Generalized Lévy walks and thermalized stochastic time series

R. Kutner

Physics Department, Warsaw University, Hoża 69, Pl-00681 Warsaw, Poland

Abstract

We developed generalized Lévy walks with varying velocity (shorter called the Weierstrass walks model) by which one can describe both stationary and non-stationary stochastic time series. Essentially, our approach is based on the continuous-time random walk formalism where the waiting-time distribution or memory kernel and sojourn probability density play a fundamental role. We considered a non-Brownian random walk where the walker moves, in general, with a velocity that assumes a different constant value between the successive turning points, i.e. the velocity is a piecewise constant function. So far models were developed that took into account only one chosen value of this velocity and therefore were unable to consider more realistic stochastic time series. Moreover, our model is a kind of Lévy walks where we assume a hierarchical, self-similar in the stochastic sense, spatio-temporal representation of waiting-time distribution and sojourn probability density. The Weierstrass walks model makes possible to analize both the structure of the Hurst exponent and the power-law behavior of kurtosis. This structure results from the hierarchical, spatio-temporal coupling between the walker displacement and the corresponding time of the walks. The analysis makes use of both the fractional diffusion and the super Burnett coefficients. We constructed the diffusion phase diagram which distinguishes regions occupied by classes of different universality. We study only such classes which are characteristic for stationary situations. We proved that even after performing the moving-averaging over the stochastic time series (being a kind of thermalization) which makes results stationary in the sense that they are already independent of the beginning moment of the random walk, it is still possible to see the non-Gaussian features of the basic stochastic process. We thus have a model ready for describing the data presented, e.g., in the form of moving-averages; the operation is often used for stochastic time series, especially financial ones. The more so, the model was inspired by properties of financial time series and tested for empirical data extracted from the Warsaw stock exchange since it offers an opportunity to study in an unbiased way several features of stock exchange in its early stage. Basing on the hierarchical representation of waiting-time distribution we introduced the efficient Monte Carlo algorithm which makes possible a numerical simulation of individual runs of stochastic time series; this facilitates the study of empirical stochastic time series.

Key words: Lévy walks, Lévy walks with varying velocity, Weierstrass walks, continuous-time reandom walk, fractional random walk, anomalous diffusion, hierarchical stochastic spatio-temporal coupling, dynamic diffusion coefficient, super Burnett coefficient, diffusion exponent, Hurst exponent, kurtosis, diffusion phase diagram, moving-average, financial time-series, Monte Carlo algorithm
PACS: 89.90.+n, 05.40+j

Preprint submitted to Kluwer Publ. *2 January 2002*

1. Introduction

The evolution of many complex systems [1,2] in natural, economical and social sciences is usually presented in the form of time series [3,4]. Theories of time series [5] apply to an increasing amount of branches of knowledge thanks to the progress in semi-analytical methods and numerical recipes. In this work we mainly consider time series which realize non-Gaussian stochastic processes observed, e.g., on speculative markets [6–10].

As basic stochastic process we consider the Weierstrass walks (WW) model [11] [1] assuming hierarchical, self-similar in the stochastic sense, spatio-temporal representation [2] of the waiting-time distribution (WTD) and sojourn probability density (SPD). Within the WW model we consider a fractional (non-Brownian) random walk of a walker having, in general, different velocities between the successive turning points. These velocities and the corresponding distances passed by the walker consitute a self-similar hierarchy of dynamical variables which is a foundation for the corresponding hierarchy of probability densities. The Weierstrass walks model is a kind of Lévy walks [13,14] which makes possible to construct a diffusion phase diagram (DPD) in terms of two partial diffusion exponents: (i) the first (β) defining the fractal nature, in the stochastic sense, of the spatial variable, and analogusly (ii) the second (α) defining the fractal character of the random time intervals. The DPD follows from the spatio-temporal coupling which is an essential feature of empirical stochastic time series, especially financial ones. The above mentioned probability densities WTD and SPD (which relate to the non-local memory kernel) play the main role in the Continuous-Time Random Walk (CTRW) formalism [15] applied here. Hence, the first stage is to construct the WTD and SPD, which obey the requirements introduced below.

In the context of financial markets [16,17] we can make several observations.

(i) There are no instantaneous variations of any economic index, since the time of each operation is finite and the amount of money flowing through the market is limited.
(ii) There exists a probabilistic spatio-temporal coupling which connects the probability of index variation with the probability that it occurs in a given time lag. To reproduce the observed scaling phenomenon and its breaking this coupling should have a hierarchical, self-similar structure.
(iii) The moving-average procedure is usually performed on the stochastic time series. This type of procedure can lead to a stationary situation, which can be accounted for the Weierstrass walks model.
(iv) We take into account here the time series exhibiting the space variable which always walks (but doesn't rest).

2. Basic quantities

Non-stationary situation. In the framework of the CTRW formalism a single variation of a stock market index (or displacement of a hypothetical walker, also called 'financial ant') can be characterized by the waiting-time distribution $\psi(x,t)$, which is the probability density of a single displacement (event) x in its duration time t; after time t the financial ant begins its next single event (walk). According to our previous work [11] this WTD is defined by the following geometric series

$$\psi(x,t) = \frac{1}{2}(1-\frac{1}{N})\sum_{j=0}^{\infty}\frac{1}{N^j}\delta(\mid x\mid - v^j t) \times \frac{1}{\tau^j}\exp(-\frac{t}{\tau^j}); \quad (1)$$

the sojourn probability density $\Psi(x,t)$ has an analogous form (but without factor $1/\tau^j$ under the sum). The above representations of WTD and SPD are an extension of the corresponding expressions of the known simple Lévy walks model [13,14]. Note that the above formula permits even very long single-step walks of the financial ant. Both ψ and Ψ distributions were already discussed in [11]. Here,

Email address: erka@fuw.edu.pl (R. Kutner).

[1] The term 'Weierstrass walks' is exploited here in a wider sense than that introduced by E.W. Montroll and M.F. Shlesinger [12].

[2] The term 'spatio' concerns here any stochastic variable, for example, the variation of price on a stock market.

these quantities constitute the basis for the construction of analogous distributions required for the stationary situation. They give also a basis for developing an efficient Monte Carlo algorithm [11].

Stationary situation. To have the system in a stationary situation the first step of the walker requires a special treatment, i.e. proper averaging over initial time of the process or performing a moving-average procedure which leads to the disappearing of the initial time influence. By extending the conditional probabilities techniques [15] one can derive the waiting-time distribution $\chi(x,t) = \Psi(x,t)/\langle t\rangle$ for the stationary situation, where $\langle t\rangle$ must be the finite (and positive) mean time needed for the particle to perform a single step, calculated within the basic, non-stationary process; this quantity is finite if and only if $\alpha(= \ln N/\ln\tau) > 1$ which is our principle constraint.

We also derive the sojourn probability density

$$\Xi(x,t) = \frac{1}{2}(1-\frac{1}{N})\sum_{j=0}^{\infty}(\frac{\tau}{N})^j \delta(\mid x \mid - v^j t) \times \frac{1}{\langle t\rangle}\exp(-t/\tau^j). \quad (2)$$

Below we used an approximated (closed form of WTDs and SPDs) up to the fourth order in the Fourier variable k.

3. Stationary Weierstrass walks

To construct the diffusion phase diagram in the stationary situation we calculate: (i) the variance of the displacement, (ii) velocity autocorrelation function and (iii) the kurtosis. To reach this aim the propagator $P_{st}(X,t)$ is required.

Propagator. In the frame of the CTRW formalism the multi-step walk consists of an infinite sequence of single steps described both by stationary and non-stationary WTDs and SPDs. Hence the propagator, $P_{st}(X,t)$, is represented by the sum of partial space-time convolutions of increasing order. This leads, in the Fourier-Laplace domain, to the closed form (being the sum of the infinite geometric series)

$$\tilde{P}_{st}(k,u) = \tilde{\Xi}(k,u) + \tilde{\chi}(k,u)\tilde{P}(k,u), \quad (3)$$

Table 1
Diffusion exponent η and fractional diffusion coefficient D_{st}.

Exponent β	Exponent η	Coefficient D_{st}
$\frac{1}{\beta} < \frac{1}{2}$	1	$\frac{1}{2}\frac{\langle x^2\rangle}{\langle t\rangle}$
$\frac{1}{\beta} > \frac{1}{2}$	$1+2\alpha(\frac{1}{\beta}-\frac{1}{2})$	$\frac{1}{\langle t\rangle}\frac{1-\frac{1}{N}}{\ln N}\frac{\pi\alpha}{\sin\{2\pi\alpha(\frac{1}{\beta}-\frac{1}{2})\}}$

where $\tilde{F}$ denotes the Fourier-Laplace transform of F, and $\tilde{P}(k,u) = \tilde{\Psi}(k,u)/\{1-\tilde{\psi}(k,u)\}$ is the propagator of the non-stationary Weierstrass walks. Both the stationary and non-stationary WTDs and SPDs obey (nonhomogeneous) scaling relations of the type considered in [11]. This leads to the non-Gaussian behavior of the financial ant.

MSD and VAF. The mean-square displacement (MSD), involving infinitely many steps of the walker or a time-dependent variance of displacement, is an important, commonly used stochastic quantity.

This quantity is derived below in the Laplace domain (for $\mid u \mid \ll 1$),

$$\langle \tilde{X}(u)^2\rangle_{st} = -\nabla_k^2 \tilde{P}_{st}(k,u)\mid_{k=0} \approx -\nabla_k^2\tilde{\Xi}(k,u)\mid_{k=0} - \frac{1}{u}\nabla_k^2\tilde{\chi}(k,u)\mid_{k=0} + \langle \tilde{X}(u)^2\rangle, \quad (4)$$

where $\langle \tilde{X}(u)^2\rangle$ is the MSD for the non-stationary Weierstrass walks found earlier [11] in a closed form. By applying the Mellin transform and residues techniques one can find the second moment for asymptotic time t (or $\mid u \mid \ll 1$):

$$\langle X(t)^2\rangle_{st} \approx 2\frac{D_{st}}{\Gamma(1+\eta)}t^{\eta}. \quad (5)$$

This result is finite if and only if the constraint

$$\frac{1}{\beta} < \frac{1}{2} + \frac{1}{2\alpha} \quad (6)$$

is obeyed.

The dependence of the fractional diffusion coefficient D_{st} and diffusion exponent $\eta(= 2H)$ on the basic exponents α and $\beta(= \ln b/\ln N$, here $b = v\tau)$ is shown in Table 1. Note that MSD of a single step performed at any time $\langle x^2\rangle = 2(1-1/N)/(1-b^2/N)$ used in Table 1 is finite if only if $1/\beta < 1/2$, otherwise it diverges.

From relation (4) we derived (also for $\mid u \mid \ll 1$) the dynamic velocity autocorrelation function (VAF)

$$\tilde{C}_{st}(u) = \frac{1}{2} u^2 \langle \tilde{X}(u)^2 \rangle_{st} \sim u^{-2\alpha(1/\beta - 1/2)} \tag{7}$$

or dynamic diffusion coefficient; the real part of VAF represents the low-frequency diffusion coefficient if we assume $u = i\omega$. Although for $1/\beta < 1/2$ the mean-square displacement $\langle X(t)^2 \rangle_{st}$ increases linearly with time, the random walk of the financial ant is still non-Brownian since its VAF doesn't exponentially decay for asymptotic time.

The formulae given here make possible to extract from the empirical time series only the Hurst and not the α and β exponents. The latest are obtained by introducing the fourth moment of displacement.

Kurtosis. By using propagator (3) one obtains the formula for the fourth moment of displacement (FMD) for $\mid u \mid \ll 1$, in the stationary situation

$$\begin{aligned} \langle \tilde{X}(u)^4 \rangle_{st} \approx \nabla_k^4 \tilde{\Xi}(k,u) \mid_{k=0} + \frac{1}{u} \nabla_k^4 \tilde{\chi}(k,u) \mid_{k=0} \\ + \langle \tilde{X}(u)^4 \rangle - \langle \tilde{X}(u)^2 \rangle \nabla_k^2 \tilde{\chi}(k,u) \mid_{k=0} . \end{aligned} \tag{8}$$

By performing a calculation analogous to that made above for $\langle X(t)^2 \rangle_{st}$ one derives the asymptotic formula for FMD

$$\langle X(t)^4 \rangle_{st} \approx 4! \frac{D_{st}^2}{\Gamma(1+\eta_2)} t^{\eta_2}, \tag{9}$$

where $D_{st}^2 u^{1-\eta_2}$ is the dynamic super Burnett coefficient [15], while D_{st}^2 is referred to as fractional super Burnett coefficient. The above introduced FMD is finite if and only if the following constraint

$$\frac{1}{\beta} < \frac{1}{4} + \frac{3}{4\alpha} \tag{10}$$

is obeyed. This is our <u>second</u> constraint; it is stronger than the previous one (6) related to eq.(5) and consitutes the upper border (red tangent straight line) of our diffusion phase diagram shown in Fig.1.

In general, coefficient D_{st}^2 and exponent η_2 depend on α and β (c.f. Table 2, where we used the notation $\Pi = \{4\alpha(\frac{1}{\beta} - \frac{1}{4}) - 1\}\{2\alpha(\frac{1}{\beta} - \frac{1}{4}) - 1\}$).

Combining expression (5) and relation (9) after its back transformation to the time domain

Table 2
Exponent η_2 and fractional super Burnett coefficient D_{st}^2.

Exponent β	Exponent η_2	Coefficient D_{st}^2
$\frac{1}{\beta} < \frac{1}{4} + \frac{1}{4\alpha}$	2	$(D_{st})^2$
$\frac{1}{\beta} > \frac{1}{4} + \frac{1}{4\alpha}$	$1 + 4\alpha(\frac{1}{\beta} - \frac{1}{4})$	$\frac{1}{\langle t \rangle} \frac{1-\frac{1}{N}}{\ln N} \frac{\pi\alpha}{\sin\{4\pi\alpha(\frac{1}{\beta} - \frac{1}{4})\}} \Pi$

one finds the time-dependent kurtosis $\kappa_{st}(t) = \langle X(t)^4 \rangle_{st} / \{\langle X(t)^2 \rangle_{st}\}^2$ for the stationary situation. We distinguish three different cases (see Fig.1): enhanced diffusion (ED), the semi-regular one (SRD) and the regular (RD) one. The particularly interesting one concerns enhanced diffusion which is defined by

$$\frac{1}{2} < \frac{1}{\beta} < \frac{1}{4} + \frac{3}{4\alpha}. \tag{11}$$

Then one has

$$\kappa_{st}(t) \approx 3! \frac{D_{st}^2}{\{D_{st}\}^2} \frac{\{\Gamma(1+\eta)\}^2}{\Gamma(\alpha + 2\eta)} t^{\alpha - 1}. \tag{12}$$

For the region of enhanced diffusion we are able to find exponent α from the asymptotic time dependence of kurtosis; hence, we are able to calculate the partial diffusion exponent β (since the diffusion exponent η is obtained directly from MSD).

Diffusion phase diagram. In Fig.1 we present the regions where both second and fourth moment displacements are finite in finite time. According to the above considerations the whole region of the diagram is devided into three phases (except the Lévy 'ocean' placed above the red tangent straight line) defined by different η and η_2 exponents. Several observations should be made for them. For $\alpha > 3$ the enhanced diffusion in the stationary situation disappears. It can be easily proved that in this range of α the mean residence times $\langle t \rangle$ and $\langle t \rangle_{st}$ and their variances: $\langle (t - \langle t \rangle)^2 \rangle$ and $\langle (t - \langle t \rangle)^2 \rangle_{st}$, respectively, are finite. In other words, ED takes place if at least one of these four quantities diverges; then additionally the mean-square displacement $\langle x^2 \rangle_{st}$ (and hence $\langle x^2 \rangle$) diverge. It should be noted that the region of ED means for investors an increased risk offering both sudden considerable returns as well as sudden considerable losses. Shortly speaking: (i) in the ED phase both the second and fourth moments have anomalous behaviour in time, (ii) in the SRD phase only the

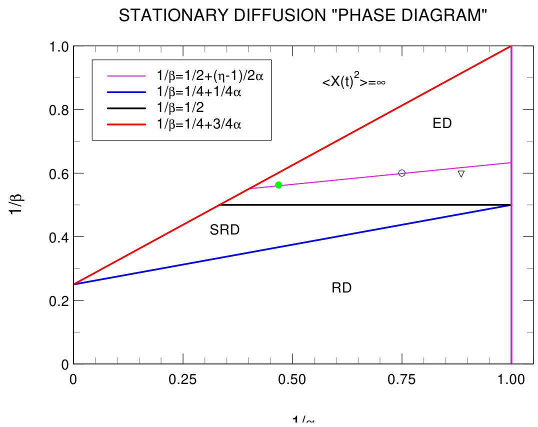

Fig. 1. Diffusion phase diagram for the stationary situation: only three different diffusion phases (defined by different exponents η and η_2) were shown.

second moment has a Gaussian form, while (iii) in the RD phase both the second and fourth moments are Gaussian and time dependent. By analysing higher moments the RD phase can be further decoupled into the hierarchy of more regular phases. However, this would require a more subtle study of the stochastic time series. Of course, all phases placed below the black border horizontal straight line $1/\beta = 1/2$ are asymptotically normal (i.e. the propagator is then asymptotically Gaussian).

4. Monte Carlo Algorithm

We present the efficient Monte Carlo algorithm which is essentially based on definition (1) of WTD. The basic step of our algorithm is to choose index j with probability $p(j)$. This crucial probability is given, according to (1), by the simple formula $p(j) = (1 - 1/N)/N^j$. Hence, one can simulate a probabilistic game of random casting of a coin. We define as a single success the situation where the coin falls to its abverse side with probability $\frac{1}{N}$. As single defeat, occurring of course with probability $1 - \frac{1}{N}$, we define the opposite situation when the coin falls to its revers side. We always cast our coin till the first defeat; then the number j of successive successes is just our seeked index. Thus τ^j, v^j and b^j are calculated.

By this effective procedure (since no Monte Carlo step is lost and always some index j is found) we are able to reach $j = j_{max}$ so large that $\frac{1}{N^{j_{max}}} \ll 1$. This is the necessary condition for observing in the numerical experiment the scaling phenomenon with a sufficient accuracy.

In the second step of our procedure we draw from the Poisson distribution, $\frac{1}{\tau^j}\exp(-\frac{\theta}{\tau^j})$, the elapsed time interval θ_j of the particle remaining in the drawn state j. This time is needed for the ant to pass the distance $v^j\theta_j$. Note that a very long one-step walk at a chosen state j could then happen.

The total performance consists of several repetitions of the above two basic steps. Note that the algorithm is a properly adapted, extended version of the known Petersburgh paradox.

It is instructive to calculate the single realizations of Weierstrass walks for different regions of the diffusion phase diagram shown in Fig.1. Note that the algorithm is able to simulate any type (stationary or non-stationary) of the Weierstrass walks.

Table 3
Diffusion exponents α, β and η_2 chosen for different enhanced diffusions with $\eta = 1.267$.

Notation	Exponent α	Exponent β	Exponent η_2
blue circle	1.333	1.667	2.867
full green circle	2.133	1.778	3.666

In Table 3 the parameters used in the Monte Carlo simulation are shown. For ED we used the diffusion exponent $\eta = 1.267$. We choose for comparison two different values of exponent α, belonging to the region of diffusion phase diagram where (i) only the mean residence time $\langle t \rangle$ is finite (c.f. Fig.1, the point marked by blue circle (bc)) and (ii) where only variance $\langle t^2 \rangle_{st}$ diverges (the same figure, point marked by full green circle (fgc)). Note that for both cases exponent $\beta < 2$.

In Fig.2 it is shown that more anomalous Weierstrass walks (red curve) follow less anomalous one (the blue curve). This time-dependent retardation effect was expected since $\tau_{bc} > \tau_{fgc}$ for $\alpha_{bc} < \alpha_{fgc}$. This behavior is caused by exponent α and has a quite general character. This exponent should be considered as important a fine characteristics of any fractional random walk in a continuous time. A similar retardation is observed, e.g., the between high-frequency WIG (Warsaw stock exchange index) and the NASDAQ-Future index (active 24 h). By our algorithm we are able to simulate the sta-

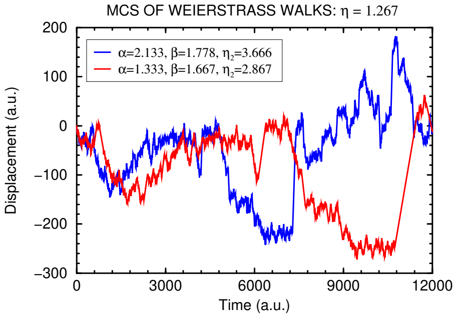

Fig. 2. Single realizations by our Monte Carlo simulation of two different Weierstrass walks for parameters which characterize enhanced diffusion (the red curve concerns point in the DPD denoted by blue circle while the blue one by full green circle).

tistical quantities needed for analysis of the empirical time series.

5. Examples

To illustrate our approach we consider two examples concerning the Elektrim company (being one of the greatest Polish telecommunication companies on the Warsaw stock exchange). Currently, we have the possibility to study in an unbiased way the stock exchange in its early stage.

5.1. *Example A: high-frequency stochastic time series*

By applying the method developed in Sec.3 we were able to determine the partial exponents α and β as well as the diffusion (or Hurst) one for high-frequency (intraday) stochastic time series concerning the evolution of the Elektrim stock price.

Fig.3 demonstrates (up to fluctuations) high-frequency empirical data, for the monthly, weekly, daily and even hourly activity of investors. Moreover, several levels of rare events hierarchy which

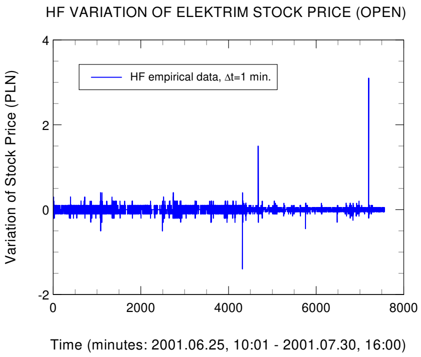

Fig. 3. The variation of stock price of Elektrim company vs. time measured at time horizon $\Delta t = 1min.$ within the time range: 2001.06.25, 10:01 - 2001.07.30, 16:00.

leads to anomalous diffusion of stock price of Elektrim company can be recognized.

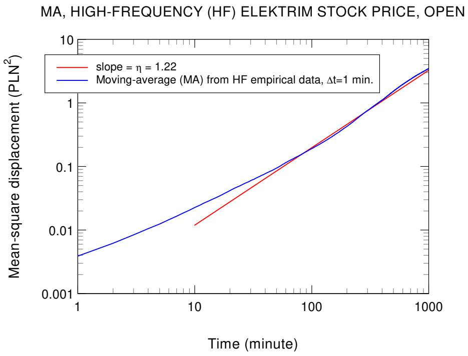

Fig. 4. Mean-square displacement (variance of stock price) vs. time calculated for high-frequency empirical data at time horizon $\Delta t = 1min.$ within the time range: 2001.06.25, 10:01 - 2001.07.30, 16:00.

In Figs.4 and 5 we demonstrate the variance (mean-square displacement) and the fourth moment of Elektrim stock price as a function of time in the log-log scale (blue curves) together with the corresponding fits (red straight lines). As a result we obtained η and η_2 shown in Table 3. Hence, from

Table 4
Exponents η, η_2, α and β obtained from empirical data for Elektrim company at time horizon $\Delta t = 1min.$ within the time range: 2001.06.25, 10:01 - 2001.07.30, 16:00.

Exponent η	Exponent η_2	Exponent α	Exponent β
1.22	2.57	1.13	1.67

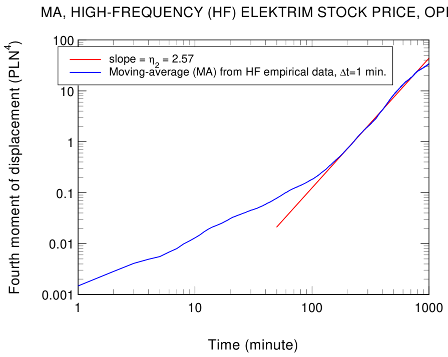

Fig. 5. Fourth moment of displacement (stock price) vs. time calculated for high-frequency empirical data at time horizon $\Delta t = 1min.$ within the time range: 2001.06.25, 10:01 - 2001.07.30, 16:00.

formula (12) and Table 1 we derive partial exponents α and β (Table 4) which makes possible to locate the company in the DPD (inverted triangle in Fig.1 located in the ED phase). The above results are (at least roughly) conirmed in Fig.6 where the crude power spectrum is shown.

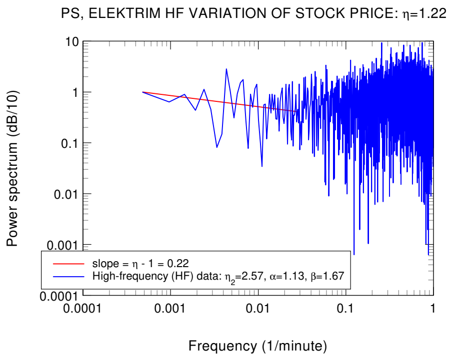

Fig. 6. Power spectra vs. frequency calculated for high-frequency empirical data at time horizon $\Delta t = 1min.$ within the time range: 2001.06.25, 10:01 - 2001.07.30, 16:00.

5.2. *Example B: daily time series*

Again, by applying the method developed in Sec.3 we were able to determine the partial exponents α and β as well as the diffusion one for empirical daily time series concerning again the dynamics of the Elektrim stock price. In Fig.7 the variation of stock price is shown which exhibits intermittent large activity of the stock exchange being a signature of the anomalous behaviour. This is an analog of the stock price variation presented in Fig.3 (shown in a different scale).

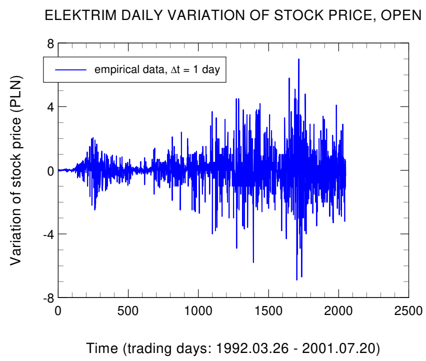

Fig. 7. The variation of stock price of Elektrim company vs. time measured at time horizon $\Delta t = 1day$ within the time range: 1992.03.26 - 2001.07.20.

In Figs.8 and 9 we present again MSD and the fourth moment of displacement now for daily time series obtained by the moving-average procedure. It is remarkable that we obtained the same value of exponents α and β as for high-frequency empirical time series (confimed also by the power spectrum very similar to that shown in Fig.6). This means that for the given company suprisingly many levels of temporal hierarchy constitute its evolution.

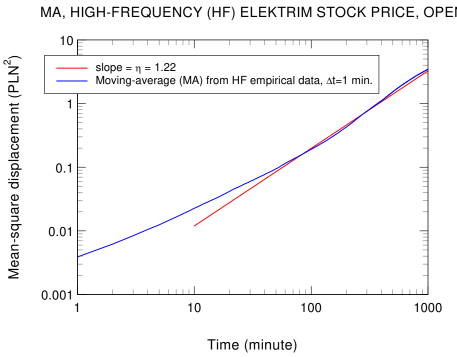

Fig. 8. Mean-square displacement (variance of stock price) vs. time calculated for daily empirical data at time horizon $\Delta t = 1day$ within the time range: 1992.03.26 - 2001.07.20.

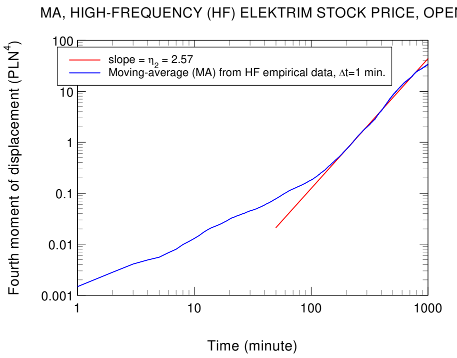

Fig. 9. Fourth moment of displacement (stock price) vs. time calculated for daily empirical data at time horizon $\Delta t = 1day$ within the time range: 1992.03.26 - 2001.07.20.

The above approach may be helpful for analysis of the statistical properties of time evolution of the stock price. For example, when the market becomes more regular then the point in DPD slows down to the semi-regular diffusion (SRD) phase and next eventually to the RD phase. Hence, the trajectory in DPD should present the most essential features of the stock price dynamics.

6. Summary

A higher order analysis is presented for the stationary situation, since beside the second order moment also the fourth order one is present in the main formulae. We considered:

(i) Fractional random walk of a financial ant having, in general, different velocities between successive turning points.

(ii) Hierarchical, self-similar in the stochastic sense, spatio-temporal coupling caused by the self-similar hierarchy of dynamical variables (displacements and velocities) and corresponding algorithm which makes possible to simulate stochastic time series.

(iii) Hurst exponent and time-dependent kurtosis.

(iv) Diffusion phase diagram for the stationary situation,

(v) Examples, concerning anomalous stock price dynamics of a company present on the Warsaw stock exchange.

Our aim was to present and illustrate the essential features of a quite general toy non-Gaussian model namely, the Lévy walks with varying velocity or Weierstrass walks model rather than to systematically analyse the anomalous dynamics of a concrete company.

References

[1] A. Bunde (ed.) *Fractals and Disorder*, Physica A 191 (1992) Nos. 1-4.

[2] R. Badii and A. Politi, *Complexity. Hierarchical structures and scaling in physics* (Cambridge Univ. Press, Cambridge 1997).

[3] A. R. Bishop, G. Grüner and B. Nicolaenko (eds.) *Spatio-Temporal Coherence and Chaos in Physical Systems*, Physica D 23 (1987) Nos. 1-3.

[4] A. Aharony and J. Feder (eds.) *Fractals in Physics. Essays in honour of Benoit B. Mandelbrot*, Physica D 38 (1989) Nos. 1-3.

[5] W. A. Fuller, *Introduction to Statistical Time Series*, (J. Wiley & Sons, New York 1976).

[6] R. N. Mantegna, Physica A 179 (1991) 232.

[7] R. N. Mantegna and H. E. Stanley, *Physics Investigation of Financial Markets*, in Proceed. of the Int. School of Physics 'Enrico Fermi', Course

CXXXIV, ed. F. Mallamace and H.E. Stanley (IOS Press, Amsterdam, 1997).

[8] F. Schweitzer and D. Helbing (eds.) *Economic Dynamics from the Physics Point of View*, Physica A 287 (2000) Nos. 3-4.

[9] J. -P. Bouchaud, M. Marsili, B. M. Roehner and F. Slanina (eds.) *Application of Physics in Economic Modelling*, Physica A 299 (2001), Nos.1-2.

[10] D. Sornette, *Critical Phenomena in Natural Science. Chaos, Fractals, Selforganization, and disorder: Concepts and Tools* (Springer, Berlin 2000)

[11] R. Kutner and M. Regulski, Physica A 264 (1999) 107; R. Kutner, Comp. Phys. Comm. (2002) in print.

[12] E. W. Montroll and M. F. Shlesinger, in *Nonequilibrium Phenomena II. From Stochastics to Hydrodynamics*, SSM Vol.XI, eds. J. L. Lebowitz and E. W. Montroll (North-Holland, Amsterdam 1984) p.1.

[13] J. Klafter, G. Zumofen, M. F. Shlesinger, eds. M. F. Shlesinger, G. M. Zaslavsky, U. Frisch (Springer, Berlin 1995) p.196.

[14] G. Zumofen, J. Klafter, M. F. Shlesinger, in *Anomalous Diffusion. From Basics to Applications*, LNP 519, eds. R. Kutner, A. Pękalski, K. Sznajd-Weron (Springer, Berlin 1999) p.15.

[15] J. W. Haus, K. W. Kehr, Phys. Rep. 150 (1987) 263.

[16] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics. Correlations and Complexity in Finance* (Cambridge Univ. Press, Cambridge UK 2000).

[17] W. Paul, J. Baschnagel, *Stochastic Processes. From Physics to Finance* (Springer, Berlin 1999).