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Risk evaluation with enhanced covariance matrix

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Abstract

We propose a route for the evaluation of risk based on a transformation of the covariance matrix. The approach uses a 'potential' or 'objective' function. This allows us to rescale data from different assets (or sources) such that each data set then has similar statistical properties in terms of their probability distributions. The method is tested using historical data from both the New York and Warsaw stock exchanges.

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1. Introduction

Optimization of portfolios has been much studied since the pioneering work of Markowitz [1,2], who proposed using the mean–variance as a route to portfolio optimization [1–16]. However, the basic construction of the portfolio has not changed much as a result. Computation of Sharp ratios [17,18] and the Markowitz analysis equate risk with the co-variance matrix. Portfolio allocations are then computed by maximizing a suitably constructed utility function [19–21]. Moreover, the approach taken by Markowitz and many other authors [1,2] is essentially only appropriate for stochastic processes that follow random walks and exhibit Gaussian distributions [3–5]. Many economists have sought to use other utility functions and invoke additional objectives [22,23] in which portfolio weights are computed via maximization of these different utility functionals. Others have introduced additional features of the probability distribution such as the third moment or skewness of the returns [22,23]. This builds in aspects of the deviation of the probability distribution from the Gaussian as well as the asymmetry. Introducing even a constant value for the skewness may yield more reliable portfolio weights than a calculation in which only the variance or second moment of the distribution is used and where the risk of extreme values is seriously underestimated. Similar comments could be made about the introduction of the kurtosis which is a first order route to addressing the issue of 'fat' tails.

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In recent years, a number of physicists have begun to study the effect of correlations on financial risk. Techniques based on random matrix theory developed and used in nuclear physics have been applied to reveal the linear dependencies between stock market data for both the US and UK markets [4,5]. More recently other workers including one of the present authors have used minimum spanning trees methods [24–26] for the same purpose. Spanning tree methods seem to yield results that are similar to those obtained using random matrix theory but with less effort and the use of less data in the sense that only a subset of the correlation matrix is actually used to construct the tree. The overall aim, in both cases, is to arrive at optimal diversified portfolios. One interesting result obtained in Ref. [25] was the identification of new classifications introduced in the FTSE index ahead of their formal introduction by the London authorities.

An important outcome of studies basing on Markowitz approach is the capital asset pricing model (CAPM) [10,27–29] that relates risk to correlations within the market portfolio [10,27–29]; of course the risk now is that all investments will collapse simultaneously. Furthermore, it is assumed that risk that achieves premiums in the long term should not be reducible otherwise arbitrage is possible [28]. This is essentially the arbitrage pricing theory (APT).

However, key issues remain unresolved. One weakness of CAPM and APT theories is that they assume efficiency in the proliferation of market information. In a real market not all investors have the same or complete information and arbitrage is possible. Merton [30] has discussed this and in so doing has extended CAPM theory to deal more effectively with small firms for which information is not always readily available.

Here we concern ourselves with a new approach to the exploitation of data sets for the computation of portfolio weights within a diversified portfolio. The method exploits the full character of the distribution function for each asset in the portfolio and seeks to maximize the impact of correlations. In the next section, we discuss the background to our approach and introduce the so-called *objective* function. Having established this we show how, from data, we can construct values for a renormalized objective function. These are then used in Section 3 to obtain both covariance matrices and weights for portfolios of stocks. The calculations are illustrated in Section 4 by examples from both the US and Warsaw stock exchanges. We also show how the approach modifies the underlying distribution of eigenvalues enhancing the correlations for larger values.

2. Objective function

Consider an asset, characterized by a price, $S(t)$ and return $x(t) = \ln S(t+1)/S(t)$. The objective function, $w(x)$ is defined in terms of the stationary probability distribution for returns, $P(x)$, viz:

$$P(x) = \frac{1}{Z} \mathrm{e}^{-w(x)/D}, \tag{1}$$

where Z is a normalization factor. Such functions are familiar to physicists and may be derived by minimizing a 'free energy' functional, $F(w(x))$, subject to constraints on the mean value of the objective function, viz:

$$F = \int_R \mathrm{d}x P(x) \left[\ln P(x) + \frac{w(x)}{D} - \lambda \right]. \tag{2}$$

Such a form for the probability distribution is also the outcome of a model that assumes x is governed by a generalized Markovian stochastic process of the form:

$$\dot{x}(t) = f(x) + g(x)\varepsilon(t). \tag{3}$$

The Gaussian process, ε, satisfies:

$$\begin{aligned} \langle \varepsilon(t)\varepsilon(t') \rangle &= D\delta(t-t'), \\ \langle \varepsilon(t) \rangle &= 0. \end{aligned} \tag{4}$$

For the moment we leave the form of the functions f and g unspecified except to say that they only depend on $x(t)$. The solution to such a stochastic process has been deduced elsewhere [31–33]. Adopting the Ito convention, the distribution function, $P(x,t)$, associated with the process is given by the Fokker

Table 1
Examples of objective values $w(x)$ and corresponding probability distributions, P for different choices of f and g

$f(x)$	$g(x)$	$w(x)/D$	$P(x) \cdot Z$
$-sgn(x)$	1	$\lvert x\rvert/D$	$e^{-\lvert x\rvert/D}$
$-x$	1	x^2/D	$e^{-x^2/D}$
$\lambda g g'$	$g(x) \neq const$	$2(1-\lambda/D)\ln g$	$\frac{1}{g^{2(1-\lambda/D)}}$
$\frac{2x}{v}(1+x^2/v)$	$1+x^2/v$	$(v+1)/2\ln(1+x^2/v)$	$\frac{1}{(1+x^2/v)^{(v+1)/2}}$

Planck equation:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\frac{D}{2} g^2(x) P(x,t)\right) - \frac{\partial}{\partial x}(f(x)P(x,t)). \tag{5}$$

The stationary solution is

$$P(x) = \frac{e^{\int dx(2f/(Dg^2))}}{Z \cdot g^2(x)} = \frac{1}{Z}\exp\left(-\frac{2}{D}\int dx \frac{Dgg' - f}{g^2}\right), \tag{6}$$

where Z is a normalization factor.

A number of different cases are evident as expressed in Table 1. Row 4 is obtained from row 3 by choosing $g = (1 + x^2/v)$, $f = gg'$ and $(v+1)/2 = 2(1-\lambda/D)$ when we see that the distribution function reduces to a Student distribution. For $D>0$ we see that $v<3$. On the other hand, we know that Student distribution is defined for $v>2$ in order the variance to be finite. Nevertheless this limitation, that stochastic process cannot be defined for $v>3$, we can normalize the distribution function, because $w(x)$ is well defined for whole spectrum of $v>2$ using Eq. (1). In developing our methodology in the next sections we shall focus on the use of the Student distribution that seems to offer a good fit to the data we consider. Tsallis and Anteneodo [34] have shown how similar multiplicative stochastic processes based on other nonanalytic choices for the function f and g can lead to q-exponentials. More recently Queiros, Anteneodo and Tsallis [35] have shown that for many financial processes where fat tailed probability functions are empirically observed these Student or Tsallis distributions are good choices.

3. Portfolio optimization

As we have noted above it is usual for a portfolio of M stocks to compute portfolio weights, p_i using the covariance matrix, $\mathbf{C}$ and defining the risk, R, as

$$R = \sum_{i,j} \mathbf{C}_{i,j} p_i p_j. \tag{7}$$

Optimizing this in the absence of risk free assets yields the weight of stock i:

$$p_i = \frac{1}{Z}\sum_j (\mathbf{C}^{-1})_{i,j}, \tag{8}$$

where $Z = \sum_{i,j}(\mathbf{C}^{-1})_{i,j}$.

It is known that a nonlinear transformation of data can change correlations e.g. correlations of $|x_i|$ decrease much slower than x_i [5]. We exploit this by introducing a particular transformation that increases correlations by renormalizing the objective values such that the total set of values, $x_i(t_j)$ for all i from 1 to M and j from 1 to N are drawn from a common distribution. To effect this change, we first compute for each asset the probability distribution by fitting the data for each asset using a Student distribution characterized by the power-law index. We then compute for each value of the return $x_i(t_j)$ the corresponding objective value,

$w_i(x_{t_j})$. These objective values are then transformed to yield a set of renormalized objective values as follows:

$$\tilde{w}_i(x_{t_j}) = w_i(x_{t_j})\frac{\hat{w}}{\bar{w}_i} = w_i(x_{t_j})\frac{(1/MN)\sum_{i,j}^{M,N} w_i(x_{t_j})}{(1/N)\sum_j^N w_i(x_{t_j})}. \tag{9}$$

In effect we are renormalizing the objective value with its mean value $\bar{w}_i$ relative to the overall mean value, $\hat{w}$, of the *entire* data set. Having computed these renormalized objective values we can now obtain the corresponding set of values for $\tilde{x}_i(t)$ by inverting the values according to a new Student distribution that characterizes the *entire* data set consisting of one value of v and $M \times N$ values. Hence using the result in row 4 of Table 1:

$$\tilde{x}_i(t_j) = \pm\sqrt{v(1 - \mathrm{e}^{2\tilde{w}_i(x_{t_j})/(v+1)})}, \tag{10}$$

where v is now the tail exponent that characterizes the pdf of the *entire* data set.

Thus we can compute for our portfolio of M stocks a new covariance matrix, $\tilde{\mathbf{C}}$ using these renormalized values of $\tilde{x}_i(t_j)$. This yields a new minimized value for the risk:

$$\tilde{R} = \sum_{k,i=1}^{M} \tilde{\mathbf{C}}_{k,i}\tilde{p}_k\tilde{p}_i. \tag{11}$$

4. Illustrative results and conclusions

We show in Figs. 1 and 2 the outcome of implementing the method for a simple portfolio of two stocks (i.e., $M = 2$). Specifically, we used data for NYSE stocks General Electric and Boeing. For each stock we used 12 500 data points extending over the time period January 1999 to December 2000. Student distributions are fitted separately to the positive and negative returns. It can be seen that the Student distributions for each stock are different prior to renormalization but are the same after renormalization. The overall changes as a result of our renormalization process are small but we show in Fig. 5 that they can lead to significant changes in the distribution of eigenvalues for large eigenvalues.

We followed up this computation by renormalizing data for two different groups of stocks. First we selected 60 stocks from the NYSE as before over the period January 1999 to December 2000 and implemented the prescription over a moving 75 day window using 1500 points for each window, what corresponds to quarter of hour returns. In this way, we could compute the various elements of the correlation matrix and the associated optimum weights for the different stocks in the portfolio as a function of time. The results are shown in Fig. 3.

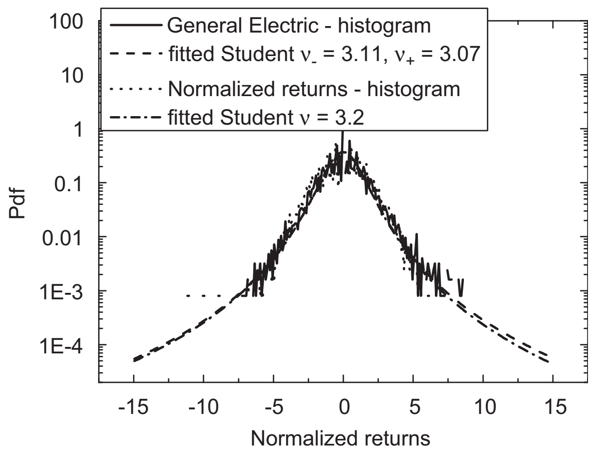

Fig. 1. Plot of the histogram of returns and normalized returns in the case of General Electric counted in NYSE in years 1999 and 2000 and corresponding Student distributions with $v_+ = 3.11$, $v_- = 3.07$ and $v = 3.2$, respectively.

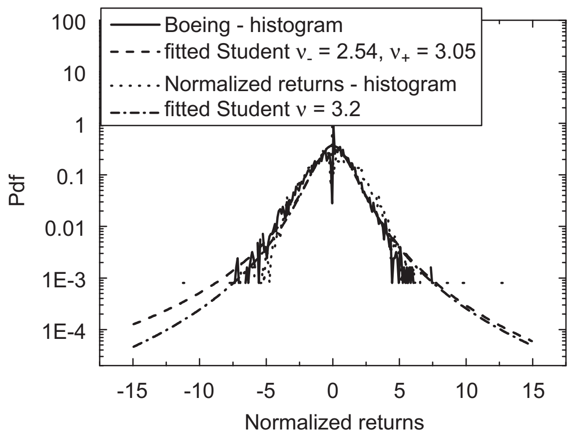

Fig. 2. Plot of the histogram of returns and normalized returns in the case of Boeing counted in NYSE in years 1999 and 2000 and corresponding Student distributions with $v_+ = 2.54$, $v_- = 3.05$ and $v = 3.2$, respectively.

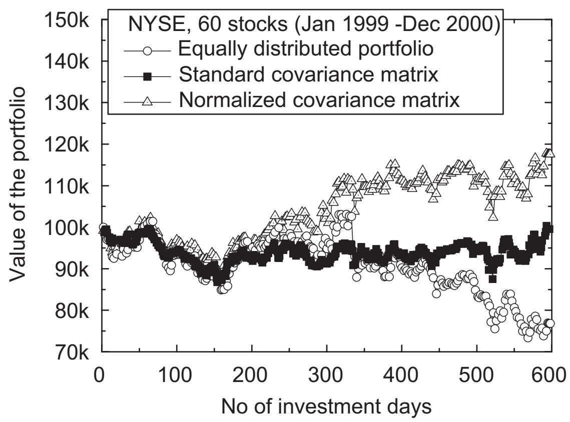

Fig. 3. Portfolios runaway of 60 stocks at New York Stock Exchange from May 1999 to December 2000. Equally distributed portfolio (open circles) and portfolio with weights calculated from standard covariance matrix equation (8) (solid squares) and portfolio with weights calculated from normalized covariance matrix are presented.

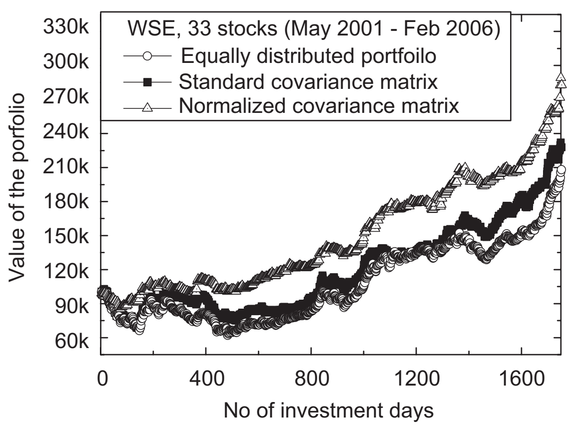

Fig. 4. Portfolios runaway of 33 stocks at Warsaw Stock Exchange from May 2001 to February 2006. Equally distributed portfolio (open circles) and portfolio with weights calculated from standard covariance matrix equation (8) (solid squares) and portfolio with weights calculated from normalized covariance matrix are presented.

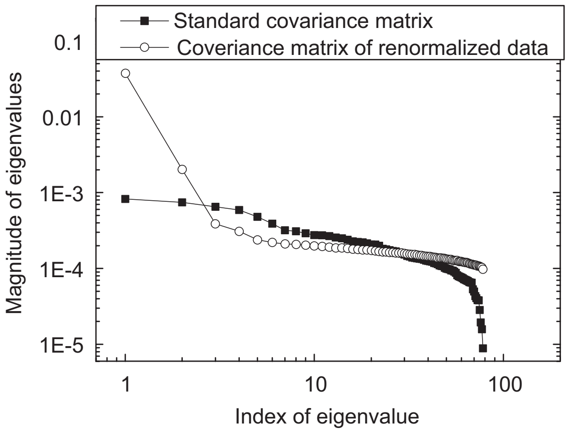

Fig. 5. Distribution of eigenvalues of covariance matrices of 78 stocks in NYSE (January 1999–December 2000). Eigenvalues of standard covariance matrix (solid squares) and of covariance matrix from renormalized data (open circles) are presented in the graph.

Fig. 4 gives the results of a similar set of calculation for a portfolio of 33 stocks from the Warsaw stock exchanges over the period May 2001 to February 2006. In order to prevent situations arising where all the money is invested in just one stock we have, in our calculations, imposed the limit $p_i < 0.15$. We have checked that a precise value of this limit is not crucial for optimization procedure.

Although we have not included transaction costs (we have changed our portfolio every day, usually by a very small amount), in both cases it does seem that using data based on our renormalization procedure we have a route to greater overall returns.

Additional insight into the procedure is provided when we compare the distribution of eigenvalues for the standard covariance matrix with the corresponding distribution for the renormalized covariance matrix. These are shown in Fig. 5. It can be seen that the transformation procedure enhances correlations as anticipated and this enhancement occurs at larger eigenvalues. One could ask why the procedure we have used reduces the risk associated with the portfolio? This is because having evaluated the risk connected to each stock then we have a better estimation of the weights in the portfolio. We claim that correlations calculated with standard method underestimate the linear dependencies between stocks, so the error of the corresponding portfolio risk is much higher. Further, we claim, that we reduce the error related to risk evaluation, so risk as a whole is smaller.

It might also be argued at this point that we could by-pass the entire background given in Section 2 and simply fit the 'best' distribution function to the data as was done, for example, by Levy and Duchin [16]. Using this approach they obtained different distributions for different stocks then also obtained different distributions for the same stock at different times. To our mind this is not a very satisfactory approach and ignores the evidence from groups led by physicists such as Stanley [5,24] that financial data exhibits universal behavior such as scaling, power-law tails, etc.

Of course, an empiricist could still insist that our approach does not yield the best fit and other choices for example for the entropy might improve our results. To answer this question requires a more extensive study that we have presented here.

The covariance matrix is now widely used for the analysis of portfolios. Our approach to the exploitation of this matrix that yields new and correct linear dependencies clearly has wide application and will, we believe, prove to be of considerable benefit to industry practitioners.

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