Optics of Anisotropic Media

Introduction

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Aim of this course

- Introduction to physical principles of polarization optics,
- Electromagnetic waves in different media,
- Application of polarization of light in physics, optoelectronics and photonics,
  - Especially in electrooptic and magnetooptic systems and optical fibers, liquid crystal cells,

Agenda

1. Maxwell equations for electromagnetic waves in different media,
2. Mathematical description of state of polarization:
    a. Trigonometric method,
    b. Jones vectors,
    c. Coherence matrix for quasi-monochromatic electromagnetic wave,
    d. Stokes vector,
    e. Poincare sphere,
3. Electromagnetic wave in anisotropic media,
4. Anisotrpic media:
    a. Types of anisotropic media,
    b. Piezo-optic and elasto-optic phenomena,
    c. Electro-optic and magneto-optic effects,
5. Transformation of state of polarization:
    a. Jones matrix, Mueller matrix,
    b. Determination of SOP changes using Poincare sphere,
6. Application of anisotropic media in technics,
    a. Methods of SOP generation and measurement,
    b. Polariscopy
    c. Compensators,
    d. Conoscopy,
    e. Lyott filters,

Recomended literature:

1. Florian Ratajczyk, **Optyka ośrodków anizotropowych ,** PWN
   (lub **Dwójłomność i polaryzacja optyczna,** OFPWr)
2. D. Goldstein, **Polarized Light**, M. Dekker, New York, 2003
3. E. Collet, **Polarization light in fiber optics**, PolaWave Group, Lincroft 2003
4. C. Brosseau, **Fundamentals of Polarized Light**, Wiley & Sons, New York ,1998
5. M.Born, E.Wolf, **Principles in Optics**, Cambridge University Press, Cambridge, 1999
6. D. J. Griffiths, **Podstawy elektrodynamiki**, PWN

Lecture 1

Mathematical basics

$$\vec{E} = [E_x, E_y, E_z]$$

Operator $\vec{\nabla} = \vec{i}\ \frac{\partial}{\partial x} + \vec{j}\ \frac{\partial}{\partial y} + \vec{k}\ \frac{\partial}{\partial z}$

grad $\varphi = \vec{\nabla} \cdot \varphi = \vec{i}\ \frac{\partial \varphi}{\partial x} + \vec{j}\ \frac{\partial \varphi}{\partial y} + \vec{k}\ \frac{\partial \varphi}{\partial z}$

Divergence of vector -> a quantity of a vector field's source at each point,

$$div\, \vec{E} = \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Curl -> infinitesomal rotation of 3-dimensional vector field

$$rot\,\vec{E} = \vec{\nabla} \times \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\vec{\nabla}\vec{E} - \nabla^2\vec{E} = \vec{\nabla}\vec{\nabla}\vec{E} - \Delta\vec{E}$$

Divergence theorem (Gauss – Ostogradsky theorem)

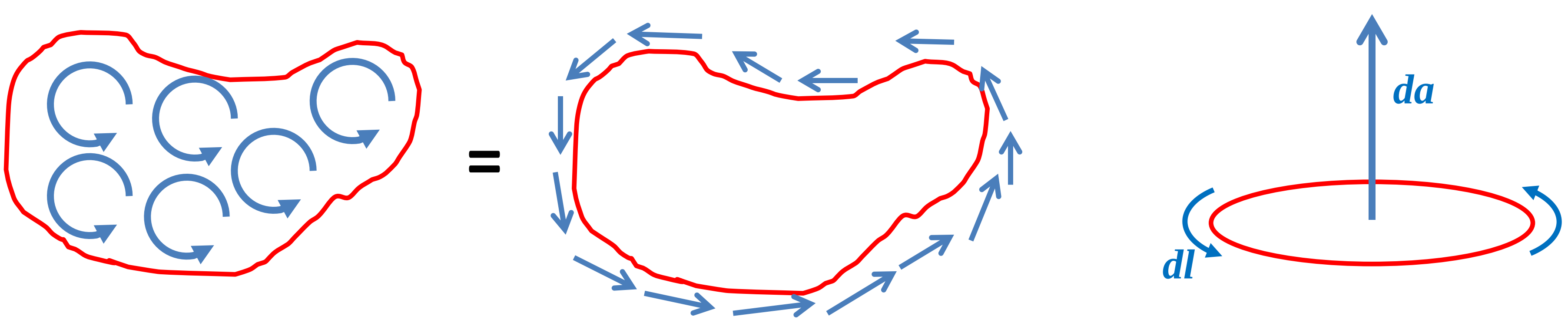
$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

Interpretation: The outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface.
Moreover, *sum of all sources minus the sum of all sinks gives the net flow out of a region*.

Kelvin-Stokes theorem (curl theorem)

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

Interpretation: The integral of the curl of the vector field over some surface is equal to the line integral of the vector field around the boundary of the surface.



Maxwell’s equations for isotropic media

Maxwell's equations

$$rot\vec{H} - \dot{D} = \sigma\vec{E}$$
$$rot\vec{E} + \dot{B} = 0$$
$$div\vec{D} = \rho$$
$$div\vec{B} = 0$$

Material equations

$$\vec{D} = \varepsilon\varepsilon_0\vec{E}$$
$$\vec{B} = \mu\mu_0\vec{H}$$

$\vec{E}$ Intensity of electric field[V/m]

$\vec{D}$ Electric displacement field vector[C/m^2]

$\vec{H}$ Intensity of magnetic field [A/m]

$\vec{B}$ Magnetic field vector [Tesla]

ρ Density of free charge

σ conductivity

Isotropic media

tensor

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$$

$$\varepsilon = n^2$$

$$\bar{\varepsilon} = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

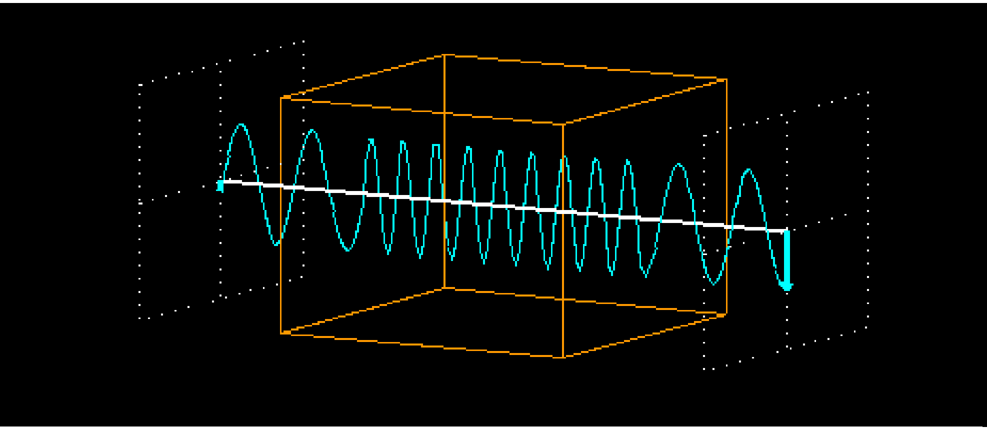
Linearly birefringent media

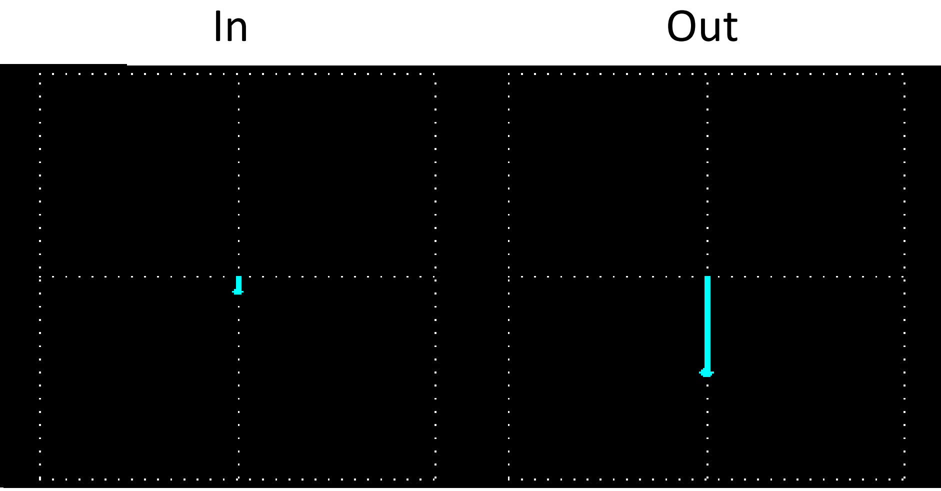
$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$$

$$\varepsilon = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

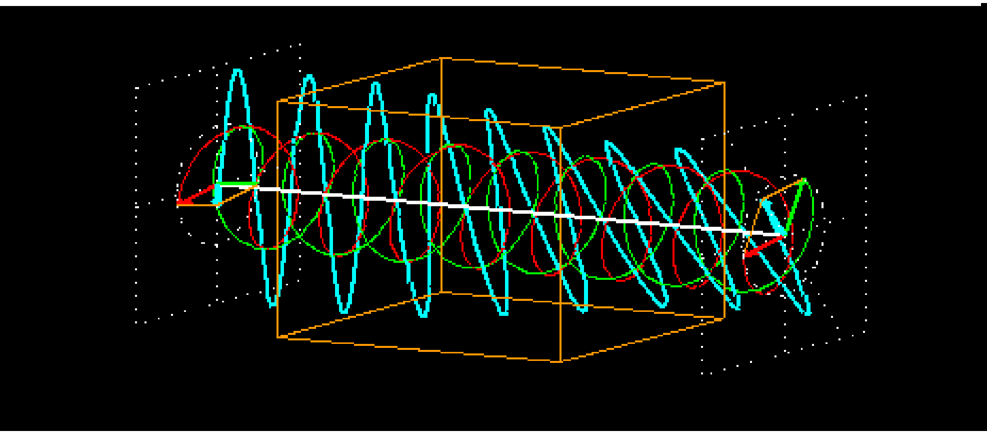
Elliptically birefringent media

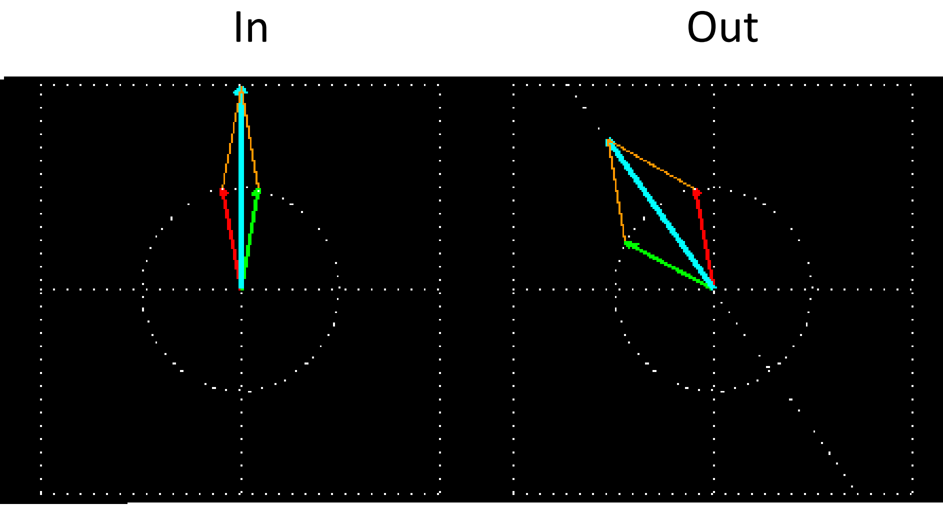
Isotropic medium





Elliptically birerfingent medium





$$E_x = E_{0x} exp\left[i\omega\left(t - \frac{z}{c_n}\right)\right]$$

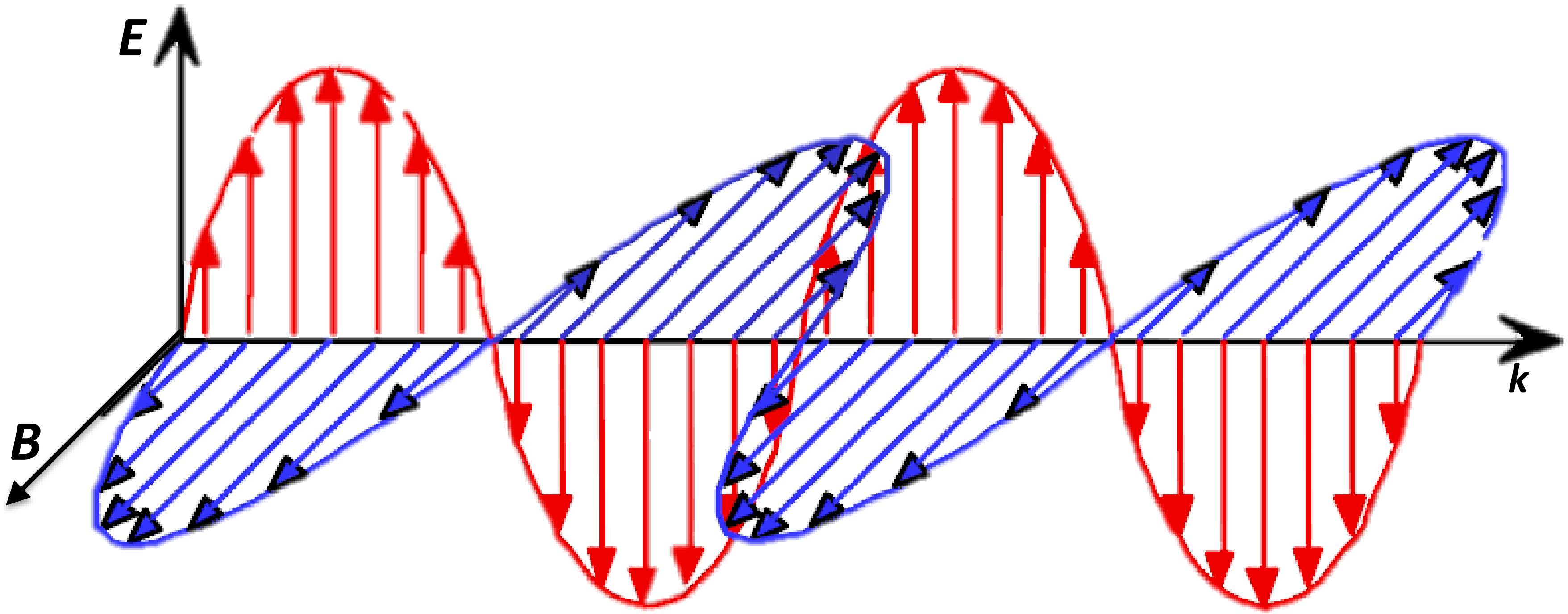
$$E_y = E_{0y} exp\left[i\omega\left(t - \frac{z}{c_n}\right)\right]$$

1. Electromagnetic wave is a transverse wave,
2. Is a plane wave (amplitude m is independent on z coordinate)
3. Vectors E i H are orthogonal to each other,
4. … and oscilates at the same phase,

6. The energy os transported perpendicularly to the wavefront
(but only in anisotropic medium!)

$$\vec{S} = \vec{E} \times \vec{H}$$

7. Refractive index is a dispersive quantity,


E
B
k

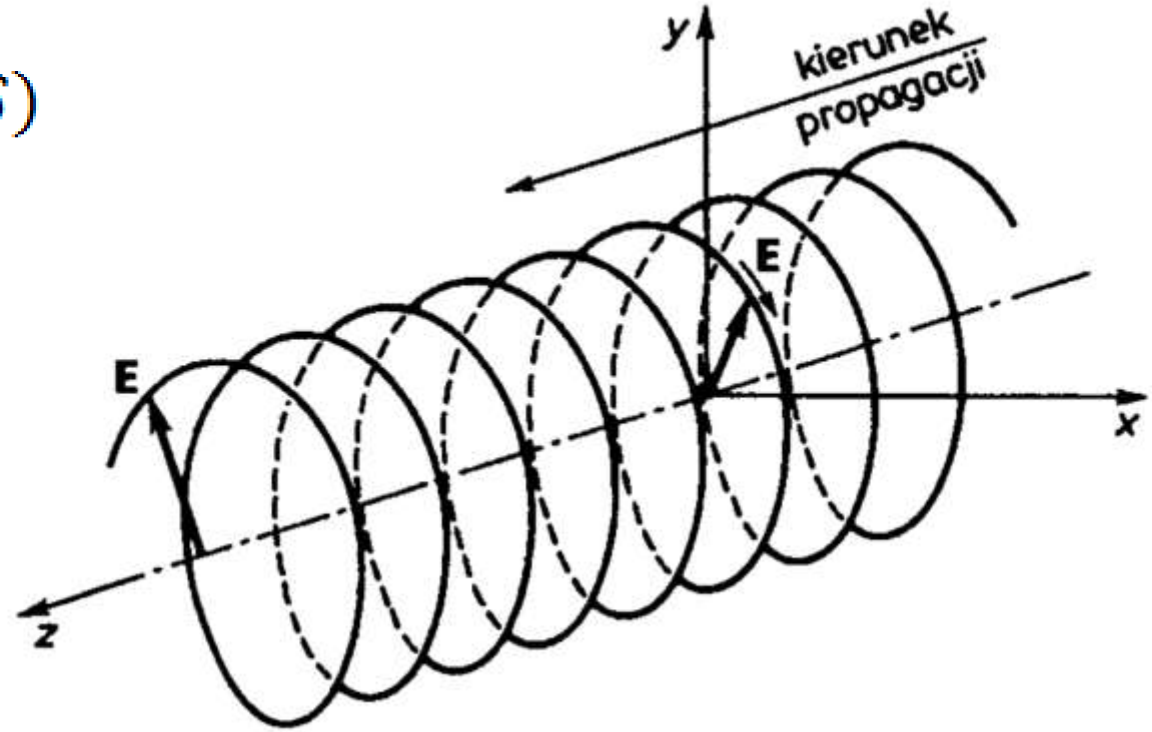
Methods of describing state of polarization for light

Trigonometric method

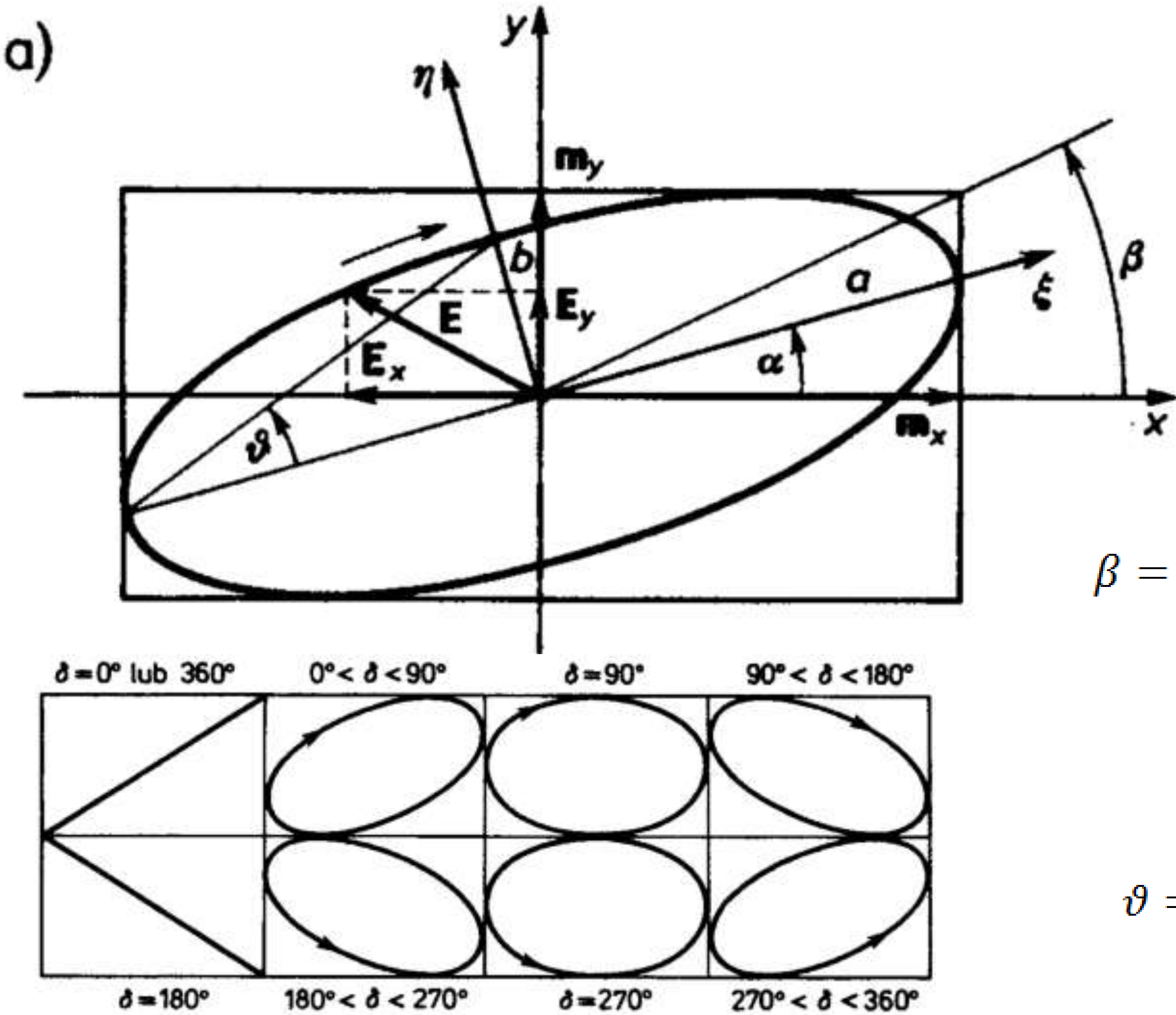
$$E_x = m_x exp\left(i\left[\omega t - \left(\frac{\omega z}{c_n} + \delta_{0x}\right)\right]\right) = m_x exp(\omega t - \delta_x)$$

$$E_y = m_y exp\left(i\left[\omega t - \left(\frac{\omega z}{c_n} + \delta_{0y}\right)\right]\right) = m_y exp(\omega t - \delta_y)$$

$$E_x = m_x \cos(\omega t)$$
$$E_y = m_y \cos(\omega t + \delta)$$



$$\left(\frac{E_x}{m_x}\right)^2 - \frac{2E_xE_y}{m_xm_y}cos(\delta) + \left(\frac{E_y}{m_y}\right)^2 = sin^2(\delta)$$



$\beta = arctg\left(\frac{m_y}{m_x}\right)$ Kąt przekątnej

α azymut

$\frac{b}{a}$ Eliptyczność

$\vartheta = arctg\left(\frac{b}{a}\right)$ Kąt eliptyczności

Jones's vector

$$[E] = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = e^{i\omega t} \begin{bmatrix} m_x \\ m_y e^{i\delta} \end{bmatrix}$$

Or in simplified form

$$[E] = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} m_x \\ m_y e^{i\delta} \end{bmatrix}$$

Jones's vector estimation

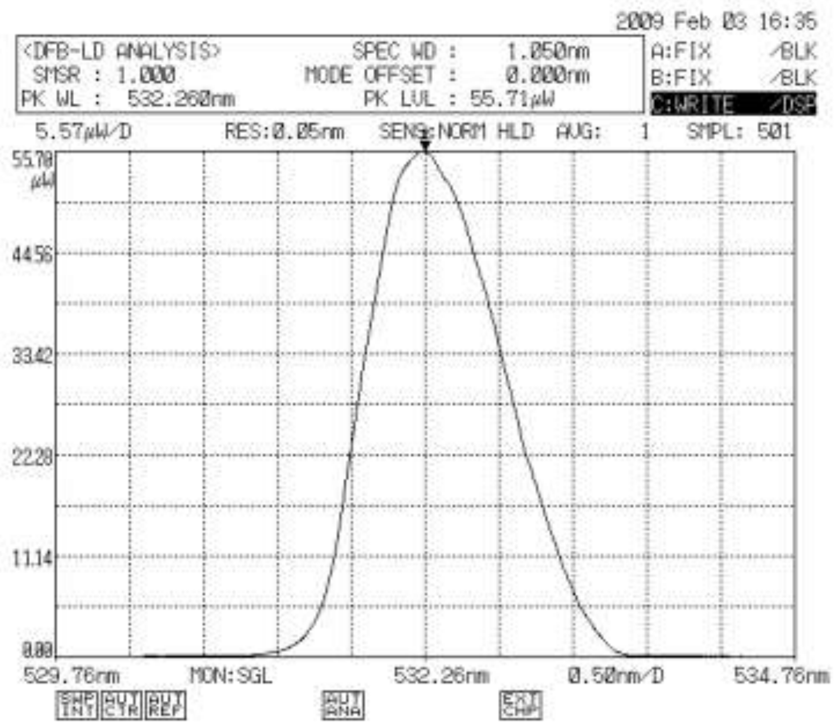
$$[E] = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} m_x \\ m_y e^{i\delta} \end{bmatrix}$$

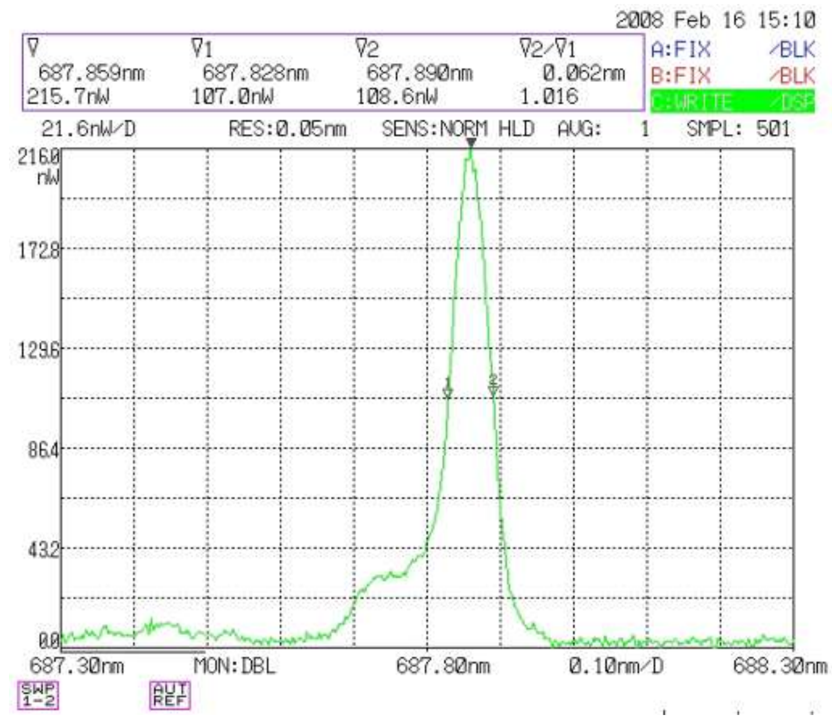
- Light intensity for X and Y axis through polarizer,
- Phase angle δ through compensator,

Symbol	Azymut	Kąt eliptyczności	Standardowy wektor Jonesa	Pełny wektor Jonesa
	0°	0°	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}m_x\\0\end{bmatrix}$
	90°	0°	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}0\\m_y\end{bmatrix}$
	45°	0°	$\frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}m_x\\m_x\end{bmatrix}$
	−45°	0°	$\frac{\sqrt{2}}{2}\begin{bmatrix}1\\-1\end{bmatrix}$	$\begin{bmatrix}m_x\\-m_x\end{bmatrix}$
polaryzacja liniowa ogólnie		0°	$\begin{bmatrix}\cos\beta\\\pm\sin\beta\end{bmatrix}$	$\begin{bmatrix}m_x\\\pm m_y\end{bmatrix}$
	–	−45°	$\frac{\sqrt{2}}{2}\begin{bmatrix}1\\-i\end{bmatrix}$	$\begin{bmatrix}m_x\\m_x e^{-i\pi/2}\end{bmatrix}$
	–	45°	$\frac{\sqrt{2}}{2}\begin{bmatrix}1\\i\end{bmatrix}$	$\begin{bmatrix}m_x\\m_x e^{i\pi/2}\end{bmatrix}$
	0°	26,57°	$0{,}89\begin{bmatrix}0{,}5\\i\end{bmatrix}$	$\begin{bmatrix}m_x\\2m_x e^{i\pi/2}\end{bmatrix}$
polaryzacja eliptyczna ogólnie	α	ϑ	$\begin{bmatrix}\cos\beta\\\sin\beta\ e^{i\delta}\end{bmatrix}$	$\begin{bmatrix}m_x\\m_y e^{i\delta}\end{bmatrix}$

Coherence matrix

For quasi-monochromatic wave





Coherence matrix

$$[K] = \begin{bmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix}$$

Light intensity: $I = Tr[K]$

Polaryzacja eliptyczna $\det [K] = 0;\ \mu_{xy} = e^{i\sigma}$	$[K] = \begin{bmatrix} m_x^2 & m_x m_y e^{i\delta} \\ m_x m_y e^{-i\delta} & m_y^2 \end{bmatrix}$
Polaryzacja kołowa prawoskrętna	$[K] = \frac{I}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Polaryzacja kolowa lewoskrętna	$[K] = \frac{I}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Polaryzacja liniowa $\quad z = 0, 1, 2, \ldots$	$[K] = \begin{bmatrix} m_x^2 & m_x m_y\ (-1)^z \\ m_x m_y (-1)^z & m_y^2 \end{bmatrix}$

- Estimation of coherence matrix elements

$$I(\theta, \delta)$$

$$\begin{aligned}
I(0^\circ, 0^\circ) &= I_{xx}, \\
I(90^\circ, 0^\circ) &= I_{yy}, \\
I(45^\circ, 0^\circ) &= 0{,}5\,(I_{xx} + I_{yy} + I_{xy} + I_{yx}), \\
I(135^\circ, 0^\circ) &= 0{,}5\,(I_{xx} + I_{yy} - I_{xy} - I_{yx}), \\
I(45^\circ, 90^\circ) &= 0{,}5\,(I_{xx} + I_{yy} - \mathrm{i}I_{xy} + \mathrm{i}I_{yx}), \\
I(135^\circ, 90^\circ) &= 0{,}5\,(I_{xx} + I_{yy} + \mathrm{i}I_{xy} - \mathrm{i}I_{yx}).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\,I_{xy} &= \mathrm{Re}\,I_{yx} = 0{,}5\,[I(45^\circ, 0^\circ) - I(135^\circ, 0^\circ)], \\
\mathrm{Im}\,I_{xy} &= \mathrm{Im}\,I_{yx} = 0{,}5\,[I(135^\circ, 90^\circ) - I(45^\circ, 90^\circ)].
\end{aligned}$$

Stokes vector

$$S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} I_{xx} + I_{yy} \\ I_{xx} - I_{yy} \\ I_{xy} + I_{yx} \\ i(I_{yx} - I_{xy}) \end{bmatrix} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \begin{bmatrix} \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\ \langle E_x E_y^* \rangle + \langle E_y E_x^* \rangle \\ i\left(\langle E_y E_x^* \rangle - \langle E_x E_y^* \rangle\right) \end{bmatrix}$$

- can be applied to characterize quasi-monochromatic and partially polarized light
- Jones vector cannot be applied here,

Stokes vector

- For monochromatic wave

$$S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} m_x^2 + m_y^2 \\ m_x^2 - m_y^2 \\ 2m_x m_y \cos\delta \\ 2m_x m_y \sin\delta \end{bmatrix} = I_s \begin{bmatrix} 1 \\ \cos 2\beta \\ \sin 2\beta \cos\delta \\ \sin 2\beta \sin\delta \end{bmatrix} = I_s \begin{bmatrix} 1 \\ \cos 2\vartheta \cos 2\alpha \\ \cos 2\vartheta \sin 2\alpha \\ \sin 2\vartheta \end{bmatrix}$$

Stokes vector

DOP

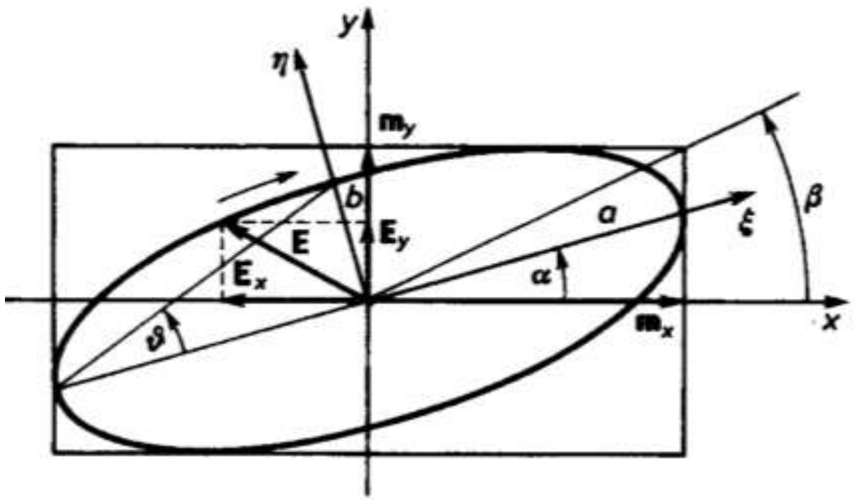
$$P = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \leq 1$$

Ellipticity

$$\frac{b}{a} = \pm \tan^{-1}\left[\frac{1}{2}\sin^{-1}\left(\frac{S_3}{\sqrt{S_1^2 + S_2^2 + S_3^2}}\right)\right]$$

Azimuth

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right)$$



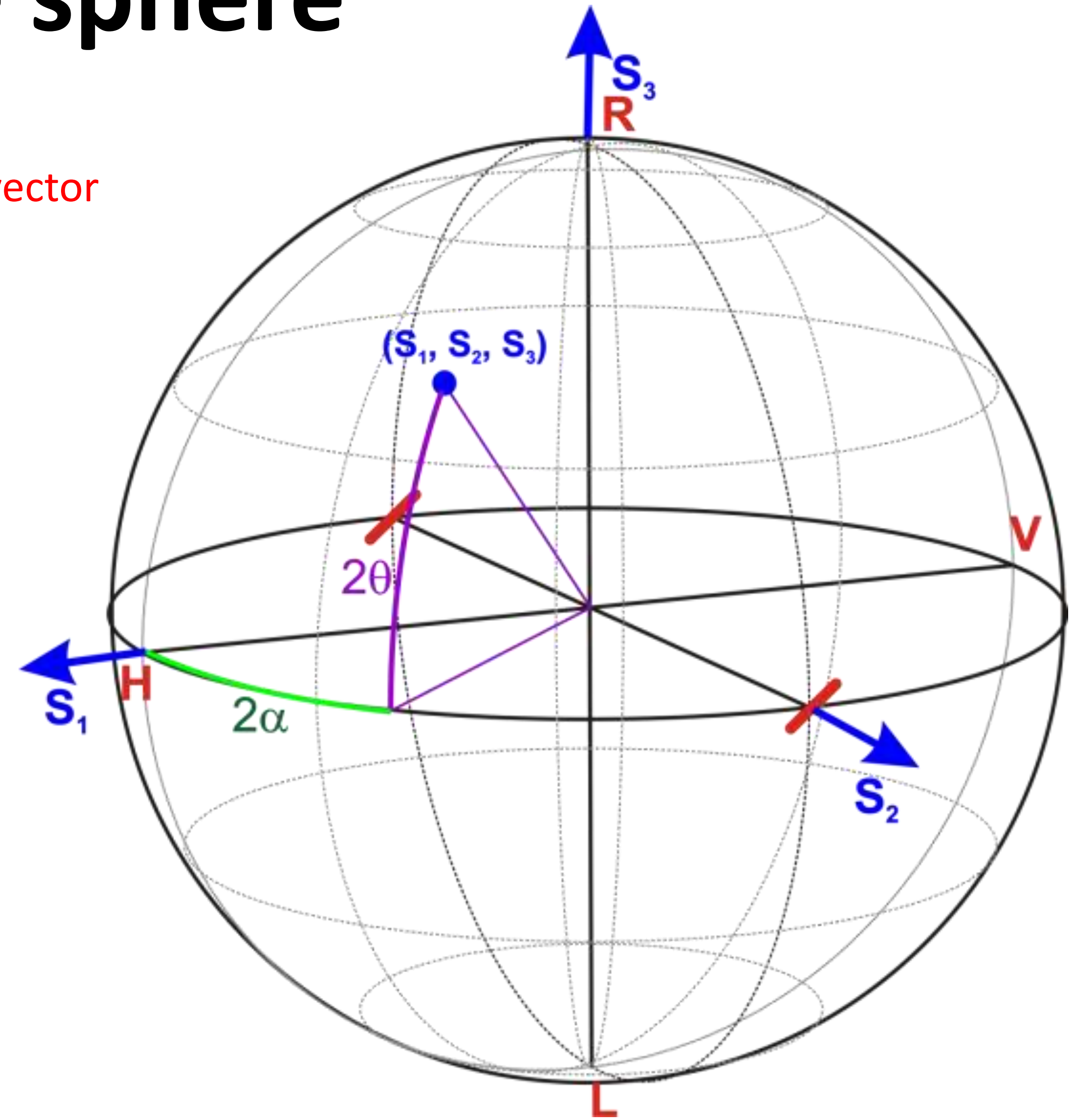
Symbol	Azymut	b/a	Wektor Stokesa	
*	–	–	$I\{1, 0, 0, 0\}$	światło naturalne
—	0°	0	$I\{1, 1, 0, 0\}$	
|	90°	0	$I\{1, -1, 0, 0\}$	
/	45°	0	$I\{1, 0, 1, 0\}$	polaryzacja
\	−45°	0	$I\{1, 0, -1, 0\}$	liniowa
ogólnie	α	ϑ	$I\{1, \cos 2\alpha, \sin 2\alpha, 0\}$	
	–	−1	$I\{1, 0, 0, -1\}$	polaryzacja
	–	1	$I\{1, 0, 0, 1\}$	kołowa
	0°	0,5	$I\{1, 0{,}6, 0, 0{,}8\}$	
	90°	0,5	$I\{1, -0{,}6, 0, 0{,}8\}$	polaryzacja
	22,5°	0,318	$I\{1, \sqrt{1/3}, \sqrt{1/3}, \sqrt{1/3}\}$	eliptyczna
ogólnie	α	$\mathrm{tg}\vartheta$	$I\{1, \cos 2\vartheta \cos 2\alpha, \cos 2\vartheta \sin 2\alpha, \sin 2\vartheta\}$	
ogólnie	α	$\mathrm{tg}\vartheta$	$\{I, pI \cos 2\vartheta \cos 2\alpha, pI \cos 2\vartheta \sin 2\alpha, pI \sin 2\alpha\}$	polaryzacja częściowa

Poincare sphere

Graphical representation of Stokes vector

$$S_1 = I \cos 2\vartheta \cos 2\alpha,$$
$$S_2 = I \cos 2\vartheta \sin 2\alpha,$$
$$S_3 = I \sin 2\vartheta$$

Equator – linear states,
Meridians – states with the same azimuths,
Parallels – states with the same ellipticity,
Poles – circular polarization



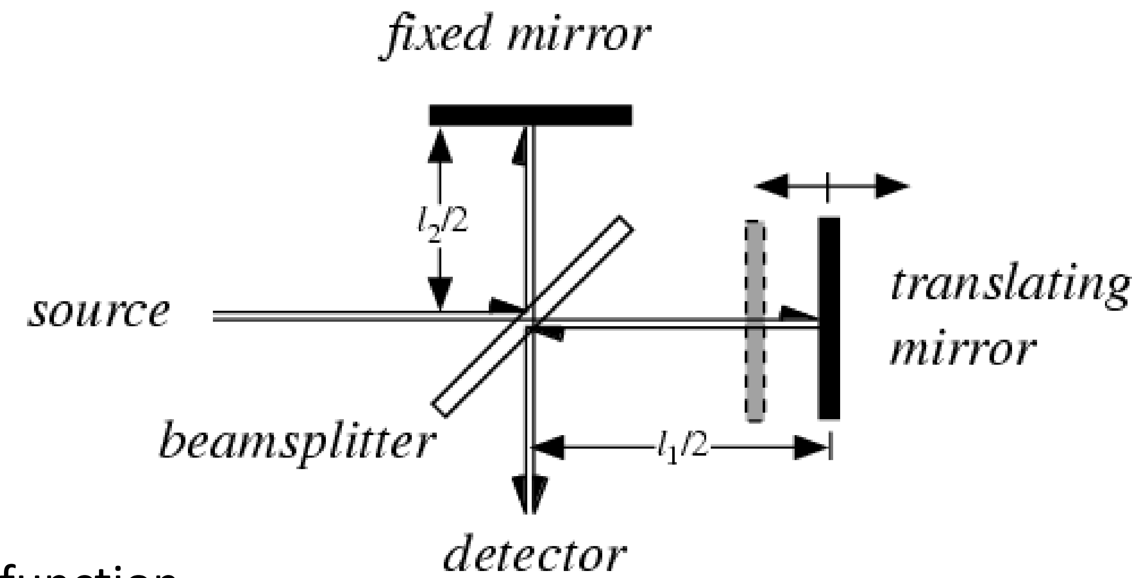
Polarization and interference

Temporar coherence

$$\Delta L = 2(l_2 - l_1) = c\tau_c$$

$$\Delta L = c\Delta t = \frac{c}{\Delta \nu} \approx \frac{\lambda^2}{\Delta \lambda}$$

Michelson's interferometer



γ_{12} - Temporar coherence function

$|\gamma_{12}|$ - Module of temporar coherence function

$$I_d = \left|\left\langle \overrightarrow{E_1} + \overrightarrow{E_2} \right\rangle\right|^2 = \overrightarrow{I_1} + \overrightarrow{I_2} + 2\sqrt{I_1 I_2}\,\gamma_{12}(\tau)$$

$$I = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left\langle EE^* \right\rangle = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left\langle (E_1 + E_2)(E_1 + E_2)^* \right\rangle = I_1 + I_2 + 2\sqrt{\frac{\varepsilon_0}{\mu_0}}\mathrm{Re}\left\{\left\langle E_1^* E_2 \right\rangle\right\}$$

$$\Gamma(\tau) = \left\langle E_1^*(t) E_1(t+\tau) \right\rangle = \lim_{T_m \to \infty} \frac{1}{T_m} \int_{-T_m/2}^{T_m/2} E_1^*(t) E_1(t+\tau)\, \mathrm{d}t$$

$$\gamma(\tau) = \left|\gamma(\tau)\right| \exp(i\beta_{12}) = \frac{\Gamma(\tau)}{\Gamma(0)}$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\left|\gamma(\tau)\right| \cos(\beta_{12} - \Delta\delta),$$

$$\Gamma(\tau) = \int_0^\infty 4G(\nu) \exp(-i2\pi\nu\tau)\, d\nu$$

$$\gamma(\tau) = \int_0^\infty \hat{G}(\nu)\exp(-i2\pi\nu\tau)\,d\nu$$

$$\hat{G}_G(\nu) = \frac{2\sqrt{\ln 2}}{\sqrt{\pi}\Delta\nu}\exp\left[-\left(2\sqrt{\ln 2}\,\frac{\nu-\bar{\nu}}{\Delta\nu}\right)^2\right]$$

$$\hat{G}_L(\nu) = \frac{2(\pi\Delta\nu)^{-1}}{1+2\left(\frac{\nu-\bar{\nu}}{\Delta\nu}\right)^2}$$

$$\gamma_G(\tau) = \exp\left[-\left(\frac{\pi\Delta\nu\tau}{2\sqrt{\ln 2}}\right)^2\right]\exp(-i2\pi\bar{\nu}\tau)$$

$$\gamma_L(\tau) = \exp(-\pi\Delta\nu\tau)\exp(-i2\pi\bar{\nu}\tau)$$

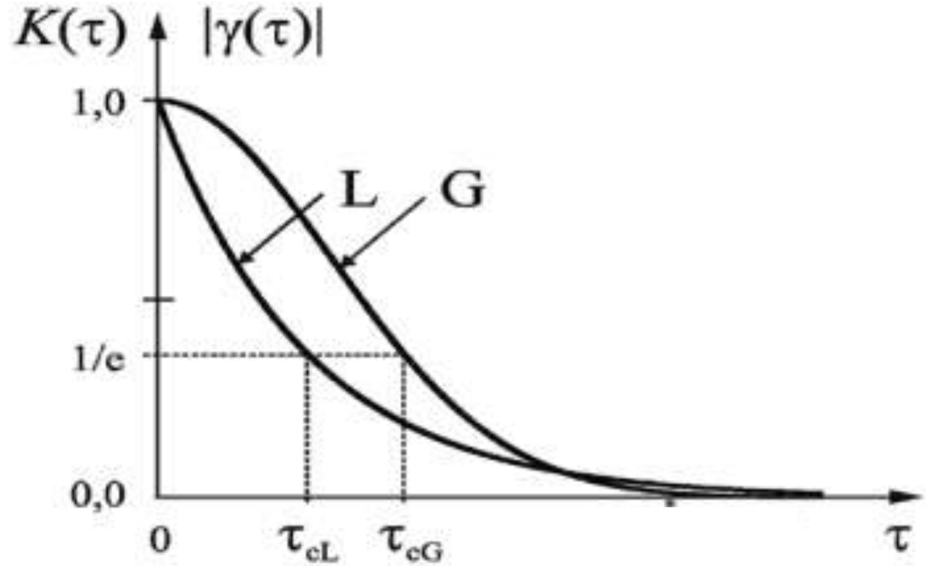
Temporar coherence of light

$$\Delta L = 2(l_2 - l_1) = c\,\tau_c$$

$$\Delta L = c\Delta t = \frac{c}{\Delta f} = \frac{\lambda^2}{\Delta\lambda}$$

$$K(\tau) = \frac{I_{max}(\tau_1) - I_{min}(\tau_2)}{I_{max}(\tau_1) + I_{min}(\tau_2)}$$

$$\mathrm{K}(\tau) = |\gamma(\tau)|$$

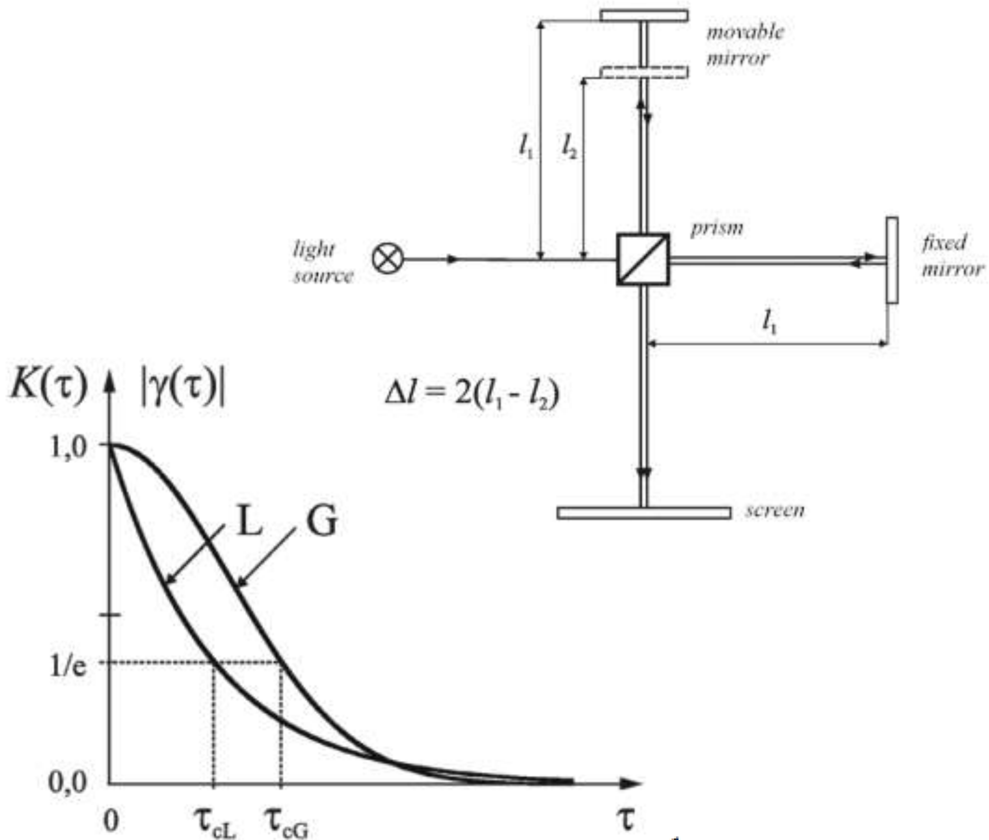


$$\mathrm{I}_{\min}(\tau_2) = 2\mathrm{I}_1 + 2\mathrm{Re}\{\Gamma(\tau_2)\} = 2\mathrm{I}_1 - 2|\Gamma(\tau)|$$

$$I_{max}(\tau_1) = 2\,I_1 + 2\mathrm{Re}\{\Gamma(\tau_1)\} = 2\,I_1 + 2|\Gamma(\tau)|$$

Koherencja czasowa

Opisuje zdolność do interferencji dwóch wiązek świetlnych z tego samego źródła, ale propagujących się w różnych kierunkach (interferometr)



Rozkład natężenia światła można wyrazić zespolonym stopniem koherencji

$$\gamma(\tau) = |\gamma(\tau)| \exp(i\beta_{12}) = \frac{\Gamma(\tau)}{\Gamma(0)}$$

$$\tau = \frac{\Delta l}{c}$$

$$\gamma_G(\tau) = \exp\left[-\left(\frac{\pi \Delta \nu \tau}{2\sqrt{\ln 2}}\right)^2\right] \exp(-i2\pi\bar{\nu}\tau)$$

$$\gamma_L(\tau) = \exp(-\pi \Delta \nu \tau) \exp(-i2\pi\bar{\nu}\tau)$$

Kontrast równy jest stopniowi koherencji

$$K(\tau) = |\gamma(\tau)|$$

Droga koherencji

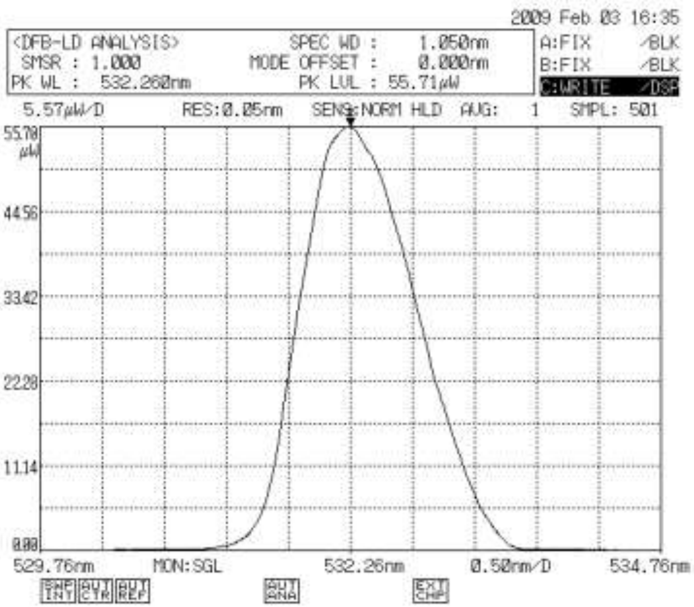
$$\Delta L = c \cdot \tau_c = \xi \frac{\lambda^2}{\Delta \lambda}$$

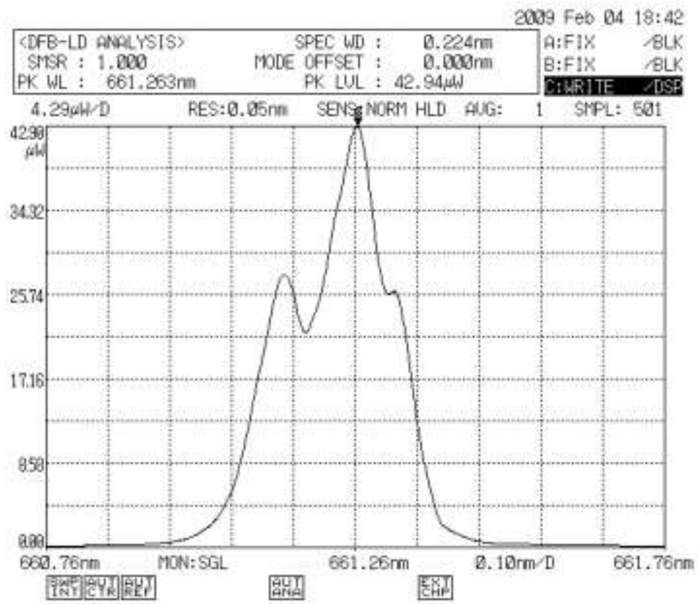
τ_c - Czasowe opóźnienie kiedy kontrast spada do $\frac{1}{e}$ wartości maksymalnej

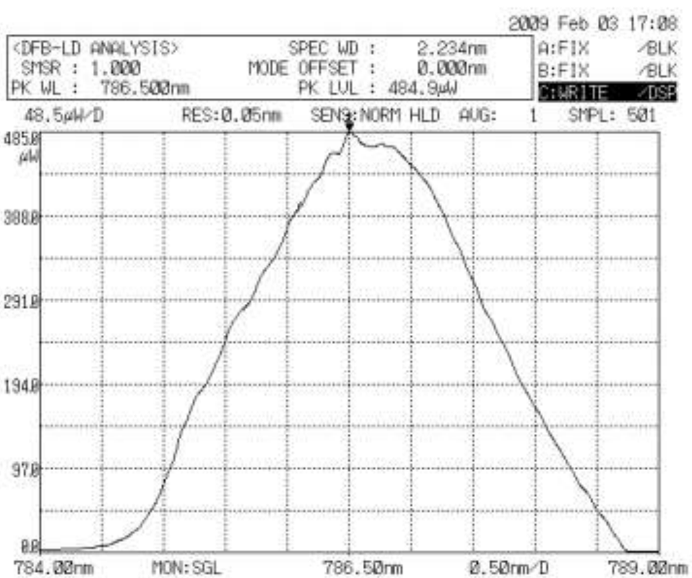
Coherence length

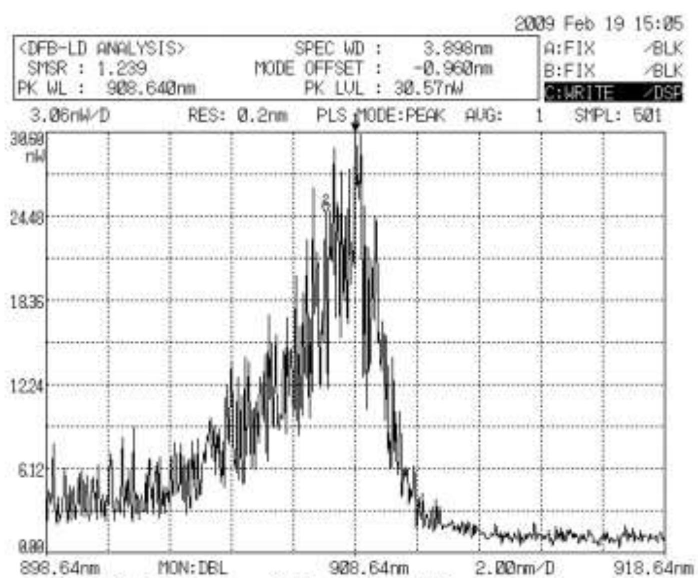
	λ	$\Delta\lambda$	$\Delta L \sim \lambda^2 / \Delta\lambda$
White light	550 nm	750 nm	1μm
LED	850 nm	30 nm ÷50 nm	24 ÷15 μm
	1300 nm	50 nm ÷100 nm	34 ÷17 μm
SLD	850 nm	150 nm	5 μm
	1300 nm	60 nm	28 μm
LD	670 nm	0.2 nm ÷2 nm	2.3 mm ÷230μm
	1300 nm	<2 nm	1mm ÷50m
	1550 nm	<2 nm	1mm ÷50m
He-Ne	632.8 nm	<<0.1 nm	0.1m ÷ 100m
	1152 nm	<<0.1 nm	0.1m ÷ 300m

Light sources







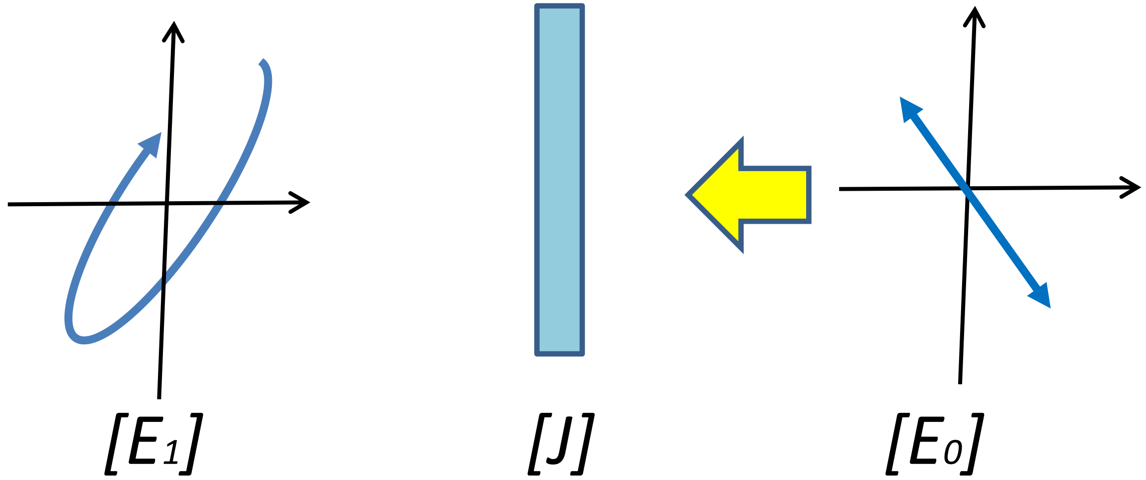


Methods of decribing state of polarization changes

Types of descriptions

- Trigonometric method (time consuming)
- Jones's matrix method,
- Muellera's matrix method,
- Double complex functions method,

Jones's matrix



$$[E_1] = [J][E_0]$$

$$[E_1] = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} [E_0]$$

Jones's matrix

$$[E_1] = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} [E_0]$$

$$E_{1x} = j_{11} E_{0x} + j_{12} E_{0y}$$

$$E_{1y} = j_{21} E_{0x} + j_{22} E_{0y}$$

Jones's matrix

General form (according to Ścierski and Ratajczyk)

- Can be applied to:
  - dichroic media,
  - Nondichroic absorbing media,

$$[J] = \begin{bmatrix} T_f \cos^2\beta_f + T_s \sin^2\beta_f \, e^{-i\gamma} & \sin\beta_f \cos\beta_f (T_f - T_s e^{-i\gamma}) e^{-i\delta_f} \\ \sin\beta_f \cos\beta_f (T_f - T_s e^{-i\gamma}) e^{i\delta_f} & T_f \sin^2\beta_f + T_s \cos^2\beta_f \, e^{-i\gamma} \end{bmatrix},$$

T_f – fast wave transmitance,
T_s – slow wave transmitance,
γ- phase difference in birefringent medium,
β_f – diagonal angle for fast wave,
δ_f – phase difference difining SOP of the fast wave,

Jones's matrix

- According to Jones

$$[J] = \begin{bmatrix} \cos^2\beta_f + \sin^2\beta_f\, e^{-i\gamma} & \sin\beta_f \cos\beta_f\, (1 - e^{-i\gamma})\, e^{-i\delta_f} \\ \sin\beta_f \cos\beta_f\, (1 - e^{-i\gamma})\, e^{i\delta_f} & \sin^2\beta_f + \cos^2\beta_f\, e^{-i\gamma} \end{bmatrix}.$$

Can be applied to:

- nonabsorbing media,
- Nondichroic waveplates,

Jones's matrix

- Examples:

For free space $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For isotropic medium with transmittance p $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$

Linear polarizer, azimuth 0 $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Linear polarizer, azimuth 90 $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Linear polarizer, azimuth 45 $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Right-handed circular polarizer $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

Left-handed circular polarizer $\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

QWR with fast axis at 0 $\begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$

HWR with fast axis at 45 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Wave plate with phase shift δ and fast axis at 0 $\begin{bmatrix} e^{i\delta}\cos^2\theta + \sin^2\theta & (e^{i\delta} - 1)\sin\theta\cos\theta \\ (e^{i\delta} - 1)\sin\theta\cos\theta & e^{i\delta}\sin^2\theta + \cos^2\theta \end{bmatrix}$

Jones's matrix

Example:

Linearly polarized light passing through QWR:

$$[E] = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-i\pi/4} \\ e^{i\pi/4} \end{bmatrix}.$$

- Can be applied only for completely polarized light.
- Jones's vectors for natural light or partially polarized does not exist,

Muellers' matrix

$$[S] = [M][S_0]$$

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} S_{00} \\ S_{01} \\ S_{02} \\ S_{03} \end{bmatrix}$$

Mueller matrix

According to Ścierski – for dichroic media

$$[M] = \begin{bmatrix} T^+ & MT^- & CT^- & ST^- \\ MT^- & M^2Z+X & CMZ+YS & SMZ-YC \\ CT^- & MCZ-YS & C^2Z+X & SCZ+YM \\ ST^- & MSZ+YC & CSZ-YM & S^2Z+X \end{bmatrix}$$

T – transmitance of the amplitude

$$\begin{aligned} T^+ &= (T_f^2+T_s^2)/2, & M &= \cos 2\vartheta_f \cos 2\alpha_f, \\ T^- &= (T_f^2-T_s^2)/2, & C &= \cos 2\vartheta_f \sin 2\alpha_f, \\ X &= T_f T_s \cos\gamma, & S &= \sin 2\vartheta_f, \\ Y &= T_f T_s \sin\gamma, & Z &= T^+ - X. \end{aligned}$$

Mueller matrix

For free space

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Absorbing medium with transmittance k

$$\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix}$$

Linear polarizer with azimuth θ

$$\frac{1}{2}\begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \cos 2\theta \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Mueller matrix

Linear polarizer with azimuth 0

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear polarizer with azimuth 90

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear polarizer with azimuth 45

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Mueller matrix

For wave-plate with azimuth θ and phase retardation δ

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \sin^2 2\theta\cos\delta & (1-\cos\delta)\sin 2\theta\cos 2\theta & -\sin 2\theta\sin\delta \\ 0 & (1-\cos\delta)\sin 2\theta\cos 2\theta & \sin^2 2\theta + \cos^2 2\theta\cos\delta & \cos 2\theta\sin\delta \\ 0 & \sin 2\theta\sin\delta & -\cos 2\theta\sin\delta & \cos\delta \end{bmatrix}$$

QWR with azimuth 0

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

QWR with azimuth 45

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

QWR with azimuth -45

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

QWR with azimuth 90

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Mueller matrix

For wave-plate with azimuth θ and phase retardation δ

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2\theta + \sin^2 2\theta \cos\delta & (1-\cos\delta)\sin 2\theta \cos 2\theta & -\sin 2\theta \sin\delta \\ 0 & (1-\cos\delta)\sin 2\theta \cos 2\theta & \sin^2 2\theta + \cos^2 2\theta \cos\delta & \cos 2\theta \sin\delta \\ 0 & \sin 2\theta \sin\delta & -\cos 2\theta \sin\delta & \cos\delta \end{bmatrix}$$

HWR with azimuth 0

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

HWR with azimuth 22,5

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

HWR with azimuth -22,5

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

HWR with azimuth 45

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Mueller’s matrix

$$[M_{\alpha_0+\alpha}] = [T_{-\alpha}][M_{\alpha_0}][T_{\alpha}],$$

$$[T_{-\alpha}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[T_{\alpha}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

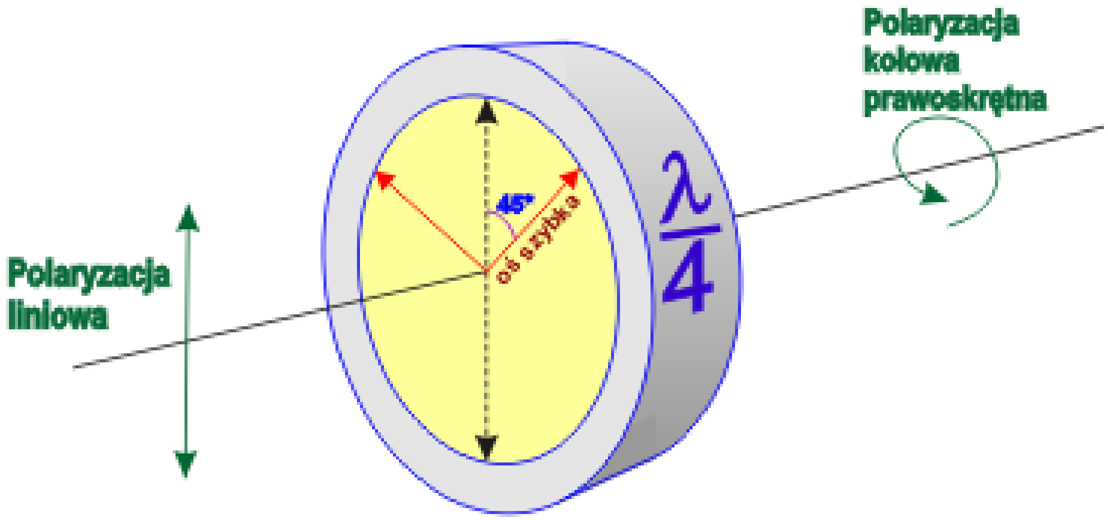
Mueller's matrix

- Example

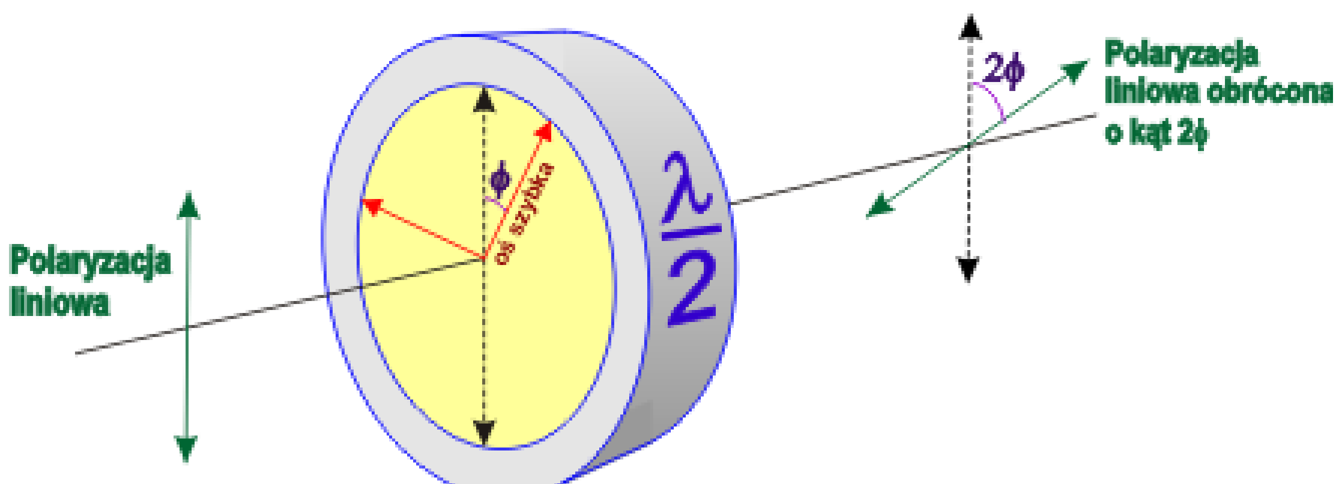
A depolarized light passing through polarizer and QWR

$$[S] = \overset{\text{ćwierćfalówka}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \overset{\text{polaryzator}}{\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \overset{\text{swiatło naturalne}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Ćwierćfalówka przesuwa fazy składowych o ćwierć długości fali.

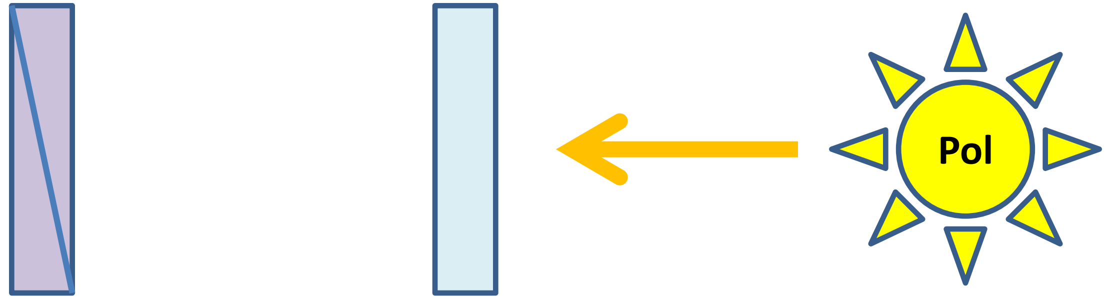


Półfalówka przesuwa fazy składowych o pół długości fali.



Task

1. How to estimate experimentally fast and slow axis of the waveplate using linear polarizer and linearly polarized light beam? Can we distinguish fast and slow axis?



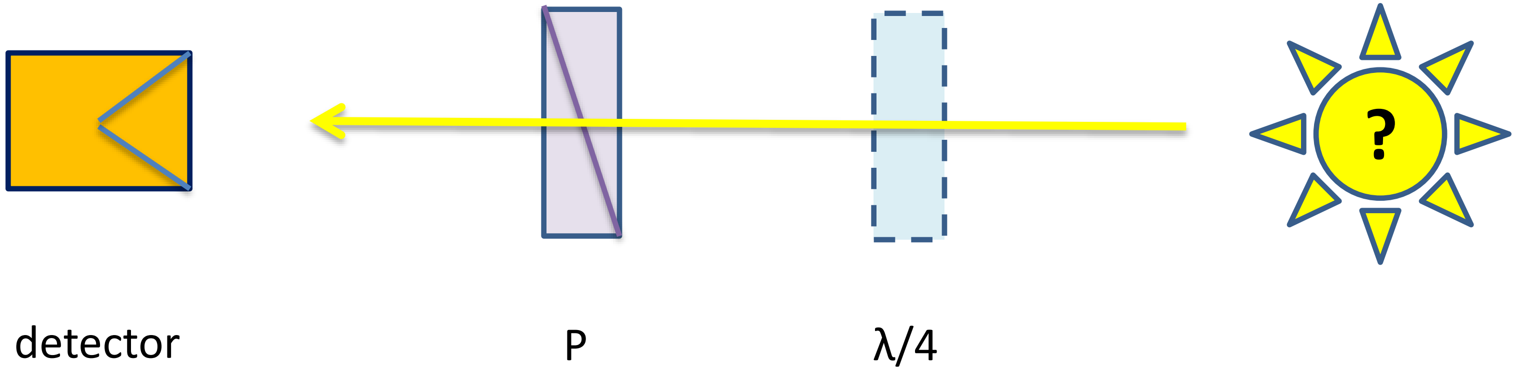
$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Methods of measuring SOP

- 6 (or 4) light intensity measurements using polarizer and QWR,
- Method based on circular polarizer,
- Rotating QWR method,
- Null-intensity method,

Method 1

QWR and linear polarizer,
6 light intensities with different configurations of P and QWR.



$$[S] = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \begin{bmatrix} I_{(0,0^\circ)} + I_{(0,90^\circ)} \\ I_{(0,0^\circ)} - I_{(0,90^\circ)} \\ I_{(0,45^\circ)} - I_{(0,135^\circ)} \\ I_{\left(\frac{\lambda}{4},45^\circ\right)} - I_{\left(\frac{\lambda}{4},135^\circ\right)} \end{bmatrix}.$$

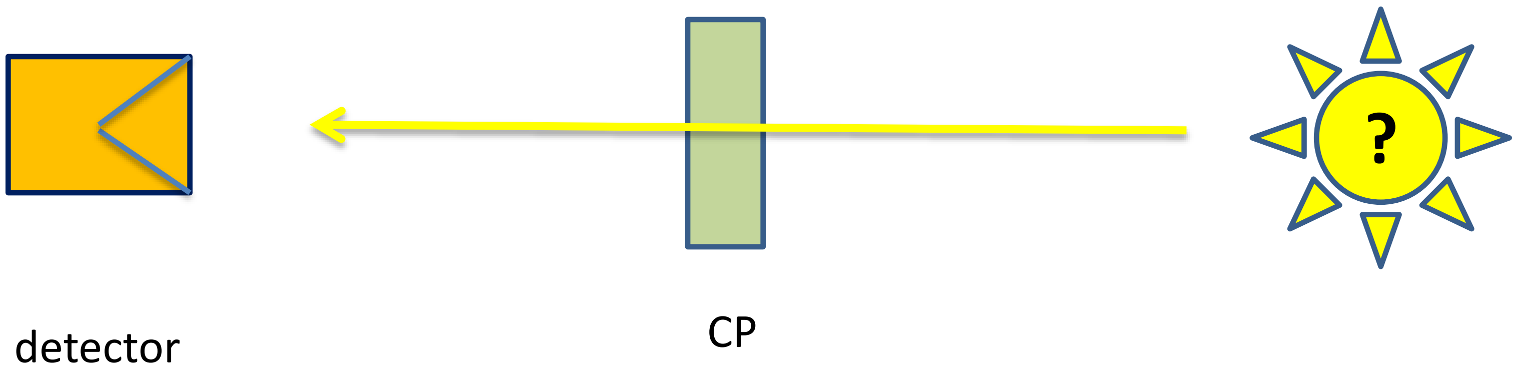
Method 1

- 6 measurements can be simplified to 4 measurements:

$$[S] = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \begin{bmatrix} I_{(0,0^\circ)} + I_{(0,90^\circ)} \\ I_{(0,0^\circ)} - I_{(0,90^\circ)} \\ 2I_{(0,45^\circ)} - I_{(0,0^\circ)} - I_{(0,90^\circ)} \\ 2I_{\left(\frac{\lambda}{4},45^\circ\right)} - I_{(0,0^\circ)} - I_{(0,90^\circ)} \end{bmatrix}.$$

Method 2

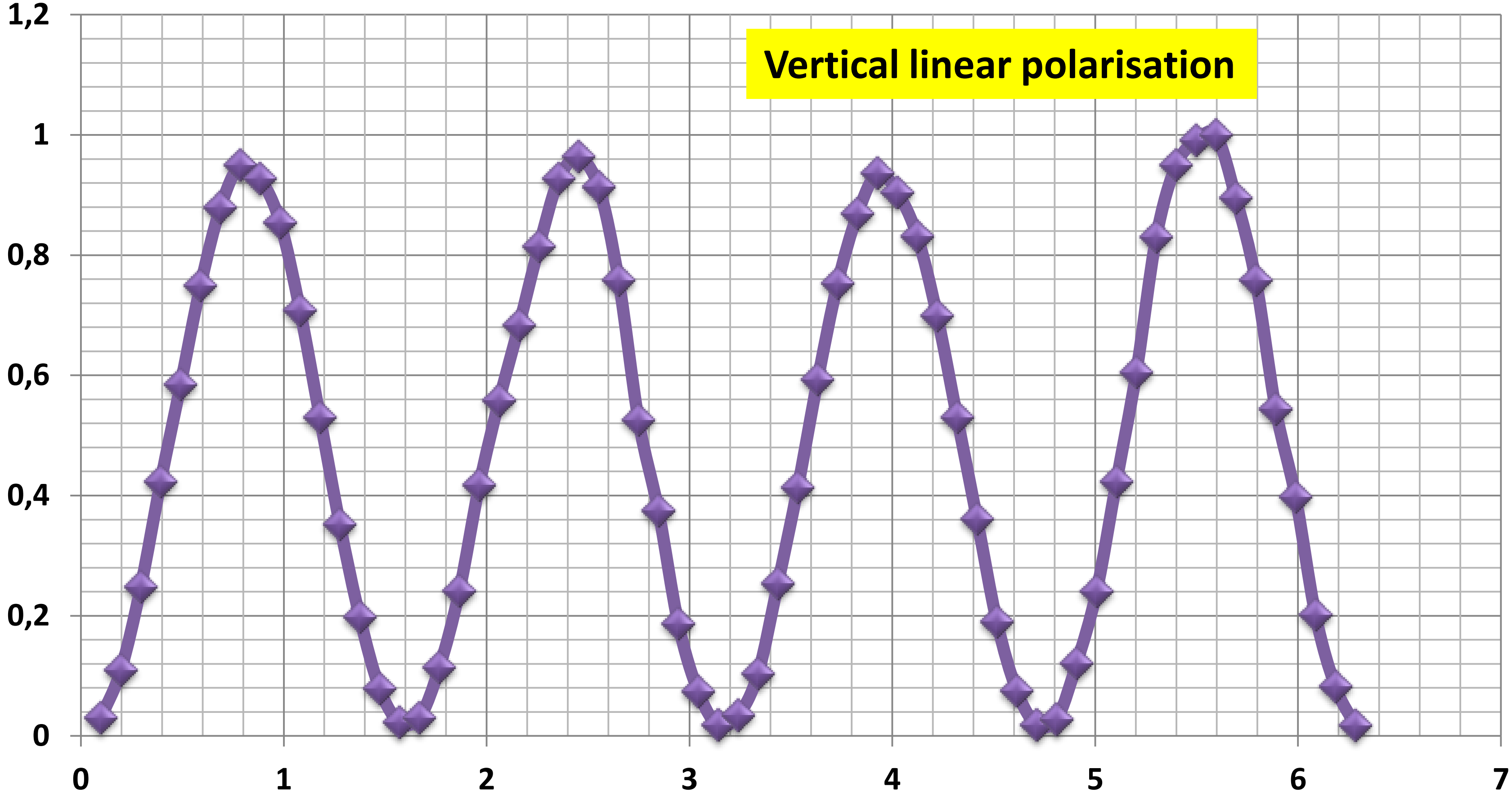
- Using circular polarizer



$$[S] = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} I_c(0°) + I_c(90°) \\ S_0 - 2I_c(45°) \\ I_c(0°) - I_c(90°) \\ S_0 - 2I_L(0°) \end{bmatrix}$$

Method 3

- Rotating QWR,
  - Commonly used in commercial devices,


Vertical linear polarisation
1,2
1
0,8
0,6
0,4
0,2
0
0
1
2
3
4
5
6
7

Method 3

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi}{T}x\right) + b_n \sin\left(\frac{2n\pi}{T}x\right)\right)$$

where:

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi}{T}x\right)dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi}{T}x\right)dx, \quad n = 1, 2, 3, \ldots$$

$$[S] = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_0 - a_4 \\ 2a_4 \\ 2b_4 \\ b_2 \end{bmatrix}$$