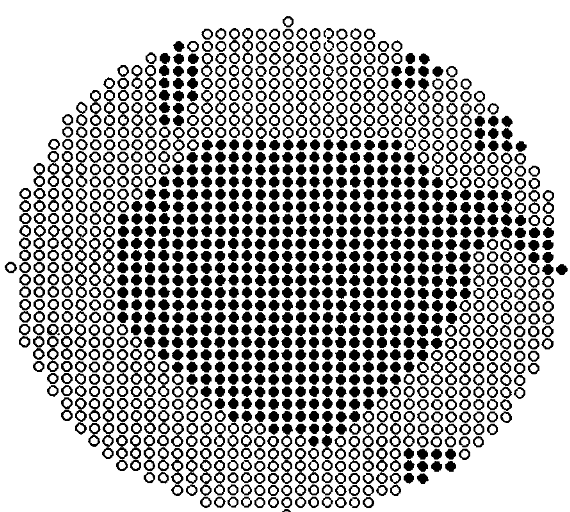

Fig. 6. Stationary state reached after 16 time steps (33% "black" minority). Initial state: 45% "black" minority randomly distributed over the circle; $I \propto 1/r^4$, $s_L = 30000$.

the probability distribution Pr (s_i) characterizing the random variable s_i is a space-independent function, one can reduce the resulting equations to the Eq. (8). We checked in corresponding computer simulations that the behavior of such "random" model was similar to that with a fixed value of the strength parameter s_i.

In all our analytical calculations as well as computer simulations we considered only the clusters that were *centered* around the leader so we could use the resulting *symmetry* and get explicit forms for the cluster radius a. It is clear that *nonsymmetric* clusters are also possible. For example, if the initial state is not symmetric with respect to the leader, the stationary state can have the form of a nonsymmetric cluster or a set of nonsymmetric clusters (Fig. 6). Such clusters are more likely for models with short-range interactions. We observed also that an increase of the value of the self-support parameter β significantly favors the appearance of an asymmetry, and in the case of $\beta = 0$ asymmetric clusters are possible only as states with an *oscillating* cluster boundary. On the other hand, a change from the *synchronous dynamics* to an *asynchronous dynamics* results in an increase of the probability of nonsymmetric clusters which can then appear even for symmetric initial conditions.

4. NOISY DYNAMICS

It is obvious that the behavior of an individual in a group does not only depend on the influence of others, nor is the influence itself strictly determined. There are many factors, both internal (personal) and external ones, that induce opinion changes. It seems necessary to model them somehow. In our model we do this by using noisy dynamics. Noise can be introduced into the dynamics (4) in different ways. One of them is the special case of formula (6) when h_i is a random variable. Another way is to write the updating rule as

$$\sigma_i' = \begin{cases} 1 & \text{with probability} \quad \dfrac{\exp(-I_i\sigma_i/T)}{\exp(-I_i\sigma_i/T) + \exp(I_i\sigma_i/T)} \\ -1 & \text{with probability} \quad \dfrac{\exp(I_i\sigma_i/T)}{\exp(-I_i\sigma_i/T) + \exp(I_i\sigma_i/T)} \end{cases} \tag{16}$$

where we abbreviated $\sigma_i(t)$ by σ_i and $\sigma_i(t+1)$ by σ_i'. Equation (16) is an analog of the Glauber dynamics in neural networks models[4] (with $-I_i\sigma_i$ corresponding to the local field) and it is related also to the multinomial logit model used in decision theory.[7] The parameter T may be interpreted as a "social temperature," i.e., a degree of randomness in the behavior of individuals. Equation (16) implies the deterministic dynamics (4) when

$T \to 0$. Both ways of introducing noise are equivalent when h_i in (6) is a random variable with distribution

$$\Pr(h_i) = \frac{1}{2T \cosh^2(h_i/T)} \tag{17}$$

having the mean value $\overline{h_i} = 0$ and variance $\overline{h_i^2} = (T\pi/\sqrt{12})^2$. To show this, we rewrite (6) as

$$\sigma_i' = \begin{cases} 1 & \text{with probability} \quad \Pr(I_i\sigma_i + h_i < 0) \\ -1 & \text{with probability} \quad \Pr(I_i\sigma_i + h_i > 0) \end{cases} \tag{18}$$

However, $\Pr(I_i\sigma_i + h_i < 0) = \int_{-\infty}^{-I_i\sigma_i} \Pr(h_i)\, dh_i$. Comparing (16) and (18), we find

$$\int_{-\infty}^{-I_i\sigma_i} \Pr(h_i)\, dh_i = \frac{\exp(-I_i\sigma_i/T)}{\exp(-I_i\sigma_i/T) + \exp(I_i\sigma_i/T)} \tag{19}$$

and, after differentiating the above equation, we obtain (17).

Now let us see the effect of noise on the cluster dynamics of our model, using the synchronous version of the updating rule (16) and long-range interactions $I \propto 1/r$. Figure 7 shows the results of the simulation with growing temperature. As temperature increases, the cluster grows slightly up to certain temperature level at which the process speeds up rapidly and the cluster bursts to the whole space. To explain such a behavior we need to calculate the social impact on individuals in different points of social space with a cluster around the leader. Considering the dynamics (16), we can conclude that the influence of noise on a single individual depends on the ratio of impact to temperature. When $I_i/T \to 0$ both probabilities in (16) tend to $\frac{1}{2}$, and we obtain a totally random dynamics; when $I_i/T \to \infty$ it becomes strictly deterministic. Because of symmetry, the impact (at zero temperature) depends only on the distance to the leader. Again we replace the sum over individuals by an integral and then the impact on an individual [at point $(0, 0)$] at a distance d from the leader [at point $(d, 0)$] can be written as

$$\begin{aligned} I_i(d) = {} & -\frac{s_L}{d} - \iint_{K'((d,0),a)} \frac{1}{r}\, dS \\ & + \left(\iint_{K'((d,0),R)} \frac{1}{r}\, dS - \iint_{K'((d,0),a)} \frac{1}{r}\, dS \right) - \beta \qquad \text{for} \quad d < a \end{aligned} \tag{20}$$

$$I_o(d) = \frac{s_L}{d} + \iint_{K'((d,0),a)} \frac{1}{r}\,dS - \left(\iint_{K'((d,0),R)} \frac{1}{r}\,dS - \iint_{K'((d,0),a)} \frac{1}{r}\,dS\right) - \beta \quad \text{for} \quad d > a \tag{21}$$

where a is the radius of the cluster. Calculating the integrals in polar coordinates, one gets the impact inside $(d < a)$ and outside $(d > a)$ the cluster, respectively:

$$I_i(d) = -\frac{s_L}{d} - 8aE\left(\frac{d}{a}, \frac{\pi}{2}\right) + 4RE\left(\frac{d}{R}, \frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta \tag{22}$$

$$I_o(d) = -\frac{s_L}{d} - 8aE\left(\frac{d}{a}, \arcsin\frac{a}{d}\right) + 4RE\left(\frac{d}{R}, \frac{\pi}{2}\right) + 2\sqrt{\pi} - \beta \tag{23}$$

Both functions are plotted in Fig. 8 for $s_L = 400$. Since the system remains in equilibrium, the impact on every individual is negative (nobody changes his/her opinion). It approaches zero at the border of the cluster, which means that individuals located in the neighborhood of that border are most sensitive to thermal fluctuations. We can, however, observe an asymmetry of the impact. Its absolute value increases much more steeply in the region inside the cluster than outside it. From this it follows that all individuals who are placed nearer to the leader and share his/her opinion are more deeply confirmed in their opinion, so they are also more resistant against noise in the dynamics. If we increase temperature, starting from $T \simeq 0$ (as in the simulation corresponding to Fig. 7), random opinion changes begin. They primarily concern the individuals near the border (where the impact is weakest). As a result, individuals with adverse opinions appear both inside and outside the cluster, but they are more numerous outside because of the weaker impact (Fig. 8). In effect we observe a growth of the minority group. This causes the supportive impact outside the cluster to become still weaker and the majority becomes more sensitive to random changes, which is a kind of positive feedback. At a certain value of temperature the process becomes avalanche-like and the former majority disappears. Thus noise induces a jump from one attractor (cluster) to another (unification). Such a transition is possible for every temperature different from zero, but its probability remains negligible until the noise level exceeds a certain critical value. Our simulations prove that it is indeed a well-defined temperature that separates two phases (i.e., two attractors). Figure 9 demonstrates that temperatures which are very close to each other yield quite a different behavior of the system. At $T = 25$ we have a cluster with an effective radius $a < R$ and with randomly flipping individuals, mainly near its border, while

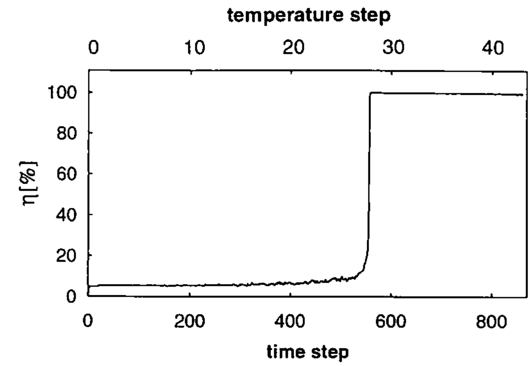

Fig. 7. Time evolution of the percentage η of the leader's followers at temperature increasing by 1 every 20 time steps. $I \propto 1/r$, $s_L = 400$. Starting point: $T = 0$, $\eta = 0.08\%$ (leader only).

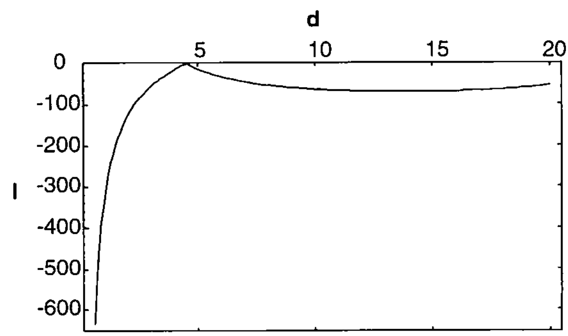

Fig. 8. Social impact I as a function of distance d to the leader. The leader's strength was set to $s_L = 400$.

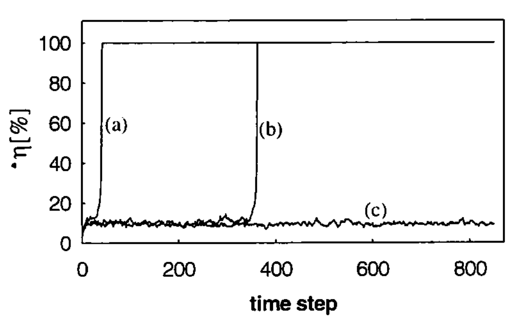

Fig. 9. Temporal evolution of the percentage η of the leader's followers at constant temperatures (a) $T = 27$, (b) $T = 26$, (c) $T = 25$. Here $I \propto 1/r$, $s_L = 400$; starting point: $\eta = 0.08\%$ (leader only).

at $T=27$ the system quickly reaches unification, i.e., $a=R$. At $T=26$ metastability of the cluster can be observed. Its mean lifetime is then of the order of a few hundred time steps and then we have a jump to unification.

We can estimate how the effective cluster radius depends on temperature. For this we assume that the individuals in the cluster share the opinion +1. Putting (22) and (23) into the noisy dynamics (16), we get the following form for the probability that an individual at distance r from the leader shares the opinion +1:

$$\Pr(\sigma=1)(r)=\begin{cases}\dfrac{\exp(-I_i(r)/T)}{\exp(-I_i(r)/T)+\exp(I_i(r)/T)} & \text{for} \quad r<a\\[2ex] \dfrac{\exp(I_o(r)/T)}{\exp(-I_o(r)/T)+\exp(I_o(r)/T)} & \text{for} \quad r>a\end{cases} \tag{24}$$

The mean number of all individuals with opinion +1 may be calculated by integrating the probability multiplied by the surface density (which is equal to 1) over the whole space:

$$\overline{n(\sigma=1)(T)}=\int_0^R \Pr(\sigma=1)(r)\, 2\pi r\, dr \tag{25}$$

This number equals the effective area of the circular cluster, so its radius is

$$\overline{a(T)}=\left[\frac{\overline{n(\sigma=1)(T)}}{\pi}\right]^{1/2} \tag{26}$$

In fact, this is not an explicit form of the function $a(T)$, because the radius of the cluster appears on the right-hand side of the above equation, namely in $I_i(r)$ and $I_o(r)$ given by (22) and (23). However, the transcendent equation (26) may be solved numerically.

As one can see in Fig. 10, at low temperatures there are two solutions and only the smaller one is stable. The cluster radius grows with increasing temperature up to a critical value T_c, when both solutions coincide. At this temperature a transition from a stable cluster to unification occurs (Fig. 7). For $T>T_c$ no solution exists. Figure 11 presents the function $a(T)$ in comparison with the results of computer simulations.

We have quite a good conformity of both curves, despite the fact that our calculation was simplified. Namely, we used expressions for impact (22), (23) which assumed the existence of a regular, circular cluster, while in fact, in the presence of noise, the border is diluted and both opinions may appear within the whole social space. This is a kind of mean-field approximation.

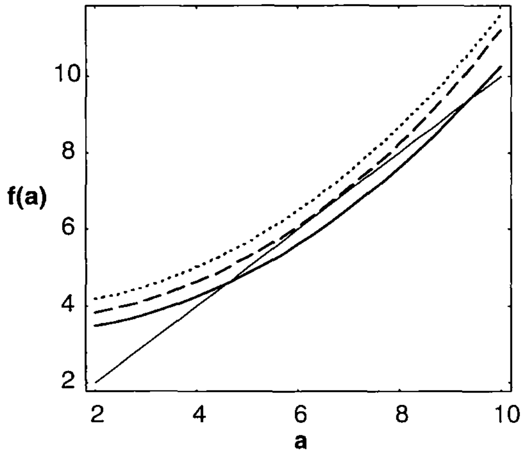

Fig. 10. Solution of Eq. (26); $f(a) \equiv$ RHS of Eq. (26) for $s_L = 400$ and $T = 10$ (solid line), $T = 25 = T_c$ (dashed line), and $T = 30$ (dotted line).

Using (26), one can also calculate T_c as a function of the leader's strength s_L. However, (22) and (23) imply that the leader's opinion does not change. In fact, when the s_L is small and the temperature sufficiently large, the probability of such a change becomes significant. Below a certain value of $s_L = s_{L\,\mathrm{min}}$ the leader cannot maintain his/her opinion even at zero temperature (12). To prevent this, we set the opinion fixed, independent of the influence of the group.

The plot of $T_c(s_L)$ (Fig. 12) is a kind of *phase diagram*, i.e., the leader's followers are the minority for parameter values below the curve and the majority above it. Upon crossing the $T_c(s_L)$ line (for constant s_L), a *phase transition* occurs in the form of a rapid jump in the percentage η of the

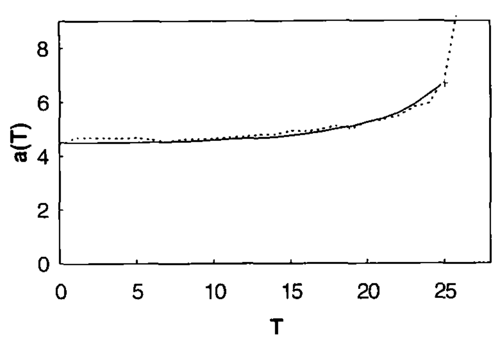

Fig. 11. Mean cluster radius a vs. temperature T; $s_L = 400$. Results of our calculations are represented by the solid line and those of computer simulations by the dotted line.

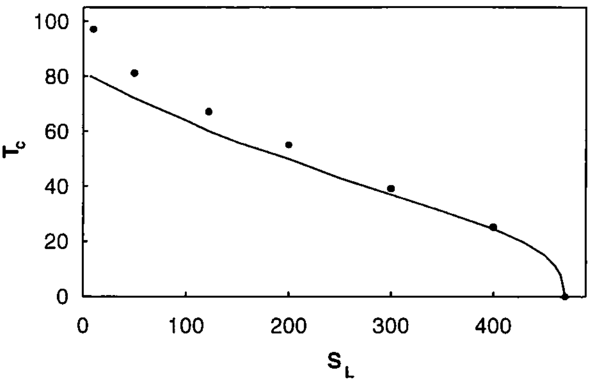

Fig. 12. Critical temperature T_c (above which no stable cluster exists) vs. leader's strength s_L. Leader's opinion was fixed (independent of the group). The line corresponds to analytical results [Eq. (26)], points to simulations.

leader's followers (as in Fig. 7). The magnitude of that jump is maximal for s_L close to but smaller than s_{Lc}, and it decreases with decreasing s_L. For small s_L we cannot see any cluster around the leader even at a low noise level; however, as the temperature approaches T_c there is still a rapid change in η. For example, if $s_L = 10$, it changes from 30% to 60%, which shows that the system still "feels" the presence of a strong individual. At high temperatures the dynamics becomes random and, on average, equal numbers of individuals share both opinions.

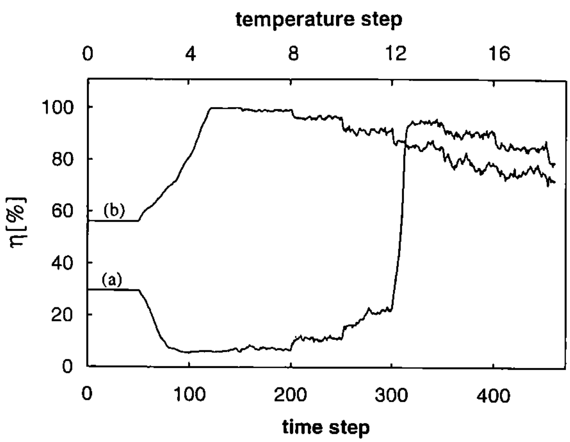

Fig. 13. Temporal evolution of the percentage η of the leader's followers with temperature increasing by 2 every 50 time steps. (a) Interactions $I \propto 1/r^2$, $s_L = 200$; (b) interactions $I \propto 1/r^4$, $s_L = 40000$. Starting point: $T = 0$.

Noise effects were also observed in systems with interactions of shorter range. The results are qualitatively similar to the case of $I \propto 1/r$. However, because of the weaker interactions, the impact on every individual is smaller than for $I \propto 1/r$, so that the system is more sensitive to noise. In the simulations shown in Fig. 13 we started from a state lying in the area of the repeller, split by self-support (Figs. 5a and 5c). In the case of interactions $I \propto 1/r^2$ when noise appears, the system quickly falls on the attractor (stable cluster). As temperature goes on increasing, the minority grows and finally the cluster loses its stability, jumping to unification. In the case of interactions $I \propto 1/r^4$ there is no relaxation to the stable cluster, but we can see a constant growth of the number of the leader's followers toward uniformity (rather then a rapid jump). The process is slower, because an individual practically feels only an impact of his/her nearest neighbors, and it requires more time for changes to propagate.

One can observe also in Fig. 13 that for $T > T_c$ the thermal fluctuations grow and as the result the percentage of the leader's followers η decreases. In the limit $T \to \infty$ one has $\bar{\eta} \to 50\%$ and the dynamics is totally random $[\Pr(\sigma_i = \pm 1) \to 1/2]$. Similar results could be seen for the interactions $I \propto 1/r$ (Fig. 7) if the temperature were appropriately higher.

5. CONCLUSIONS

The model of opinion formation in a social network with a strong leader exhibits interesting clustering phenomena which can be analytically described. Besides the "trivial" state of uniform opinion distribution, the system may remain in the state of a stable or unstable cluster centered around the leader (unstable clusters can be stabilized due to the effect of self-support). There is a critical value of the leader's strength above which no cluster can exist; below this value the system exhibits bistability, which leads to a hysteresis phenomenon. In the presence of noise (nonzero "social temperature") all clusters seem to be metastable states; however, at low temperatures their lifetimes may be practically infinitely long. An increase of temperature can cause the following effects: an increase of the effective radius of the stable cluster, a loss of stability of the cluster and transition to the uniform state, and jumps from an unstable cluster to a stable one or to uniformity. All these effects result in rapid changes of the minority to majority proportion.

The model presented in this paper is very simplified and the quantitative results are not directly applicable to real social systems, but still it might be useful for the description of some sociological phenomena as in the case of the original Nowak and Latané model. For example, in the history of human civilization there have been many leaders with a strong personality (e.g., Luther, Gandhi, Hitler) who were nearly alone with their "ideas" at the beginning. However, in the course of time they managed to influence their societies to such a degree that their opinions came to be shared by more and more individuals and in some cases even "social homogenization" took place. One also observes that for such a transition to the homogeneous state the role of "social noise" is sometimes very important; e.g., Hitler came to power in a politically destabilized Germany during the time of the Weimar Republic. On the other hand, it is remarkable that existing homogeneous societies were (and are) very resistant against further changes, so that one might speak of social hysteresis. All these features (appearance of clusters, homogenization, the influence of social noise, social hysteresis) were observed in our computer simulations, and were described analytically.

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