Dynamics of simple economic growth models

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Abstract

The dynamical systems methods are used to analyse models of growth in macroeconomics. The simple example of a two-sector model with physical and human capital is presented to demonstrate the effectiveness of the qualitative analysis of differential equations. Some advantages of using the language of dynamical systems in economics is also given.

1 Introduction

Macroeconomics is the part of economics which is concerned with the study of the economy as a whole. The main questions are connected with unemployment and inflation and estimation of proper policies to prevent recessions. Apart from these short-run problems there are long-run problems of the economic development and growth. Why are some countries rich and others poor? Why do countries grow? These questions have been attracting attention since Adam Smith's times.

The last fifty years saw the great progress in our understanding of economic growth. Robert Solow developed a framework for estimating the main factors in economic growth. The new growth theory extends the Solow model to explain the nature of technical progress [2].

Accumulation of knowledge — the effectiveness of labour is knowledge or technology. It could be abstract as well as applied knowledge produced by the Research and Development sector. There is a whole spectrum of knowledge, ranging from basic scientific knowledge (mathematics, quantum mechanics) to knowledge about specific goods (how to repair the car). Another determinant of knowledge accumulation is *learning by doing*. Producing goods we are improving the production process.

Human capital — apart from physical capital there is human capital. Human capital consists of the abilities, skills, and knowledge of particular workers. If, for example, an engineer's full effort is being devoted to one activity, that precludes the use of his or her skills in another.

2 The Solow model

Solow [3] saw the tree main variables: capital K, labour L, and effectiveness of labour A to be responsible for output Y which is given by the production function

$$Y(t) = F(K(t), A(t)L(t)). \tag{1}$$

We assume that the production function has constant returns to scale

$$F(cK, cAL) = cF(K, AL).$$

and the production function can be rewritten in the intensive form

$$y = f(k) \tag{2}$$

where $y = Y/AL$ — output per unit of effective labour and $k = K/AL$ — the amount of capital per unit of effective labour.

It is assumed that the intensive-form production function, $f(k)$, satisfies the following conditions

$$f(0) = 0, \quad f'(k) > 0, \quad f''(k) < 0;$$

$$\lim_{k\to 0} f'(k) = \infty, \quad \lim_{l\to\infty} f'(k) = 0.$$

Example
The Cobb-Douglas production function has the form

$$F(K, AL) = k^{\alpha}(AL)^{1-\alpha}, \quad 0 < \alpha < 1$$

The intensive form of this production function is

$$f(k) = k^{\alpha}.$$

Let us assume that the stock of labour and knowledge change over time in such a way that labour and knowledge grow exponential at constant rates

$$\dot{L}(t) = nL(t)$$
$$\dot{A}(t) = gA(t).$$

Output is divided between consumption and investment. It is assumed that there is an exogenous, constant fraction of output devoted to investment; in addition, existing capital depreciates at rate δ. Then

$$\dot{K} = sY(t) - \delta K(t).$$

It is more convenient to focus on the capital stock per unit of effective labour, k, rather than the capital stock, K, then

$$\dot{k} = sf(k(t)) - (n + g + \delta)k(t).$$

It is the key equation of the Solow model. The graphical solution is presented on Fig. 1. For $k = k^*$, the economy converges to *balanced growth path*, on which each variable of the model grows at a constant rate. Therefore AL, K, and Y grow at rate $n + g$ while K/L and Y/L grow at rate g.

3 New growth theory

Let us consider the human capital influence on the economic growth [1]. Output is given

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} [A(t)L(t)]^{1-\alpha-\beta}, \tag{3}$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$.

It is convenient to define the new variables as the amount of capital per unit of effective labour, and output per unit of effective labour

$$k = \frac{K}{AL}, \quad h = \frac{H}{AL}, \quad y = \frac{Y}{AL}.$$

then the production function has the form

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

and we obtain the system

$$\dot{k} = s_K k(t)^{\alpha} h(t)^{\beta} - (n + g)k(t) \tag{4}$$
$$\dot{h} = s_H k(t)^{\alpha} h(t)^{\beta} - (n + g)h(t). \tag{5}$$

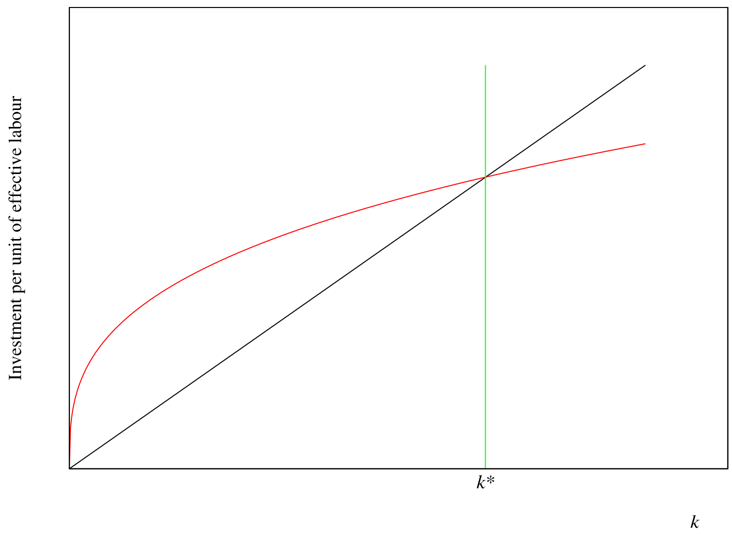

Figure 1: The solution of the Solow model

We define new variables as the rate of growth of physical and human capital respectively

$$u = \frac{\dot{k}}{k}, \quad v = \frac{\dot{h}}{h}$$

then we obtain the following system

$$\dot{u} = [(\alpha - 1)u + \beta v](u + n + g) \tag{6a}$$

$$\dot{v} = [(\beta - 1)v + \alpha u](v + n + g) \tag{6b}$$

System (6) has four critical points

$$\begin{array}{lll} \text{Point I}: & u_0 = 0, & v_0 = 0 \\ \text{Point II}: & u_0 = -n - g, & v_0 = -n - g \\ \text{Point III}: & u_0 = -n - g, & v_0 = \dfrac{\alpha}{\beta - 1}(n + g) \\ \text{Point IV}: & u_0 = \dfrac{\beta}{\alpha - 1}(n + g), & v_0 = -n - g. \end{array}$$

We must linearize our system at a critical point. The linearization matrix at the critical point u_0, v_0 is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{7}$$

where

$$
\begin{aligned}
a_{11} &= (\alpha-1)(2u_0+n+g)+\beta v_0 \\
a_{12} &= \beta(u_0+n+g) \\
a_{21} &= \alpha(v_0+n+g) \\
a_{22} &= (\beta-1)(2v_0+n+g)+\alpha u_0.
\end{aligned}
$$

The type of critical point is determined by eigenvalues of the system linearization matrix

$$\lambda^2-(\operatorname{tr} A)\lambda+\det A=0.$$

For these four critical points in our model we have
Point I: $(u_0=0, v_0=0)$ stable node:

$$
\begin{aligned}
\operatorname{tr} A &= (\alpha+\beta-2)(n+g)<0 \\
\det A &= (1-\alpha-\beta)(n+g)^2>0 \\
(\operatorname{tr} A)^2-4\det A &= (n+g)^2(\alpha+\beta)(4-3(\alpha+\beta))>0.
\end{aligned}
$$

Point II: $(u_0=-n-g, v_0=-n-g)$ unstable node:

$$
\begin{aligned}
\operatorname{tr} A &= 2(n+g)(1-\alpha-\beta)>0 \\
\det A &= (n+g)^2(1-\alpha-\beta)^2>0 \\
(\operatorname{tr} A)^2-4\det A &= 0.
\end{aligned}
$$

Point III: $(u_0=-n-g, v_0=\alpha(n+g)/(\beta-1))$ saddle:

$$
\begin{aligned}
\operatorname{tr} A &= \frac{\beta(\alpha+\beta-1)}{\beta-1}<0 \\
\det A &= \frac{(\alpha+\beta-1)^2}{\beta-1}(n+g)^2<0.
\end{aligned}
$$

Point IV: $(u_0=\beta(n+g)/\alpha-1)), v_0=-n-g)$ saddle:

$$
\begin{aligned}
\operatorname{tr} A &= \frac{\alpha(\alpha+\beta-1)}{\alpha-1}<0 \\
\det A &= \frac{(\alpha+\beta-1)^2}{\alpha-1}(n+g)^2<0.
\end{aligned}
$$

These critical points and some trajectories are depicted on Fig. 2. This phase portrait illustrates the dynamics of system (6). The only stable solution (stable node) corresponds to the balanced growth path, which means that each variable of the model is growing at a constant rate.

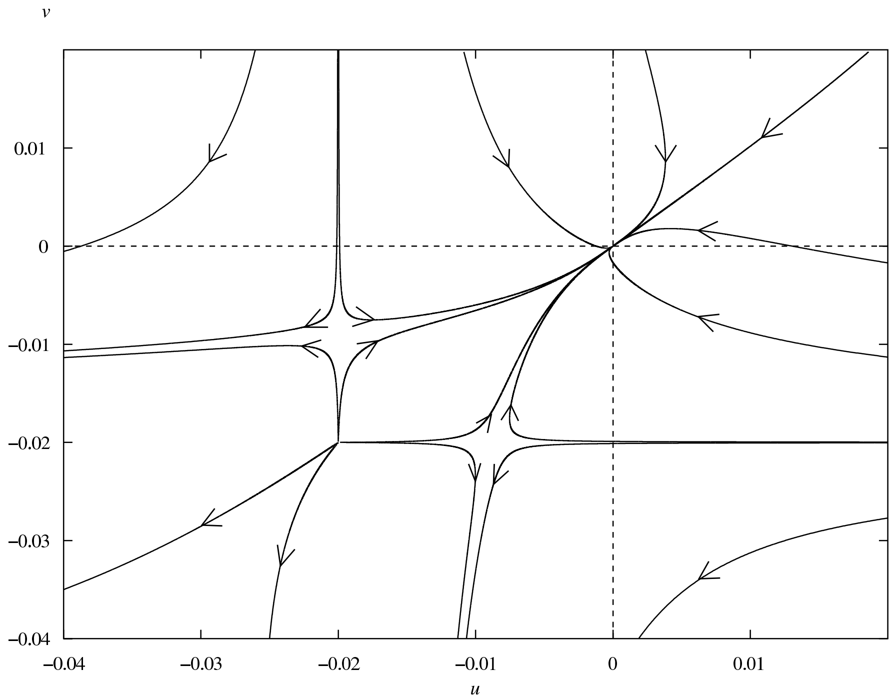

Figure 2: The phase portrait of system (6)

References

[1] Gregory N. Mankiw, David Romer, and David N. Weil. A contribution to the empirics of economic growth. *Quarterly Journal of Economics*, 107:407–437, May 1992.

[2] David Romer. *Advanced Macroeconomics.* McGraw-Hill, New York, 1996.

[3] Robert M. Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70(1):65–94, February 1956.