

# Phase transitions in social networks

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**Abstract.** We study a model of network with clustering and desired node degree. The original purpose of the model was to describe optimal structures of scientific collaboration in the European Union. The model belongs to the family of exponential random graphs. We show by numerical simulations and analytical considerations how a very simple Hamiltonian can lead to surprisingly complicated and eventful phase diagram.

**PACS.** 89.75.-k Complex systems – 02.50.-r Probability theory, stochastic processes, and statistics

## 1 Introduction

During the last years, there has been noticed a significant interest in the field of complex networks and a lot of interdisciplinary initiatives have been taken aiming at investigations of these systems [1–4]. In parallel with empirical studies of real world networks [5,6], theoretical models [7–9] and abstractive mathematical tools [10,11] have been developed in order to understand complex mechanisms hidden behind the network functionality.

Among many studied network models like random graphs [12], or growing networks [7] there exists a class of models, called exponential random graphs, which has attracted an attention of the physics community just recently. The class was first studied in the 1980s by Holland and Leinhardt [13], and later has been developed extensively by Strauss and others [14–17]. The idea diffused from social statistics communities to physical society in recent years, when a number of physicists have made theoretical studies of specific models belonging to this family [9,18–21].

Exponential random graph model is defined to be not a single network but a set of possible networks (ensemble). The probability of a particular graph  $G$  in this ensemble is proportional to  $e^{-H(G)}$ , where

$$H(G) = \sum_i \theta_i m_i(G), \quad (1)$$

and  $H(G)$  is called graph Hamiltonian,  $m_i$  is a collection of graph observables that reflect relevant constraints on studied graph properties and  $\theta_i$  is the set of fields conjugate to  $m_i$ .

A variety of graph Hamiltonians has been studied so far including simple random graphs [9], a network with

reciprocity [9], the so-called two-star model [21], Strauss's model of a network with clustering [22], and others. Theoretical analysis of exponential random graph models has been developed by a number of authors. In most of cases linear models can be solved exactly in the limit of large system size. For nonlinear Hamiltonians mean-field theory and perturbation theory [19,20,9] have been applied in order to find phase transitions in the network structures.

In this paper we would like to show how a very simple Hamiltonian can lead to surprisingly complicated and eventful phase diagram where wealth of structural phase transitions can not be forecast at first glance. Due to complexity of observed structures our methodology is mostly concentrated on Monte Carlo simulations. A simple mathematical apparatus is also expounded in order to reveal details of the observed phenomena. The mentioned calculations, although not so powerful, allow to understand when and why a particular transition occurs.

## 2 Motivation and model description

The model is defined on network that is composed of  $N$  nodes and  $L$  links, where each node acts as a single scientist or a single scientific group and a link between two nodes means that there exists scientific collaboration between them.

The original purpose of the model was to describe optimal structures of scientific collaboration in the Sixth Framework Programme for Research and Technological Development (FP6), which is the European Union action aiming to stimulate and support scientific activities conducted at national and international level. One of main purposes of the European Commission financing scientific projects in FP6 was to strengthen co-operation between

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project partners [23]. In their proposal, applicants had to show that one of the aims of the planned project is intensification of co-operation between participants. They also had to argue that without such an interaction a goal of the project will be not achieved. The most popular observable which allows to measure effects of the co-operation in social network is *clustering coefficient* introduced by Watts and Strogatz in 1998 [24]. The clustering coefficient  $c_i$  of a single node  $i$  is the proportion of the number of links between the nodes within its neighborhood  $e$  divided by the number of links that could possibly exist in the neighborhood

$$c_i = \frac{2e}{k_i(k_i - 1)}, \quad (2)$$

where  $k_i$  represent degree of the considered node. If  $k_i < 2$  then  $c_i = 0$ . The global clustering coefficient  $C$  is just an average of  $c_i$  over all nodes.

The other obvious purpose of the funded project is to achieve the highest possible productivity. In our model productivity of each scientist  $i$  (or local scientific group) depends on the number of collaborators  $k_i$ . The more collaborators work with a given scientist, the more papers/ideas the scientist can produce. On the other hand, however, a large number of collaborators means the necessity of parallel concentration on different scientific threads which leads to the decrease of productivity. In consequence, productivity  $p_i$  of a single project participant can be modelled by a logistic-like equation

$$p_i = k_i e^{-k_i/h}, \quad (3)$$

where  $h$  is an optimal (desired) number of collaborators (note that the highest productivity occurs for  $k_i = h$ ). Although one could expect that in reality  $h$  should be described by a kind of the Lotka distribution [25], here we concentrate on the simplest case where  $h$  is the same for all elements of the system. Productivity  $P$  of the whole network is just an average over all  $N$  nodes normalized to unity

$$P = \frac{e}{Nh} \sum_i p_i. \quad (4)$$

Hamiltonian of the described model can be written as follows:

$$H(G) = -\theta P(G) - \alpha C(G). \quad (5)$$

Monte Carlo procedure is defined by the following algorithm: we choose randomly two nodes and try to remove (add) existing (non existing) link between them. If the change leads to the decrease of the initial system energy  $E_0$ , i.e.  $\Delta E = E_{m/p} - E_0 < 0$ , where  $E_{m/p}$  is the system energy after link removal/addition, we accept such a replacement. Otherwise, when  $\Delta E \geq 0$ , we accept it with the probability  $e^{-\Delta E}$ , i.e. we apply the typical Metropolis algorithm.

### 3 Analytical considerations

In order to show a large variety of structural transitions observed in networks described by the Hamiltonian (5),

in our Monte Carlo simulations we have decided to fix one parameter  $\alpha = 10^9$ , and check behavior of the system as a function of the second parameter  $\theta$ . At the moment please note, that since one may think of parameters  $\alpha$  and  $\theta$  as external fields coupled to corresponding observables, our choice of the large value of  $\alpha$  makes the considered system less susceptible to random effects. Further in the paper, the assumption of very large  $\alpha$  allows us to estimate the critical value of  $\theta$  from the simple condition  $\Delta E = 0$ . The condition means that the stability of the initial stable structure is no longer preserved and a new network configuration can emerge (in other words we accept changes in the network structure only if  $\Delta E$  drops below zero).

Figure 1 shows possible scenarios of structural transitions in our model. Arrows represent directions of changes of the control parameter  $\theta$  given  $\alpha = 10^9$ . As one can see the number of possible transition paths is impressive. Later we show that the path chosen by the system depends mainly on the network size  $N$ . Moreover, beside simple paths like *ABCA* or *ABEFGHA* there may exist much more complicated paths like *ABEFGHIDFGHA*, i.e. we have to change parameter  $\theta$  from  $+\infty$  to  $-\infty$  and backward from  $-\infty$  to  $+\infty$  two or more times to return to the same structure we started!

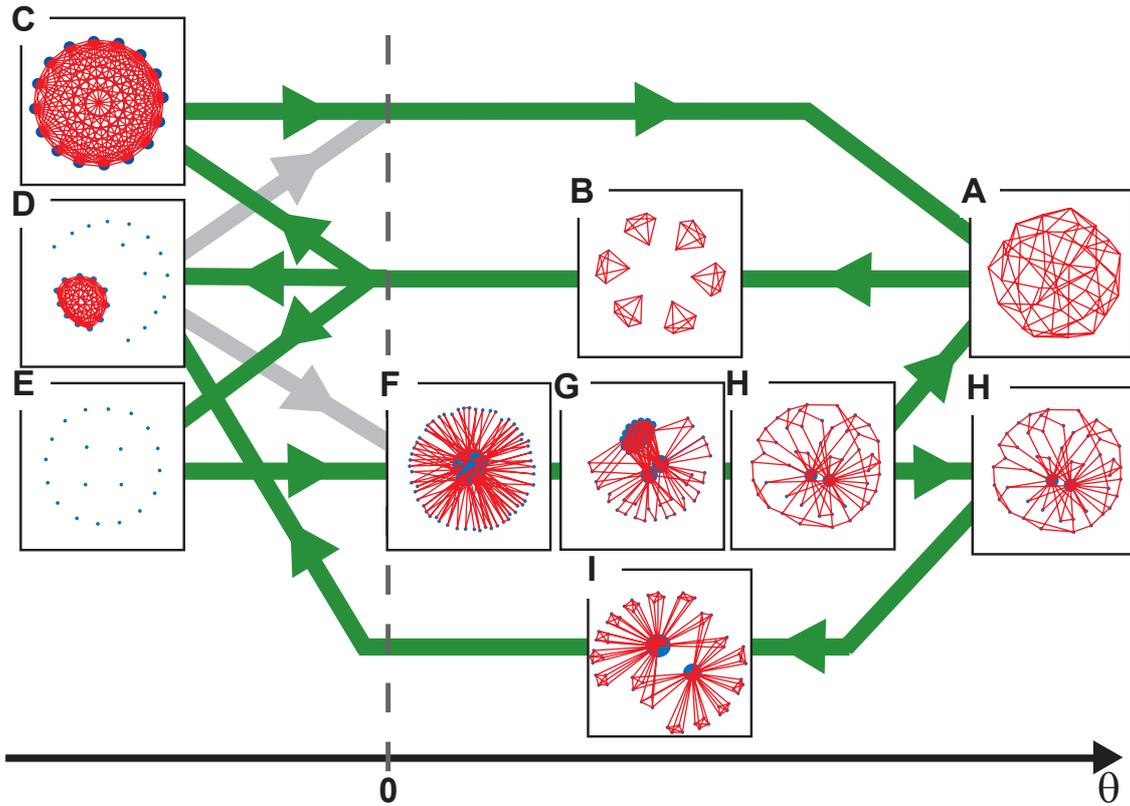
To find critical value of the parameter  $\theta$  at which a particular structural transition occurs one has to analyze a change in energy  $\Delta E$  induced by a link addition or removal taking into consideration a currently existing network structure. As an example let us analyze transition  $A - B$ . The structure  $A$  corresponds to a regular random graph, where a node degree distribution is given by the delta function  $P(k) = \delta(k - h)$ . In this structure productivity is maximal  $P = 1$ , whereas the number of triangles contributing to the clustering coefficient is negligible small  $C \simeq 0$  (assuming that graph is sparse i.e.  $h \ll N$  that is sociologically justified). The transition takes place when for a particular value of the parameter  $\theta$  energetically favorable is to add a link which creates the first triangle (i.e. a decrease of productivity is sufficiently rewarded with an increase of clustering). The described situation is schematically presented in Figure 2a. Energies corresponding to both structures depicted in the figure are respectively given by

$$\begin{aligned} E_0 &= -\theta \cdot 1 - \alpha \cdot 0, \\ E_p &= -\theta \left[ (N-2)he^{-1} + 2(h+1)e^{-\frac{h+1}{h}} \right] \frac{e}{Nh} \\ &\quad - \alpha \left[ \frac{4}{Nh(h+1)} + \frac{2}{Nh(h-1)} \right]. \end{aligned} \quad (6)$$

First, inserting the values of energy into the condition  $\Delta E = E_p - E_0 = 0$ , and next expanding exponential functions in Taylor series up to the second order one gets the critical value of the control parameter for the considered transition  $A - B$

$$\theta_{A-B} \stackrel{h^2 \gg 1}{\approx} \frac{2(3h-1)\alpha}{h-1}. \quad (7)$$

As one can see the transition does not depend on the system size  $N$  and for large  $h$  it also does not depend on  $h$ .



**Fig. 1.** (Color online) Diagram representing possible transitions between network configurations as a function of parameter  $\theta$ . Gray arrows means that the configuration  $D$  is composed of two other configurations ( $C$  and  $E$ ), and the two parts of the system follow different paths. The figure shows networks with different sizes  $N$  just to emphasize particular character of a given configuration.

Much more complicated system behavior is observed when the control parameter  $\theta$  crosses zero and becomes negative (see Fig. 1). Productivity contribution to energy changes sign and all nodes having at the moment degree  $k = h$  turn out to be in unstable configuration (see schematic explanation given in Figure 3). For such nodes when decrease in clustering is sufficiently rewarded by decrease in undesirable productivity the stable configuration  $B$  will be destroyed: some nodes will decrease their degrees whereas others will increase them (cf. Fig. 2b). As one can see in Figure 1 the considered network may follow one of three paths resulting in one of three configurations  $C$ ,  $D$ , or  $E$ .

Unfortunately, due to probabilistic character of the Monte Carlo procedure it is hard to calculate analytically which direction of changes will be taken by the system of a given size  $N$ . To check what is really happening during the transition we perform numerical simulations, results of which are summarized in Figure 4. The figure shows a fraction of isolated nodes as a function of  $N$ . As one can see for small system sizes degrees of all nodes drop to zero and the system transforms into the empty graph (configuration  $E$  in Figure 1). Above some critical value of  $N$  a part of nodes condensate together and a fully connected subgraph accompanied by isolated nodes appears (configuration  $D$  in Figure 1). Finally in the thermodynamical

limit all nodes condensate and the complete graph emerges (configuration  $C$  in Fig. 1).

Now, let us analyze the transition  $C - A$ , i.e. transition from the complete graph to the regular random graph. Because one can not add a link to the complete graph the only situation to analyze is removal of a link. The described situation is schematically presented in Figure 2c. A simple calculation gives

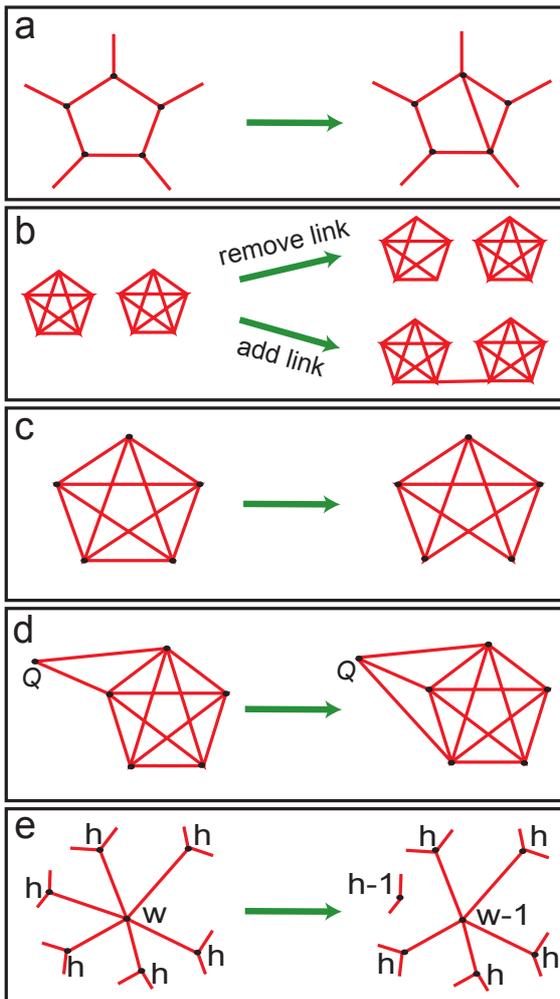
$$\begin{aligned} E_0 &= -\theta N(N-1)e^{-\frac{N-1}{h}} \frac{e}{Nh} - \alpha, \\ E_m &= -\theta \left[ (N-2)(N-1)e^{-\frac{N-1}{h}} + 2(N-2)e^{-\frac{N-2}{h}} \right] \frac{e}{Nh} \\ &\quad - \alpha \left[ 1 - \frac{2}{N(N-1)} \right], \end{aligned} \quad (8)$$

which leads to

$$\theta_{C-A} \approx \frac{e^{N/h} h^2 \alpha}{N^2} \quad (9)$$

in the limit of large  $N$ . As one can see this transition depends on the system size  $N$ , and in the thermodynamical limit  $N \rightarrow \infty$  the critical value  $\theta_{C-A}$  tends to infinity.

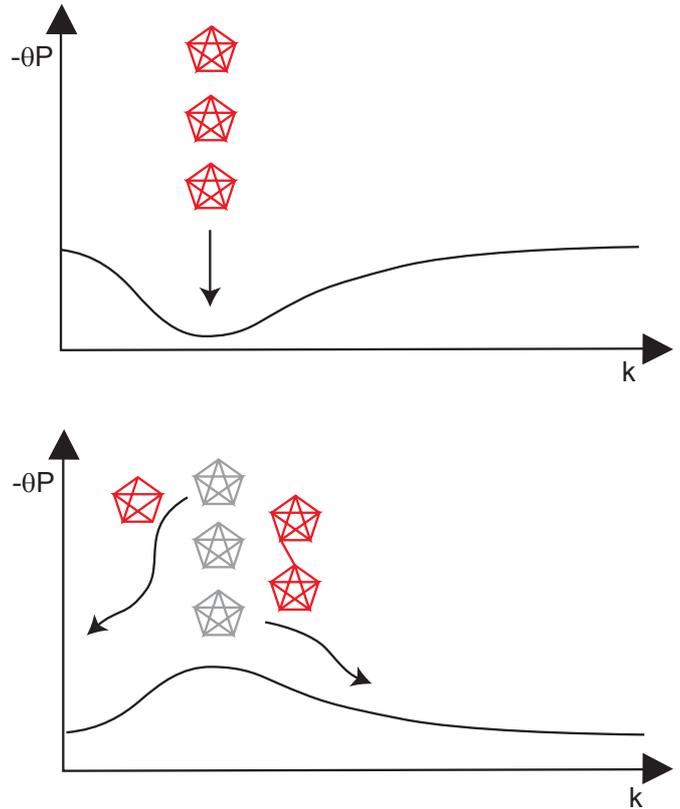
Now, let us discuss the behavior of the system if the initial configuration is  $E$  (i.e. the empty network). The first transition  $E - F$  occurs when the parameter  $\theta$  equals zero. Since the productivity does not influence energy of the system at this point, links can appear randomly (they do not



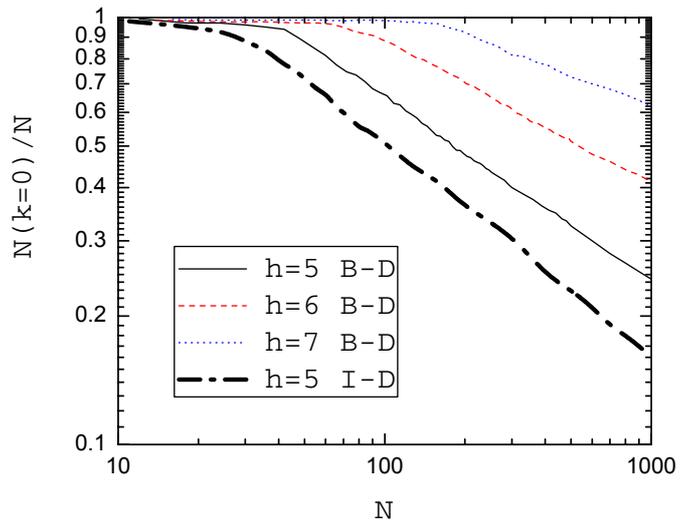
**Fig. 2.** (Color online) Schematic situations occurring during particular transitions used for considerations performed in the text. a) transition  $A - B$ ; b) transition  $B - D$ ; c) transition  $C - A$ ; d) transition  $E - F$ ; e) transition  $H - A$ .

increase energy so they are acceptable in the Monte Carlo procedure). First triangles appear in the network and the clustering coefficient increases. Such a dynamics leads to the configuration in which the fully connected subgraph is surrounded by a number of peripheral nodes with degree  $k = 2$  (configuration  $F$  in Fig. 1). Because of complicated situations in intermediate time steps a rigorous analytical explanation of the transition  $E - F$  is beyond our abilities. Nevertheless, below we analyze a simplified situation, which allows one to understand why the transition occurs.

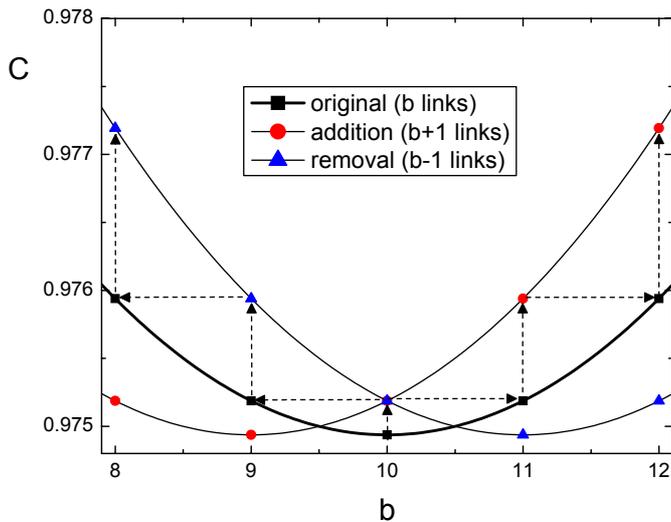
Thus, let us consider a fully connected subgraph with an additional node  $Q$  having  $b$  links (see Fig. 2d). Just like before, one can analyze what happens with the clustering coefficient if we add (remove) one link. Figure 5 shows the solution of this problem when the size of the fully connected subgraph is  $N_F = 20$ . For a given  $b$  clustering coefficient of the considered structure is marked by the thick line. Thin lines show a new clustering coefficient after addition (circles) or removal (triangles) of one link.



**Fig. 3.** (Color online) Productivity contribution to energy for  $\theta > 0$  (upper figure) and for  $\theta < 0$  (lower figure) as a function of node degree  $k$ . The later situation shows that to decrease energy some nodes will reduce their degree (destroying existing clusters) and others will increase it (connecting two clusters together).



**Fig. 4.** (Color online) Normalized number of nodes with degree  $k = 0$  as a function of system size  $N$ . Three thin curves represent a transition  $B - D$ , while a thick one represents a transition  $I - D$ . The thick curve shows that for the same value of the parameter  $h$  the number of isolated nodes is lower after transition  $I - D$  than after transition  $B - D$ .

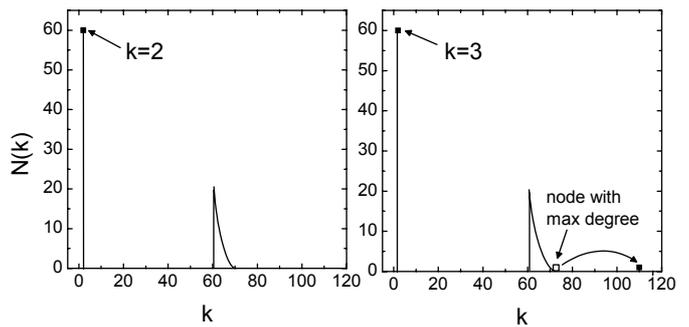


**Fig. 5.** (Color online) Clustering coefficient  $C$  for the system depicted in Figure 2d. Arrows show evolution of the system (addition or removal of successive links).

At the beginning, let us assume that we have  $b < 10$ . To increase clustering coefficient we have to remove one link which leads to the configuration with  $b - 1$  links attached to the peripheral node  $Q$ . Further, it is easy to see that the process will follow towards removing next links belonging to  $Q$  until the node will have only two links. The node degree can not drop below  $k_Q = 2$ , because for  $k_Q < 2$  local clustering coefficient  $c_Q$  suddenly drops from 1 to 0 which drastically decreases global clustering coefficient of the whole structure. On the other hand, if we assume that  $b > 10$ , then energetically favorable is to add another  $b + 1$  link to the node  $Q$ . Again, one can see that the node  $Q$  will try to connect to all other nodes, i.e. the node  $Q$  becomes a member of the fully connected subgraph.

Starting from the configuration  $F$  if one further increases  $\theta$  the productivity  $P$  starts to matter. It means that above some critical value of this parameter an addition of the third link to one out of peripheral nodes with degree  $k = 2$  can compensate energy increase coming from decrease of the clustering coefficient. The same reasoning explains successive network reorganizations when adding next links (up to  $k = h$ ) to peripheral nodes is energetically favorable.

At the moment, let us note that nodes belonging to the fully connected subgraph in the configuration  $F$  have different degrees. Their degrees are composed of  $N_F - 1$  internal links and links coming from peripheral nodes with degree  $k = 2$  (the peripheral links are randomly distributed among the nodes creating the fully connected subgraph). It means that in a finite system there always exists a node (or a group of nodes) with the largest degree. It is easy to check that addition of a new link to such a node leads to the minimal decrease of the clustering coefficient. It means that above a certain value of  $\theta$  new links are added to this node making its degree rapidly growing. Figure 6 shows in a schematic way the node degree distribution below and at the critical value of  $\theta$  for the transition.



**Fig. 6.** Schematic node degree distribution before and after transition  $F - G$ . The node with the highest degree is exaggerated to show its dynamics during the transition.

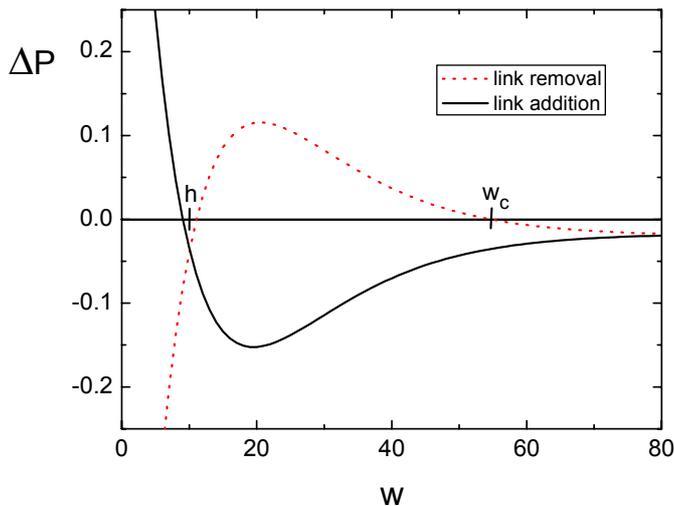
As we have described above, if one further increases the control parameter  $\theta$  the peripheral nodes suddenly increase their degrees from  $k = 3$  to  $k = 4$ . It generates a similar mechanism as described in the previous paragraph, i.e. consecutive node with the largest degree (except the one which has already increased its degree to  $N$ ) stepwise increases its connectivity. The network configuration arising along the transition corresponds to the configuration  $G$  presented in Figure 1.

Another crucial point in the system evolution is the transition  $G - H$ , where the fully connected subgraph is destroyed in the similar manner as the complete graph in the transition  $C - A$ . After the transition  $G - H$  our network consists of several hubs and a large number of loosely connected peripheral nodes.

The configuration  $H$  is presented twice in Figure 1 in order to show two possibilities of the system evolution: it can be stable when  $\theta \rightarrow \infty$ , or hubs can be destroyed, and the transition  $H - A$  takes place. To analyze stability of the configuration  $H$  is enough to consider a simplified structure presented in Figure 2e (note that the simplified star-like structure neglects effects of clustering which may occur in the original configuration  $H$ ). In such a structure, the change of a local productivity resulting from addition ( $\Delta P_{local} = P_p - P_0$ ) or removal ( $\Delta P_{local} = P_m - P_0$ ) of a single link between the hub with a large degree  $w$  and one of peripheral nodes with degree  $h$  (as we have already stated the maximal connectivity of the peripheral nodes along the transition path  $F - G - H$  is  $k = h$ ), where

$$\begin{aligned} P_0 &= \left[ he^{-1} + we^{-\frac{w}{h}} \right] \frac{e}{Nh}, \\ P_m &= \left[ (h-1)e^{-\frac{h-1}{h}} + (w-1)e^{-\frac{w-1}{h}} \right] \frac{e}{Nh}, \\ P_p &= \left[ (h+1)e^{-\frac{h+1}{h}} + (w+1)e^{-\frac{w+1}{h}} \right] \frac{e}{Nh}, \end{aligned} \quad (10)$$

is presented in Figure 7. A given process (link addition or removal) occurs only if the change of productivity is positive. It means that if  $w$  is small the only possibility is to add a new links. When  $h < w < w_c$  (c.f. Fig. 7) links can only be removed. It means that degree of the hub should decrease from  $w$  to  $h$ , and the transition  $H - A$  takes place. On the other hand, when  $w > w_c$ , both processes (addition and removal of links) are no longer possible. It means that the system remains in the stable configuration



**Fig. 7.** Change of productivity in the star-like structure depicted in Figure 2e as a function of degree of the central highly connected node  $w$ , for  $h = 10$ .

$H$ . Unfortunately, because the analysis neglects clustering it does not allow us to calculate the precise critical value of  $\theta_{H-A}$ .

Finally, let us analyze the transition  $H - I$  (Fig. 1), which occurs when starting from the stable configuration  $H$  one decreases the control parameter  $\theta$ . On the basis of our previous considerations one can predict that the configuration  $I$  emerges when the contribution of clustering coefficient to the system energy crosses a critical value. In a similar way like during the transition  $A - B$  nodes with low degrees have tendency to form triangles. On the other hand, however, because hubs are stable (we have shown it in the previous paragraph) instead of many separated clusters like in the configuration  $B$  the system evolves towards the stable configuration  $I$ .

If the parameter  $\theta$  is sufficiently negative the configuration  $I$  is destroyed just like the configuration  $B$ , and the mechanism of this transition is the same: which configuration ( $C$ ,  $D$ , or  $E$ ) appears depends on the system size  $N$  (c.f. thick dotted curve in Fig. 4). The only difference is that for a given  $N$  the number of isolated nodes is lower in comparison with the transition  $B - D$ .

As we have stated before, in fact, our system in the phase  $D$  consists of two configurations  $C$  and  $E$ . It means that at least to some value of the parameter  $\theta$  the part of the system that is equivalent to configuration  $E$  follows a path  $D - F - G - H$ , and the second part, equivalent to the configuration  $C$ , follows the path  $C - A$ . At some point, up to now separated parts of the network combine together. In figure 1, in order to make the whole picture as clear as possible, we marked paths accessible for these two components of the configuration  $D$  by gray arrows.

Let us also notice that a path our system follows can be really complicated. For example, let us consider a network in the configuration  $A$ . After transitions  $A - B - D$  the number of isolated nodes is high, which allows to create hubs with very high degrees  $w > w_c$  as a result of the transition  $F - G$ . It means that the configuration  $H$  is

stable, and the system follows the path  $H - I - D$ . Now, the number of isolated nodes is much lower (see Fig. 4). Their number is often too small to once more time create hubs with degrees  $w > w_c$  during the transition  $F - G$ . Thus, after a series of transitions  $D - F - G - H$  the system returns to the initial state  $A$ . It means, that the return to the same state is possible after a complicated path  $A - B - D - F - G - H - I - D - F - G - H - A$ , in which the chain of transitions  $D - F - G - H - I$  may be repeated several times.

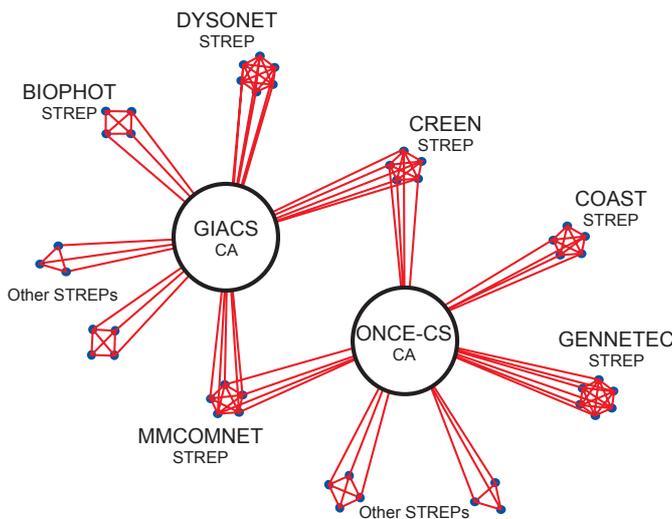
Finally, we have to stress that a part of phase transitions we observed are visible only in finite size systems, i.e. if the system is large enough a particular phase transition can change its character and can lead to development of different structure. Although usually physicists use term *phase transition* in the context of systems in thermodynamical limit, our delinquency can be justified because the model we study has been proposed to social systems where the number of elements is always limited.

## 4 Discussion

We presented a simple model that is characterized by a surprisingly complicated and eventful phase diagram with plenty of metastable states. Nevertheless, since the model was sociologically motivated, let us discuss the observed network configurations in the context of scientific collaboration.

First, the configuration  $B$  seems to be the easiest to interpret. If each node represents scientific group then we see here separated projects consisting of several scientific groups where every group collaborates with each other. The realism of the situation can be questionable since real projects can be composed of different number of participants, but let us remind that in the model we have assumed that the optimal number of collaborators  $h$  is fixed for all groups. To make the model more realistic we should draw the parameter from the Lotka-like distribution [25], but it would certainly complicate obtained results and at the moment we were much more interested in the description of the observed structural phase transitions.

The second configuration which seems to reflect real-world observations is the configuration  $I$ . Figure 8 shows the real (although simplified) case of EU projects being currently in progress. Let us explain that complex systems research in Europe is funded through two European Commission actions: NEST (New and Emerging Science and Technology, and FET-IST (Future and Emerging Technology Information Society Technology) [23]. It is mainly being done through small projects called STREPS (Strategic Targeted Research Projects). Apart from them the European Commission is currently funding two Coordination Actions to support complex systems science: ONCE-CS (funded by IST-FET) and GIACS (funded by NEST). In Figure 8 they are presented as main nodes which serve as knowledge transfer units between particular projects. Part of projects funded mainly by NEST collaborate with GIACS, and projects funded mainly by IST-FET are supported by ONCE-CS. Of course there



**Fig. 8.** (Color online) Schematic view on network of EU projects. Two highly connected coordination actions are surrounded by a plenty of small research projects.

are projects which take support from both coordination actions because aims of both CA's are slightly different. Let us stress that the above picture is very simplified. The main simplification is that CA's are represented by single nodes in Figure 8, which is evidently not true - coordination actions are projects consisting of many participants as well as STREPs.

A careful reader can ask what could be the interpretation of negative value of the control parameter  $\theta$ . In fact, it corresponds to the situation where groups composed of small number of participants are undesirable. The sociological explanation (although not connected with scientific collaboration) can be expressed as follows: *or you commune with one global social group or you will be separated*, which can be recognized as the fascist ideology.

Finally, let us discuss the situation when the parameter  $\alpha$  is varied while  $\theta$  is fixed. In the case, most of configurations described above are not observed. For example, for  $\theta > 0$  one can observe only one phase transition (with hysteresis), namely the transition between configuration *A* and *B*. For  $\theta < 0$  the situation is even more trivial: configuration *C* occurs for all positive  $\alpha$ , while for negative  $\alpha$  a random graph without triangles appears. Moreover, because the explanation of negative clustering coefficient is much more difficult than negative productivity, the analysis of fixed  $\alpha$  and varied  $\theta$  seems to be easier and much more reasonable.

## 5 Conclusions

In this paper we have presented a model of social collaboration. Although the model is expressed by a simple Hamiltonian the richness of observed structural phase transitions is impressive. Most of them we can only analyze qualitatively and further studies have to be performed

to clarify reasons for which a given structure appears. We uncover many aspects of the studied model but in fact much more questions arise during our investigations.

Although simplifications of the model do not allow to render in detail the real-world space of scientific projects, we have shown that some configurations formed in the system remind existing structures of European projects.

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