Analytical approach to the model of scientific revolutions

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The model of scientific paradigms spreading throughout a community of agents with memory is analyzed using the master equation. The case of two competing ideas is considered for various networks of interactions, including agents placed at Erdős-Rényi graphs or complete graphs. The pace of adopting a new idea by the community is analyzed, along with the distribution of periods after which a new idea replaces the old one. The approach is extended for the chain topology to the more general case when more than two ideas compete. Our analytical results agree with the numerical simulations.

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I. INTRODUCTION

There is a tendency to separate certain periods in the history of civilizations, such as the Renaissance or Enlightenment, which qualitatively differ from each other by dominating trends in science, art, or customs. Technological innovations and scientific discoveries constantly emerge, and therefore some kind of equilibrium, i.e., "end of history" [1], will unlikely be reached. Changes (evolutionary and revolutionary ones) occur because of the interactions and exchange of innovative ideas [2–5] at the level of individuals, communities, or even civilizations. Eventually, ideas spread throughout the communities [6,7]. Some of the ideas gain broad (even global) acceptance and popularity, replacing old ones [8]. Similar phenomena can be observed in the models of opinion formation dynamics [5,9–12].

The process of adoption of an innovative technology [13] or a new scientific concept by individuals and communities differs from the adoption of, e.g., a new trend in the arts. The obsolete technologies and discarded scientific theories, once abandoned, are not likely to be accepted by individuals again. To model such a process, agents should be given some kind of memory. Another important fact is that the will of individuals to adopt a new scientific concept depends on its global popularity. For example, the spreading of technological innovations is usually slowed down by incompatibility with existing standards. In the field of art, the situation is different. Old ideas can reemerge and become popular again. For example, Renaissance artists were inspired by antique philosophy or architecture.

Recently, a model has been introduced by Bornholdt *et al.* that attempts to describe scientific revolutions [14]. The model combines interactions at the level of individuals with the influence of the whole community. Despite its simplicity, it manages to reconstruct some key features of the dynamics of scientific paradigms spreading, including an asymmetry between the rate of adopting a new idea by the community and the speed of its decline when new competing ideas emerge. The model was based on numerical simulations, and no analytical treatment was presented. In this paper, the master equation and Markov process theory [15] were applied to analyze the dynamics of the system in the case of a small level of agents' creativity for various topologies of agent interactions, such as the chain, the complete graph, Erdős-Rényi (ER) graphs, the star graph the square lattice, and the Barabási-Albert (BA) graphs. For the chain topology, the approach was extended in an attempt to describe the system dynamics for higher levels of creativity.

II. THE MODEL OF SPREADING OF IDEAS

The rules of the model [14] are very simple. N agents occupy nodes of a network. Every agent follows some paradigm (idea), labeled by a natural number. In each time step, a random agent i (with paradigm s_i) is selected, along with one of its neighbors j (with paradigm s_j). If the agent i has never followed the paradigm s_j, the agent adopts the paradigm with probability N_{s_j}/N, where N_{s_j} denotes the number of agents representing paradigm s_j. Additionally, new paradigms, which have never been present in the community, can appear. With probability α, a random agent is selected, which changes its paradigm into one that has never been present in the community.

The most important feature of the model is the *memory* of the agents, who do not adopt the same paradigm twice. One can find analogy between this model and evolutionary dynamics models: innovations can be regarded as mutations that allow affected individuals to outperform their rivals. The lack of any evident *fitness* parameter, which describes how well a species has adapted to the environment, is not necessarily a drawback of such an interpretation, as the *fitness* of a species is always known *a posteriori* [16].

As stated, the dynamics of the system is the outcome of the interactions at two levels: the local "contagion" process and the "global pressure." The local interactions are fairly natural. Definitely, the most effective exchange of ideas occurs when people communicate directly with each other. However, the reasons why the global popularity factor was introduced need more clarification.

We define a scientific revolution as a global change in perceiving the world. In terms of our model, a revolution occurs when the majority of the community abandons their idea in favor of a new one. In an idealistic picture, the only criterion for the adoption of a new idea by an agent would be its objective correctness, verified by an experiment. However, in reality, this is rather rare. First, the result of an experiment can usually be explained by several competing theories. Second, an individual is not always competent enough, or simply not willing, to assess the correctness of the theory. On the contrary, we believe that in most cases, we have to draw opinions on the theories that we are unable to verify ourselves. This situation is certainly the case if we consider the population of the whole world. Nevertheless, if we limit our interest to the population of scientists, we may observe similar phenomena. The process of specialization has gone so far that scientists working in different fields are usually unaware of recent developments outside their fields. Last, in the case of the natural sciences, determining whether a theory is correct or not is generally impossible. A theory is regarded as "correct" until it is experimentally discarded. However, the experiments may also be dubious or indecisive. In the humanities, the criteria of correctness of an idea are much less clear, or they do not exist at all. As a result of all these factors, an individual confronted with a new idea would rather check how popular such an idea is instead of try to verify its correctness. As an example supporting our reasoning, the medieval people commonly believed that the earth is flat, although the theory that it is spheric had already existed in antiquity; the circumference of the earth was measured by Eratosthenes in the third century BC.

If, instead of analyzing scientific ideas, we consider technological innovations, similar reasoning will lead us to the analogical conclusions. The knowledge regarding new technologies is transmitted through peer-to-peer interactions. However, it is not always the case that once we learn about a better technology, we immediately switch to it. We have to consider such factors as the compatibility of the new technology with the existing standards, the availability of technical support, or the reliability of the new technology. Clearly, all these criteria favor popular and widespread technologies.

For simplicity, the dependence of the pressure factor on global popularity was assumed to be linear in the model. In reality, such linearity does not have to (and probably does not) occur.

The evolution of the system has some general features, independent of the interactions network topology. For a very small probability α, two paradigms at most coexist (other cases are neglectable because of their much smaller probability). This case will be analyzed for various networks in Sec. III.

For higher values of probability α, other effects have to be considered. In such case, usually more than two paradigms coexist, which "compete" with each other. However, one may suppose that within a relatively wide range of α, two paradigms can still be separated at every moment: the "old" paradigm, which is the most popular but currently at a decline, and the "new" paradigm, which is the second most popular and one that will prevail after some time (and then enter the stage of decline). This case will be analyzed in Sec. IV.

III. THE CASE OF TWO COMPETING PARADIGMS

A. General case

When the creativity level of the agents α is small enough, two paradigms at most coexist, referred to as paradigm 0 (at the stage of decline) and paradigm 1 (at the stage of expansion). The evolution consists of two distinct periods.

(1) All agents share the same paradigm 0. The length of this *stage of stagnation* is a random variable of the exponential distribution,

$$P(T_{\mathrm{stag}}) = \alpha(1-\alpha)^{T_{\mathrm{stag}}}, \tag{1}$$

and the mean value,

$$\langle T_{\mathrm{stag}}\rangle = \sum_{T_{\mathrm{stag}}=0}^{\infty} T_{\mathrm{stag}}\alpha(1-\alpha)^{T_{\mathrm{stag}}} = \frac{1-\alpha}{\alpha} \approx \frac{1}{\alpha}. \tag{2}$$

(2) After an innovative paradigm 1 appears, it starts spreading across the community. The *time of expansion* of paradigm 1 is denoted as T. It is a random variable whose distribution depends on the interaction network topology. After time T, all agents share paradigm 1, and the state of the system is equivalent to the initial one.

In our approach the state of the system is characterized by one variable, the number n of agents sharing paradigm 1. The problem is reduced to the problem of the expansion of paradigm 1 throughout the community, starting from one agent with paradigm 1 at time $t = 0$. The generic master equation has only two terms:

$$\frac{\partial}{\partial t}P(n,t) = P(n-1,t)W_{n,n-1} - P(n,t)W_{n+1,n}, \tag{3}$$

where transition rates from state n to state $n+1$ (for $n \in [1,N-1]$) are equal to

$$W_{n+1,n} \equiv W_n = \frac{n}{N^2}\sum_{i=1}^{N}\frac{1-s_i}{k_i}\sum_{j=1}^{N}a_{ij}s_j, \tag{4}$$

where s_i denotes the state of the ith agent ($s_i = 0$ means the agent follows the old paradigm, and $s_i = 1$ means the agent follows the new paradigm), k_i is the degree of node i, and a_{ij} is the adjacency matrix. For $n = N$, the defined transition rate is automatically equal to 0, as $\forall_i s_i = 0$ then.

It can be easily proved that, if all the transition rates are different ($k \neq j \Rightarrow W_k \neq W_j$), the solution of Eq. (3) with the initial condition $P(n,0) = \delta_{n1}$ is

$$P(n,t) = \sum_{k=1}^{n} C_k^n e^{-W_k t}, \tag{5}$$

where

$$C_k^n \equiv \prod_{i=1}^{n-1} W_i \prod_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{W_j - W_k}. \tag{6}$$

Note that, as $W_N = 0$ and $\forall_{1\leqslant n<N} W_n > 0$, the distribution evolves into

$$\lim_{t\to\infty} P(n,t) = \delta_{nN} \tag{7}$$

(all the agents share paradigm 1), which is an expected limit.

The approximation of only two competing paradigms makes sense if the mean stagnation time $\langle T_{\mathrm{stag}}\rangle = 1/\alpha$ is greater than the mean expansion time $\langle T\rangle$, i.e.,

$$\alpha < \frac{1}{\langle T\rangle}. \tag{8}$$

This upper limit of α has to be estimated for each type of network separately. In general, the $P(T)$ distribution can be expressed by the $P(n,t)$ probability as

$$P(T=t) = W_{N-1}P(N-1,t-1) \approx \frac{1}{N}P(N-1,t), \tag{9}$$

and, considering Eq. (5), we obtain

$$\langle T\rangle \approx \int_0^\infty tP(T=t)dt \approx \frac{1}{N}\sum_{k=1}^{N-1}\frac{C_k^{N-1}}{W_k^2}. \tag{10}$$

Alternatively, we can sum up the mean times of transition between the subsequent states to calculate the mean expansion time:

$$\langle T\rangle = \left\langle \sum_{n=1}^{N-1} T_{n\to n+1}\right\rangle = \sum_{n=1}^{N-1}\langle T_{n\to n+1}\rangle = \sum_{n=1}^{N-1}\frac{1}{W_n}. \tag{11}$$

B. Chain topology

Consider the case when the agents occupy the nodes of a chain. For simplicity, periodic boundary conditions will be assumed.

This specific topology makes the problem quite simple. In the first approximation, analyzing the master equation (3) is not necessary. The average number of agents sharing paradigm 1 can be derived from the recursive equation

$$\begin{aligned}&\langle n(0)\rangle = 1,\\ &\langle n(t+1)\rangle = \langle n(t)\rangle + \frac{2}{N}\frac{1}{2}\frac{\langle n(t)\rangle}{N} = \langle n(t)\rangle\left(1+\frac{1}{N^2}\right),\end{aligned} \tag{12}$$

which has the solution

$$\langle n(t)\rangle = \left(1+\frac{1}{N^2}\right)^t \approx e^{t/N^2}. \tag{13}$$

On average, after time

$$\langle T\rangle = N^2 \ln N, \tag{14}$$

paradigm 1 will stop spreading as it will be shared by the whole community. The situation will be stable until another innovation appears. From this condition the range of α can be estimated for which this approximation makes sense. Considering Eq. (8), we obtain

$$\alpha < \frac{1}{N^2\ln N}. \tag{15}$$

For a more exact analysis, master equation (3) should be considered. Let us make the simple observation that the subgraph consisting of agents sharing the new paradigm is connected. Therefore, the transition rates are equal to

$$W_n = \frac{n}{N^2}\sum_{i=1}^{N}\frac{1-s_i}{2}(s_{i+1}+s_{i-1}) = \frac{n}{N^2}(1-\delta_{nN}). \tag{16}$$

In this equation, we used the periodic boundary conditions, so agents at positions 1 and $N+1$ are equivalent.

To solve the problem, we initially neglect the δ_{nN} term and treat the n variable as if it could grow to infinity, $n = 1,2,\ldots,\infty$. Eventually, the transition rates from state n to $n+1$ are equal to $W_n = \frac{n}{N^2}$, and the master equation has the following form:

$$\frac{\partial}{\partial t}P(n,t) = P(n-1,t)\frac{n-1}{N^2} - P(n,t)\frac{n}{N^2}. \tag{17}$$

Owing to the simple form of the transition rates, Eq. (17) can be solved using the method of characteristic function G:

$$G(s,t) \equiv \langle e^{\text{ins}}\rangle. \tag{18}$$

This approach has such an advantage over the use of (5) that the solutions are automatically in a compact form. Master equation (17), with the initial condition $P(n,0)=\delta_{n0}$, leads to the partial differential equation with the initial condition

$$\frac{\partial}{\partial t}G(s,t) + \frac{1}{iN^2}(e^{is}-1)\frac{\partial}{\partial s}G(s,t) = 0, \quad G(s,0) = e^{is}, \tag{19}$$

which can be solved as

$$G(s,t) = \frac{1}{1-e^{t/N^2}(1-e^{-is})}. \tag{20}$$

After a short algebra, the following can be proved:

$$G(s,t) = \sum_{n=1}^{\infty}\frac{1}{e^{t/N^2}-1}(1-e^{-t/N^2})^n e^{\text{isn}}, \tag{21}$$

so

$$P(n,t) = e^{-t/N^2}(1-e^{-t/N^2})^{n-1}. \tag{22}$$

This is valid for $n<N$. In order to consider the limitation on the n variable [the δ_{nN} term in Eq. (16)], one has to consider the accumulation of probability at point $n=N$:

$$P(n=N,t) = \sum_{m=N}^{\infty} e^{-t/N^2}(1-e^{-t/N^2})^{m-1} = (1-e^{-t/N^2})^{N-1}. \tag{23}$$

Eventually (see Fig. 1),

$$P(n,t) = \begin{cases} e^{-t/N^2}(1-e^{-t/N^2})^{n-1}, & 1\leqslant n<N,\\ (1-e^{-t/N^2})^{N-1}, & n=N.\end{cases} \tag{24}$$

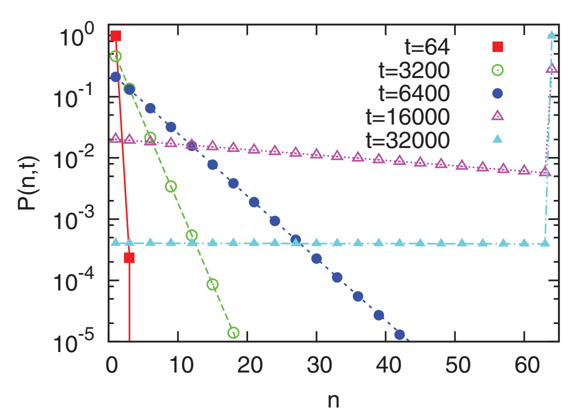


FIG. 1. (Color online) Chain graph topology, $N=64$ nodes, $\alpha < 1/\langle T\rangle$. Evolution of the system starting from $P(n,t=0)=\delta_{n1}$; probability $P(n,t)$ at various moments t. Points are obtained from the numerical solution of master equation (17). Lines show analytical predictions Eq. (24).

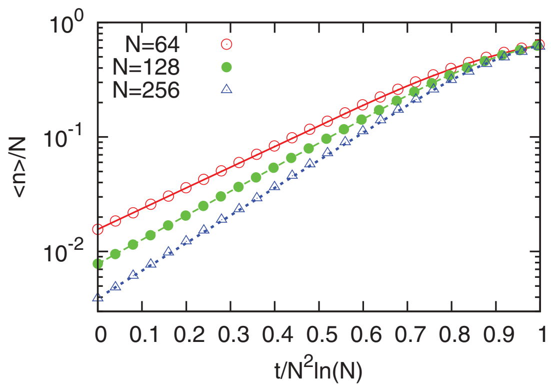


FIG. 2. (Color online) Chain graph topology, $N = 64, 128, 256$ nodes, $\alpha < 1/\langle T\rangle$. Evolution of the system starting from $n = 1$ innovative agent: $\langle n\rangle$ vs time. Points show simulated data, and lines show analytical predictions Eq. (25).

The mean value of n resulting from distribution (24) is equal to

$$\langle n\rangle = e^{t/N^2}[1 - (1 - e^{-t/N^2})^N] \tag{25}$$

(see Fig. 2), which for small t reduces to (13).

From the distribution $P(n,t)$, a more exact approximation of $\langle T\rangle$ than Eq. (14) can be obtained. According to Eq. (9),

$$P(T = t) \approx \frac{1}{N} e^{-t/N^2}(1 - e^{-t/N^2})^{N-2} \tag{26}$$

(see Fig. 3), and

$$\begin{aligned}\langle T\rangle &= \sum_{t=0}^{\infty} t P(T = t)\\ &\approx \frac{1}{N}\int_0^{\infty} t e^{-t/N^2}(1 - e^{-t/N^2})^{N-2} dt\\ &= N^2 H_{N-1},\end{aligned} \tag{27}$$

where H_n is the nth harmonic number. As harmonic numbers grow approximately as fast as the natural logarithm, the approximated solution (14) is very close to (27). In fact, for $N \gg 1$,

$$H_N \approx \ln N + \gamma, \tag{28}$$

where $\gamma \approx 0.5772$ denotes the Euler-Mascheroni constant.

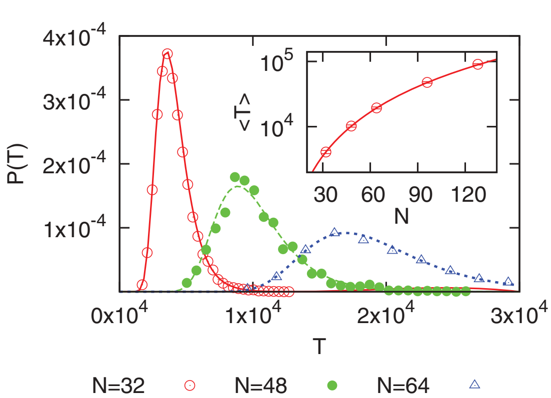


FIG. 3. (Color online) Chain graph topology, $\alpha < 1/\langle T\rangle$. Distribution of the expansion periods lengths T for different system sizes N. Points show simulations, and lines show analytical predictions Eq. (26). The inset shows the mean time of expansion $\langle T\rangle$ vs system size N: simulations (points) compared with the analytical predictions (line) [Eq. (27)].

C. Complete graph topology

Consider the situation in which interaction is possible between every pair of agents, i.e., $\forall_{(i,j)} a_{ij} = 1$. Referring to the generic master equation (3), the transition rates are equal to

$$W_n = \frac{n}{N^2}\sum_{i=1}^{N}\frac{1 - s_i}{N - 1}\sum_{j=1}^{N} s_j = \frac{n^2(N - 1)}{N^2(N - 1)} \approx \frac{n^2(N - 1)}{N^3}. \tag{29}$$

Eventually, the master equation obtains the following form:

$$\begin{aligned}\frac{\partial}{\partial t}P(n,t) &= P(n - 1,t)\frac{(n - 1)^2(N - n + 1)}{N^3}\\ &\quad - P(n,t)\frac{n^2(N - n)}{N^3}.\end{aligned} \tag{30}$$

If all the transition rates are different [$j \neq k \Rightarrow W_j \neq W_k$, which is satisfied if equation $N = a(1 + b + b^2)$ does not have trivial solutions a,b among natural numbers], the solution can be written in the form of sum (5):

$$P(n,t) = \sum_{k=1}^{n} C_k^n e^{-k^2(N-k)t/N^3}, \tag{31}$$

where

$$\begin{aligned}C_k^n &\equiv \prod_{i=1}^{n-1} i^2(N - i)\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{j^2(N - j) - k^2(N - k)}\\ &= (n - 1)!^2\frac{(N - 1)!}{(N - n)!}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{j^2(N - j) - k^2(N - k)}.\end{aligned} \tag{32}$$

The expansion time T distribution $P(T)$ can be derived from the $P(n,t)$ distribution:

$$P(T = t) \approx \frac{1}{N}P(N - 1,t) = \frac{1}{N}\sum_{k=1}^{N-1} C_k^{N-1} e^{-k^2(N-k)t/N^3}. \tag{33}$$

The mean expansion time can be derived analytically, using (11):

$$\begin{aligned}\langle T\rangle &= \sum_{n=1}^{N-1}\frac{1}{W_n} = \sum_{n=1}^{N-1}\frac{N^3}{n^2(N - n)}\\ &\approx \frac{\pi^2}{6}N^2 + 2N\ln N = O(N^2).\end{aligned} \tag{34}$$

The analytical predictions are in agreement with the simulations (Figs. 4–6).

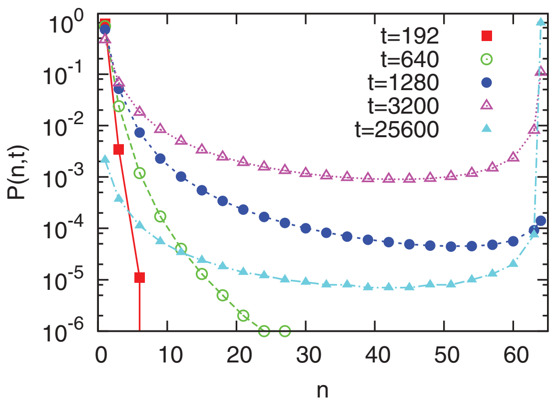


FIG. 4. (Color online) Complete graph topology, $N = 64$ nodes, $\alpha < 1/\langle T\rangle$. Evolution of the system starting from $P(n,t = 0) = \delta_{n1}$; probability $P(n,t)$ at various moments t. Points are obtained from the numerical solution of master equation (30). Lines show analytical predictions Eq. (31).

D. Erdős-Rényi graph topology

Consider the situation when the network of interactions is an Erdős-Rényi graph [17]. For each pair of nodes, the edge between them exists with probability p.

The degree distribution of an ER graph is a binomial distribution,

$$P(k) = \mathrm{Bin}(N,p) \equiv \binom{N-1}{k} p^k (1-p)^{N-1-k}. \quad (35)$$

The transition rates of master equation (3) are equal to

$$W_n = \frac{n}{N^2}\sum_{i=1}^{N}(1-s_i)\sum_{j=1}^{N}\frac{a_{ij}s_j}{k_i} = \frac{n}{N^2}\sum_{i=1}^{N}(1-s_i)\frac{k_i^+}{k_i^+ + k_i^-}, \quad (36)$$

where the random variable k_i^+ denotes the number of i's neighbors following paradigm 1 and k_i^- is that following paradigm 0. In the mean field approach, the transition rates can be estimated by

$$W_n = \frac{n(N-n)}{N^2}\left\langle \frac{k^+}{k^+ + k^-}\right\rangle, \quad (37)$$

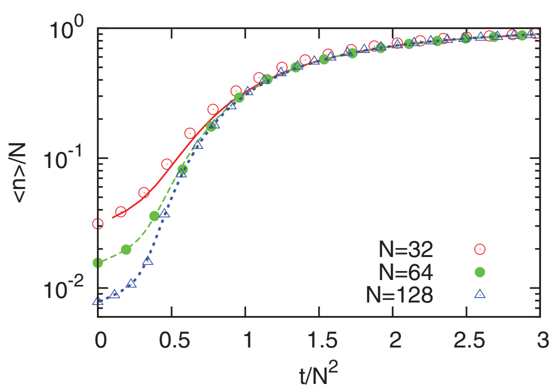


FIG. 5. (Color online) Complete graph topology, $N = 32$, 64, 128 nodes, $\alpha < 1/\langle T\rangle$. Evolution of the system starting from $n = 1$ innovative agent: $\langle n\rangle$ vs time. Points show simulated data, and lines show analytical predictions [obtained from Eq. (31)].

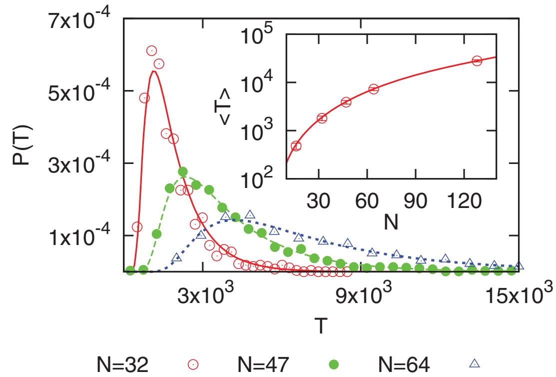


FIG. 6. (Color online) Complete graph topology, $\alpha < 1/\langle T\rangle$. Distribution of the expansion period lengths T for different system sizes N. Points show simulations, and lines show analytical predictions Eq. (33). The inset shows the mean expansion time $\langle T\rangle$ vs the system size N: simulations (points) are compared with the theoretical predictions (line) [Eq. (34)].

where $\langle\cdot\rangle$ denotes the averaging on the whole population of agents. Treating k^+ and k^- as independent random variables with binomial distributions, we obtain

$$\begin{aligned}\left\langle \frac{k^+}{k^+ + k^-}\right\rangle &= \sum_{k^+=1}^{n}\sum_{k^-=0}^{N-n-1}\frac{k^+}{k^+ + k^-}P(k^+)P(k^-)\\ &= \sum_{k^+=1}^{n} k^+\binom{n}{k^+}p^{k^+}(1-p)^{n-k^+}\\ &\quad\times\sum_{k^-=0}^{N-n-1}\frac{\binom{N-n-1}{k^-}}{k^+ + k^-}p^{k^-}(1-p)^{N-n-1-k^-}\\ &= (1-p)^{N-n-1}\sum_{k^+=1}^{n}k^+\binom{n}{k^+}p^{k^+}(1-p)^{n-k^+}\\ &\quad\times\left(\frac{1-p}{p}\right)^{k^+}\int_0^{p/(1-p)}\xi^{k^+-1}(1+\xi)^{N-n-1}d\xi\\ &= \frac{n}{N-1}[1-(1-p)^{N-1}].\end{aligned} \quad (38)$$

Eventually,

$$W_n = \frac{n^2(N-n)}{N^3}[1-(1-p)^{N-1}]. \quad (39)$$

For $p = 1$, the transition rates reduce, as expected, to the ones obtained for the complete graph topology.

The master equation has the following form:

$$\begin{aligned}\frac{\partial}{\partial t}P(n,t) = \bigg(&P(n-1,t)\frac{(n-1)^2(N-n+1)}{N^3}\\ &- P(n,t)\frac{n^2(N-n)}{N^3}\bigg)[1-(1-p)^{N-1}].\end{aligned} \quad (40)$$

Similar to the case of complete graph topology, the solution can be written in the form of sum (5):

$$P(n,t) = \sum_{k=1}^{n} C_k^n e^{-k^2(N-k)[1-(1-p)^{N-1}]t/N^3}, \quad (41)$$

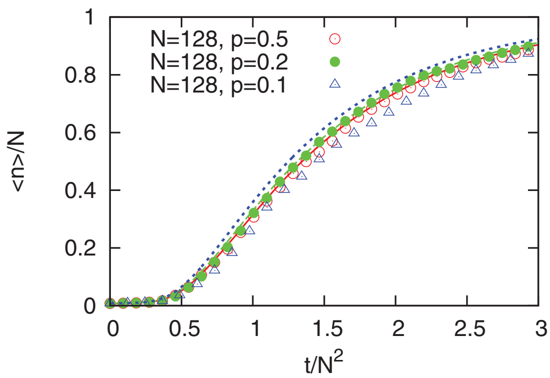


FIG. 7. (Color online) ER graph topology, $N = 128$ nodes, $p = 0.5,0.05,0.02, \alpha < 1/\langle T \rangle$. Evolution of the system starting from $n = 1$ innovative agent: $\langle n \rangle$ vs time. Points show simulated data, and lines show analytical predictions [obtained from Eq. (41)].

where

$$C_k^n \equiv \prod_{i=1}^{n-1} i^2(N-i) \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{j^2(N-j)-k^2(N-k)}$$
$$= (n-1)!^2 \frac{(N-1)!}{(N-n)!} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{j^2(N-j)-k^2(N-k)}. \tag{42}$$

Now, the expansion time distribution $P(T)$ is

$$P(T=t) \approx \frac{1}{N} P(N-1,t)$$
$$= \frac{1}{N} \sum_{k=1}^{N-1} C_k^{N-1} e^{-k^2(N-k)[1-(1-p)^{N-1}]t/N^3}. \tag{43}$$

As shown in Fig. 7, our approach predicts a decline in the rate of growth of the new paradigm cluster with a decrease in the network density (parameter p), which is, however, seriously underestimated. We suppose that some nontrivial correlations exist between the agents' states resulting from the dynamics, which was not considered.

E. Star topology

1. Central agent innovative

Consider the star topology of interactions i.e., there exists a *central* agent connected to all the other $N-1$ *peripheral* agents, which are the only connections. Moreover, we require that at time $t = 0$, the central agent follows innovative paradigm 1. This case is interesting because in this variant, the new idea spreads at the highest pace. The transition rates

$$W_n = \frac{(N-n)n}{N^2} \tag{44}$$

are the highest possible for this model among all the possible topologies [e.g., compare Eq. (44) with Eqs. (16), (29), (37), and (55)].

The peripheral agents are not connected to each other, and they are only influenced by the "mean field" of the paradigms. Therefore, one can consider the change in time of the average state of a peripheral agent. Let $\pi(t)$ denote the probability that, at time t, a peripheral agent follows paradigm 1. The evolution of $\pi(t)$ follows the recursive equation

$$\pi(0) = 0,$$
$$\pi(t+1) = \pi(t) + [1-\pi(t)]\frac{1}{N-1}\frac{\langle n(t)\rangle}{N}$$
$$= \pi(t) + \frac{1}{N}[1-\pi(t)]\left(\pi(t) + \frac{1}{N-1}\right), \tag{45}$$

which can be solved in the approximation of continuous time:

$$\frac{d\pi}{dt} \approx -\frac{1}{N}(\pi-1)\left(\pi + \frac{1}{N-1}\right), \tag{46}$$

$$\pi(t) = \frac{1+\frac{1}{N-1}}{(N-1)\exp\left(-\frac{t}{N-1}\right)+1} - \frac{1}{N-1}. \tag{47}$$

Thus, the average number of agents sharing paradigm 1 is equal to

$$\langle n(t)\rangle = (N-1)\pi(t) + 1 \approx \frac{N}{1+Ne^{-t/N}} \tag{48}$$

(see Fig. 8). From $\pi(t)$, the expansion time distribution $P(T)$ can be derived as follows:

$$P(T=t) \approx \frac{d}{dt}P(T \leqslant t) = \frac{d}{dt}[\pi(t)]^{N-1}$$
$$\approx \frac{Ne^{t/N}(e^{t/N}-1)^{N-2}}{(e^{t/N}+N-1)^N} \tag{49}$$

(see Fig. 9).

We were not able to find an analytical formula for $\langle T \rangle$. However, as $P(T)$ is unimodal with a well-defined maximum, one can assume that $\langle T \rangle$ grows as fast with N as

$$\arg\max P(T) = N \ln\left(\frac{\sqrt{N^4-2N^3+N^2-4N+4}}{2} + \frac{N^2-N}{2}\right) = O(N\ln N). \tag{50}$$

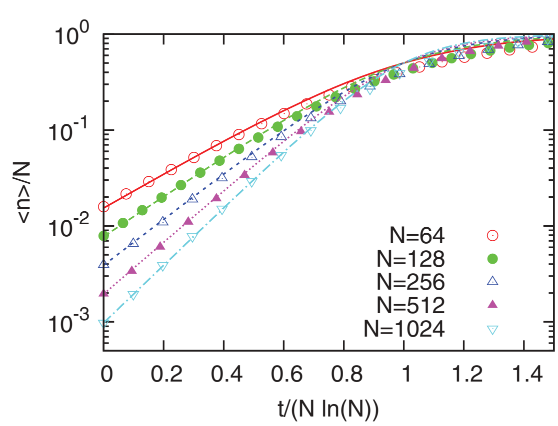


FIG. 8. (Color online) Star topology with central agent innovative, $N = 64,128,256,512,1024$ nodes, $\alpha < 1/\langle T \rangle$. Evolution of the system starting from $n = 1$ innovative agent (central): $\langle n \rangle$ vs time. Points show simulated data, and lines show analytical predictions Eq. (48).

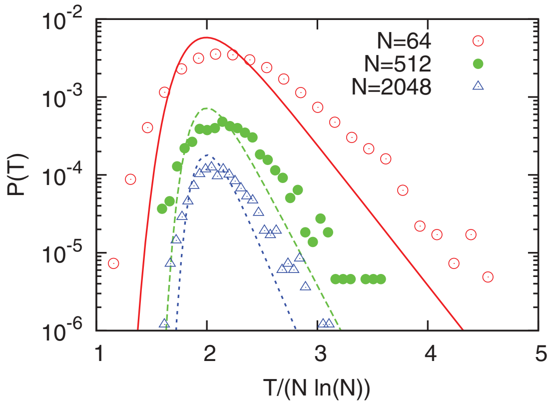


FIG. 9. (Color online) Star topology with central agent innovative, $\alpha < 1/\langle T\rangle$. Distribution of the expansion period lengths T for different system sizes N. Points show simulations, and lines show analytical predictions Eq. (49).

Indeed, as shown in Fig. 11, the mean expansion time can be well approximated by

$$\langle T\rangle \approx kN \ln N, \tag{51}$$

where parameter k, as obtained by fitting to simulated data, is equal to $k = 2.149 \pm 0.007$. As the transition rates (44) are the highest possible, the mean expansion time (51) must be the shortest for this model among all the possible topologies.

2. All agents equally innovative

The system is similar to the one described above. The only difference is that, at time $t = 0$, any agent can be the innovative one. With probability $(N-1)/N \approx 1$, a peripheral agent becomes the innovative one. After time T_0, which is a random variable with exponential probability distribution,

$$P(T_0 = t) = \left(1 - \frac{1}{N^2(N-1)}\right)^{t-1} \frac{1}{N^2(N-1)} \approx \frac{1}{N^3} e^{-t/N^3}, \tag{52}$$

the central agent adopts paradigm 1. Then the dynamics becomes as described in the case of the innovative central agent. Thus, the probability distribution of the expansion time can be well approximated by a convolution of two probability distributions:

$$P(T = t) \approx \int_0^\infty \frac{Ne^{\tau/N}(e^{\tau/N} - 1)^{N-2}}{(e^{\tau/N} + N - 1)^N} \frac{1}{N^3} e^{(-t+\tau)/N^3} d\tau \approx \frac{1}{N^3} e^{-t/N^3} = P(T_0 = t). \tag{53}$$

The mean and the standard deviation of this exponential probability distribution (Fig. 10) are equal to

$$\langle T\rangle = \sigma(T) = N^3. \tag{54}$$

As shown in Fig. 11, the analytical results for both variants of star topology agree well with simulations.

F. Square lattice topology

Consider the square lattice topology. Periodic boundary conditions are assumed, so each agent has four neighbors.

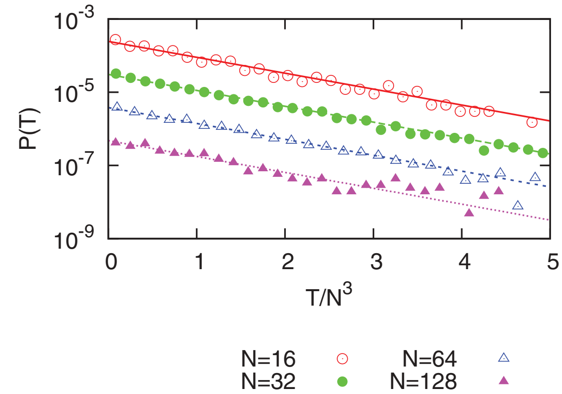


FIG. 10. (Color online) Star topology with all agents equally innovative, $\alpha < 1/\langle T\rangle$. Distribution of the expansion period lengths T for different system sizes N. Points show simulations, and lines show analytical predictions Eq. (53).

The first approximation assumes that the cluster of agents sharing paradigm 1 grows uniformly in each direction. At any moment, it is circle shaped, with the radius of the circle equal to $r = \sqrt{n/\pi}$. Therefore, in this approximation, the transition rates in the generic master equation (3), are equal to

$$W_n = \frac{\sqrt{\pi} n^{3/2}}{2N^2}(1 - \delta_{nN}). \tag{55}$$

Similar to the case of complete graph topology, the solution of the master equation

$$\frac{\partial}{\partial t} P(n,t) = P(n-1,t)\frac{\sqrt{\pi}(n-1)^{3/2}}{2N^2}(1 - \delta_{n-1,N}) - P(n,t)\frac{\sqrt{\pi} n^{3/2}}{2N^2}(1 - \delta_{nN}) \tag{56}$$

can be expressed in the form of the sum of products (5). Comparing the results of such an approximation with the simulations (Fig. 12), this approach significantly overestimates the pace of the growth of the new paradigm cluster.

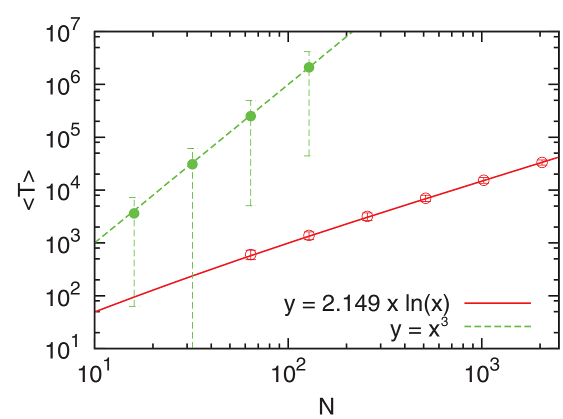


FIG. 11. (Color online) Star topology. Mean time of expansion $\langle T\rangle$ vs system size N. Open circles indicate an innovative central agent; solid circles indicate that all agents are equally innovative. Error bars correspond to the standard deviations of the samples. Lines show analytical predictions Eqs. (51) and (54).

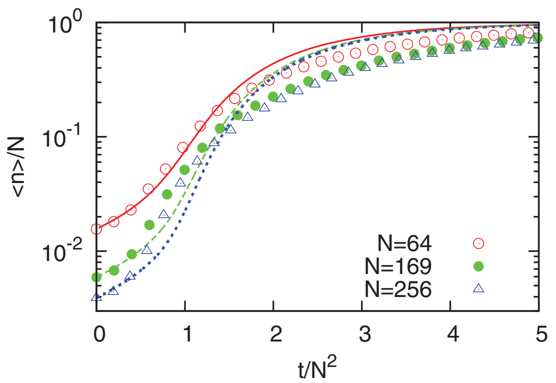


FIG. 12. (Color online) Square lattice topology, $N = 64$, 169, 256 nodes, $\alpha < 1/\langle T\rangle$. Evolution of the system starting from an $n = 1$ innovative agent: $\langle n\rangle$ vs time. Points show simulated data, and lines show analytical predictions [obtained from Eq. (5) with transition rates (55)].

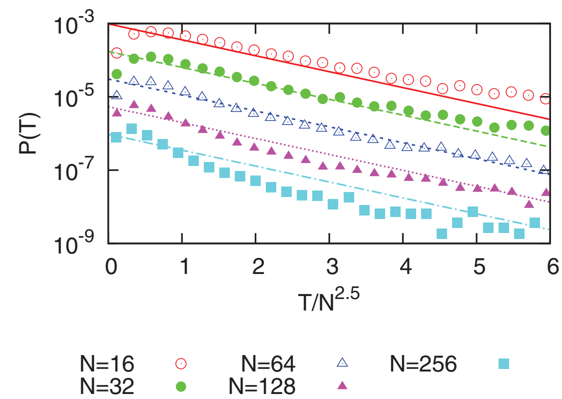


FIG. 13. (Color online) BA graph topology, $\alpha < 1/\langle T\rangle$. Distribution $P(T)$ of the expansion period lengths T for different system sizes N. Points show simulations, and lines show analytical predictions Eq. (58).

G. Barabási-Albert graph topology

The Barabási-Albert (BA) graphs [17] are constructed as follows: starting from a clique of m_0 nodes, a new node is added in each time step, which links to $m \leqslant m_0$ nodes already present. The probability of creating such a link to node i is proportional to its temporary degree $k_i(t)$. For a sufficiently large number of added nodes, such an algorithm results in the degree distribution of the network following the power law

$$P(k) \propto k^{-\gamma}, \tag{57}$$

where the exponent $\gamma = 3$. The maximum node degree in the graph scales with its size as $k_{\max} \approx m\sqrt{N}$.

We concentrated on the case of the parameters m_0,m both being equal to 1. In this case the BA graphs are trees (connected graphs without cycles), making them easier to analyze. We conjectured that the expansion period length T is dominated by the time to infect the hub [similar to the star graph topology; see Eq. (53)]. Therefore, the $P(T)$ probability distribution should be exponential,

$$\begin{aligned} P(T=t) \approx P(T_{\text{hub}}=t) &= \left(1 - \frac{\tilde{n}}{2N^{2.5}}\right)^t \frac{\tilde{n}}{2N^{2.5}} \\ &\approx \frac{\tilde{n}}{2N^{2.5}} \exp\left(-\frac{\tilde{n}t}{2N^{2.5}}\right), \end{aligned} \tag{58}$$

with the mean equal to

$$\langle T\rangle = \int_0^\infty t P(T=t)dt = \frac{2N^{2.5}}{\tilde{n}}, \tag{59}$$

where $\tilde{n}$ is a random variable denoting the number of nodes infected before the hub. The rough approximation of $\langle\tilde{n}\rangle$ as the average degree of the (hub's) nearest neighbor $\langle k_{nn}\rangle$,

$$\langle\tilde{n}\rangle \approx \langle k_{nn}\rangle = \frac{\langle kP(k)\rangle}{\langle k\rangle} = \frac{\gamma-1}{\gamma-2} = 2, \tag{60}$$

agrees surprisingly well with the results of the simulations (Fig. 13). The differences between the real $P(T)$ distribution and the approximated, exponential one Eq. (58) are, however, large enough to reflect in the $\langle T\rangle$ scaling with the system size. The $\langle T\rangle$ value scales with the system size as a power function $\langle T\rangle \propto N^\beta$, but the exponent β is not equal to 2.5, as predicted by Eq. (59), but rather $\beta = 2.179 \pm 0.004$ (Fig. 14).

In principle, it should be possible to approach the problem from a different, microscopic perspective and to try to find the $\langle n(t)\rangle$ function. The simplest way would be to treat the system as a well-mixed system of heterogeneous (due to different node degrees) agents. Thus, considering the evolution of the probability ρ_k that a node of degree k is in state 1, one would obtain the set of equations

$$\dot{\rho}_k = \frac{1}{N_k}\frac{N_k}{N}(1-\rho_k)\frac{\sum_k P(k)k\rho_k}{\sum_k P(k)k}\sum_k P(k)\rho_k. \tag{61}$$

However, the solution of these equations does not agree with the simulations. The basic reason is probably the fact that the ρ_k functions do not combine into some macroscopic variable in a natural way. In the above equation, there are both weighted and the nonweighed means of ρ_k: $\frac{1}{\langle k\rangle}\sum_k P(k)k\rho_k$ and $\sum_k P(k)\rho_k$. This is in contrast to, for example, the Voter [18], Ising [19], or zero-temperature Gluaber dynamics [20] models on complex networks, where the weighted sum of spins is a proper order parameter. As a result, should one wish to develop the microscopic description of the analyzed dynamics, one would probably have to go beyond the mean field approximation and consider the correlations between the neighboring nodes' states.

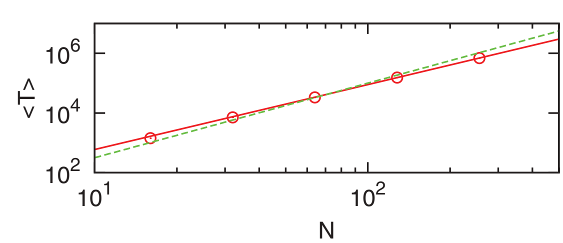
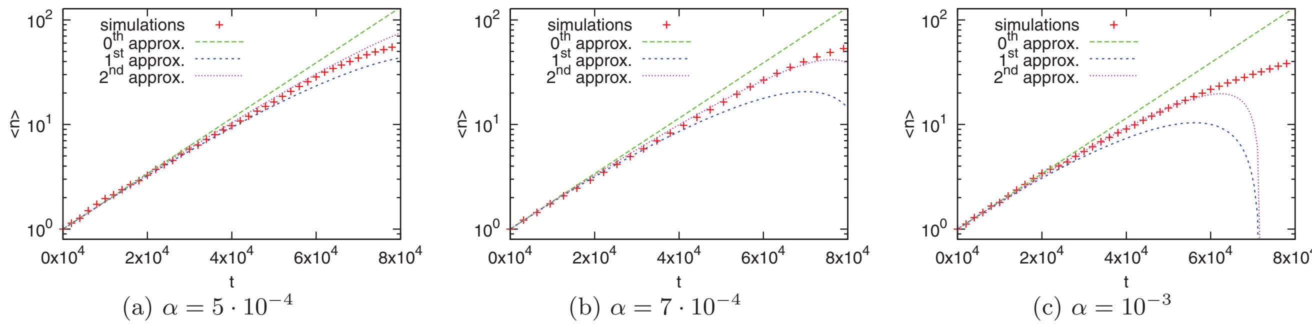


FIG. 14. (Color online) Barabási-Albert networks (trees), mean time of expansion $\langle T\rangle$ vs network size N. Points show simulations, the dashed line shows expected behavior Eqs. (59) and (60), and the solid line shows the least squares fit: $\langle T\rangle = CN^\beta$, where $\beta = 2.179 \pm 0.004$ and $C = 3.93 \pm 0.07$.



(a) $\alpha = 5 \cdot 10^{-4}$ (b) $\alpha = 7 \cdot 10^{-4}$ (c) $\alpha = 10^{-3}$

FIG. 15. (Color online) Chain topology, $N = 128$, various levels of creativity α. Comparison of approximations is Eqs. (13), (66), and (69). The red crosses refer to the simulated data.

IV. THE CASE OF MANY COMPETING IDEAS

If the mean stagnation time $\langle T_{\mathrm{stag}} \rangle = 1/\alpha$ is shorter than the mean expansion time $\langle T \rangle$, i.e.,

$$\alpha = \frac{1}{\langle T_{\mathrm{stag}} \rangle} > \frac{1}{\langle T \rangle}, \tag{62}$$

then it is most probable that more than two paradigms coexist in the community at any moment. This case is much more difficult to describe analytically. In what follows we present our results for chain topology, which is probably the simplest one.

For α higher than $1/N^2 \ln N$, Eq. (12) has to be extended by terms describing the appearance of new clusters of ideas, which slows down the process of expansion of paradigm 1. In the first approximation, the only important new clusters are assumed to be those appearing inside the cluster of paradigm 1, and they do not overlap with each other. Their growth is described by Eq. (13). Thus, the recursive equation for $\langle n(t) \rangle$ is now

$$\begin{aligned}\langle n(t+1) \rangle &= \langle n(t) \rangle \left(1 + \frac{1}{N^2}\right) - \sum_{\tau=0}^{t} \alpha \frac{\langle n(\tau) \rangle}{N} \langle \Delta n_{\mathrm{new}}(t-\tau) \rangle \\ &= \langle n(t) \rangle \left(1 + \frac{1}{N^2}\right) - \frac{\alpha}{N^3} \sum_{\tau=0}^{t} \langle n(\tau) \rangle \exp\left(\frac{t-\tau}{N^2}\right).\end{aligned} \tag{63}$$

Substituting the sum with the integral and stating that $\langle n(t) \rangle = \exp(t/N^2) f(t)$ leads to the following equation for $f(t)$:

$$f'(t) + \frac{\alpha}{N^3} \int_0^t f(\tau) d\tau = 0, \tag{64}$$

which, assuming the same initial conditions as in Eq. (12) (single innovation at time $t = 0$), has the solution

$$f(t) = \cos(\lambda t), \tag{65}$$

where $\lambda \equiv \sqrt{\alpha/N^3}$. Thus, the complete formula for the first approximation of $\langle n(t) \rangle$ is

$$\langle n(t) \rangle = \exp\left(\frac{t}{N^2}\right) \cos(\lambda t). \tag{66}$$

As expected, for $\alpha \to 0$, this approximation converges to the previous one Eq. (13).

A better approximation can be obtained by substituting the term $\exp\left(\frac{t-\tau}{N^2}\right)$ by $\langle n(t-\tau) \rangle$ in Eq. (63), as new paradigms can also be "attacked" by paradigms appearing after them. The equation

$$\langle n(t+1) \rangle = \langle n(t) \rangle \left(1 + \frac{1}{N^2}\right) - \frac{\alpha}{N^3} \sum_{\tau=0}^{t} \langle n(\tau) \rangle \langle n(t-\tau) \rangle \tag{67}$$

does not have a simple analytical solution, but by substituting the sum with the integral and stating $\langle n(t) \rangle = \exp(t/N^2) \cos(\lambda t)[1 + g(t)]$, where $g(t) \ll 1$, an integral equation can be obtained,

$$\begin{aligned}0 = &-\lambda \sin(\lambda t) + g'(t) \cos(\lambda t) + \lambda^2 \int_0^t \cos(\lambda \tau)[1 + g(\tau)] \\ &\times \cos[\lambda(t-\tau)][1 + g(t-\tau)] d\tau,\end{aligned} \tag{68}$$

which can be solved provided that all the terms in the integral apart from the product $\cos(\lambda \tau) \cos[\lambda(t-\tau)]$ are neglected. Eventually, the second approximation of $\langle n(t) \rangle$ obtains the following form:

$$\langle n(t) \rangle = \exp\left(\frac{t}{N^2}\right) \cos(\lambda t) \left[1 - \ln|\cos(\lambda t)| - \frac{1}{4} \lambda^2 t^2\right]. \tag{69}$$

The comparison with the simulations (Fig. 15) shows that the latest approximation Eq. (69) is better than the previous ones Eqs. (13) and (66).

V. CONCLUSIONS

We have developed an analytical approach based on a master equation that describes a model of paradigm evolution [14] and compared our results with the outcome of the work of Bornholdt *et al.* as well as with our numerical simulations. The outcome suggests that the asymmetry between the paces of growth and decline of the dominant idea observed in [14] is a generic property of the model and should be observed for any topology of interactions.

Our analytical methodology can be used to consider various topologies of interaction networks. The crucial parameter of the dynamics is the creativity of the agents, described by the α parameter. In the case in which agents are almost noninnovative, the evolution consists of subsequent *periods of stagnation* (i.e., a single paradigm is present in the community) and *periods of expansion* (i.e., an innovative paradigm spreads across the community and replaces the old one). The mean length of the stagnation period is equal to $\langle T_{\rm stag}\rangle = 1/\alpha$, regardless of the interaction network topology. The mean length of the expansion period $\langle T\rangle$ strongly depends on the topology. If $\alpha \ll \langle T\rangle^{-1}$, the mean time between shifts of dominant paradigms can be approximated by $\langle T_{\rm stag} + T\rangle \approx \langle T_{\rm stag}\rangle = 1/\alpha$, which is the scaling observed in the simulated data by Bornholdt *et al.* [14] (note that a different time scale was used in [14]).

Our approach is mainly based on the approximation of two competing paradigms, which is justified if the level of creativity α is small enough, i.e., $\alpha < 1/\langle T\rangle$. For each type of interaction network topology, this range has to be calculated separately. Six different topologies were considered: the chain, the complete graph, the ER graphs, the star graph the square lattice, and the BA graphs.

In the chain topology, finding compact forms of the analytical solutions is possible. The mean expansion time $\langle T\rangle$ scales with the system size N as $N^2 \ln N$, and during the stage of expansion, the mean size of the cluster of the new idea grows like a damped exponential function; see Eq. (25). The analytical results agree with the simulated data.

In the complete graph topology, the proposed approach also results in a good agreement with the simulations. However, compact forms of the functions describing the system evolution Eqs. (31) and (33) probably do not exist.

In the case of ER graph topology, our approach overestimates the rate of the growth of the new idea cluster (Fig. 7). The reasons for this result are probably the correlations between the node degree and the state of the agent located at that node, which were not considered.

Comparison of two variants of star topology brought interesting results. In the first variant, when we require that the innovation first appears in the central node, the rate of the expansion of the new idea is the fastest possible among all the topologies, and the mean expansion time $\langle T\rangle$ (which is the shortest possible) scales as $N \ln N$. However, if we remove this requirement and let the innovation appear in any node with equal probability, $\langle T\rangle$ grows to N^3 (higher than that in a chain, where the mean distance between nodes is much larger), and almost the whole time is taken by "convincing" the central agent of the new idea.

In the case of the square lattice, the method only qualitatively reproduces the results of the simulations (Fig. 12). The problem probably lies in the apparently too rough estimation of the shape of the cluster of the new idea as a circle.

The BA networks, characterized by the strong heterogeneity of the nodes, seem to be the most difficult to treat analytically. The research reveals the surprising feature of the expansion time T scaling: $\langle T\rangle \propto N^{2.179}$. The $P(T)$ distribution may be, however, quite well estimated by the exponential PDF with the mean $1/N^{2.5}$.

The dynamics described by the model we analyzed is a kind of a contagion process, including both local and global interactions. Our research shows that the interaction topology plays a crucial role in the dynamics. In the heterogeneous systems (in our paper represented by the star graphs and the BA graphs), hubs play an important, albeit ambivalent, role. Initially, a hub is reluctant to change its state, as it interacts with many other agents, only a small part of whom can be "infected." However, after the hub has changed its state, the propagation of the new idea is boosted drastically. Our results suggest that, overall, the existence of hubs slows down the process of the propagation: the mean expansion time is longer (with respect to the scaling with the system size) for the strongly heterogeneous networks than for the homogeneous ones.

For a higher level of creativity α, when most of the time more than two ideas coexist, the dynamics of the system can be found starting from the results obtained for the case of lower levels of α and using a method similar to the perturbation method. This approach proved to be useful in the simplest case, the chain topology. We considered the function describing the mean number of agents following the expanding paradigm 1. Thus, the unperturbed function (13),

$$\langle n(t)\rangle = e^{t/N^2}, \tag{70}$$

should be modified by two factors. The first one,

$$\cos\left(\frac{\alpha}{N^3}t\right) < 1, \tag{71}$$

describes the "attack" on paradigm 1 by the paradigms appearing after it. The second one,

$$1 - \ln\left|\cos\left(\frac{\alpha}{N^3}t\right)\right| - \frac{1}{4}\left(\frac{\alpha}{N^3}\right)^2 t^2 \approx 1 + \frac{1}{4}\left(\frac{\alpha}{N^3}\right)^2 t^2 > 1, \tag{72}$$

describes the attack on the paradigms attacking paradigm 1. Both these terms, as expected, converge to 1 if the creativity of the agents α converges to 0.

Our analytical approach allows for a better understanding of the system dynamics described by the model [14] and explains some of the relationships previously observed in the simulated data. The proposed methodology can be used to analyze the dynamics of paradigms spreading in other networks [17]. It is especially interesting because real networks of human contacts (including scientific collaboration networks) exhibit some nontrivial properties, such as scale-free behavior [21]. Investigations of such networks are planned in the future.

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