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Signal processing in complex networks: many faces of structural stochastic multiresonance

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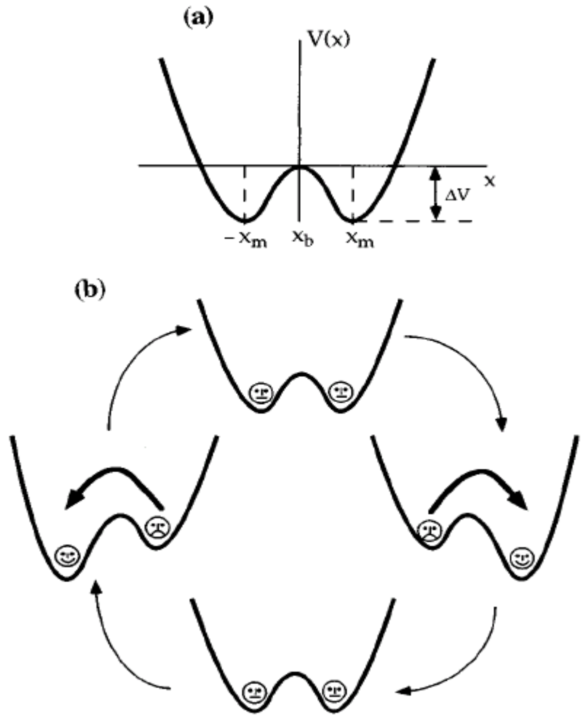
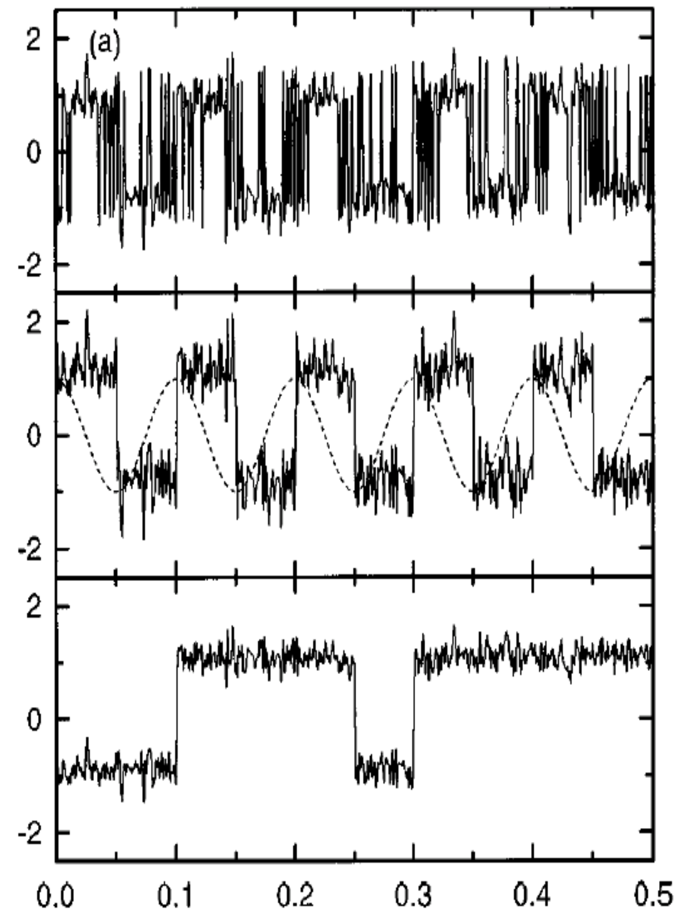
Stochastic resonance consists in the maximization of the response of a (nonlinear) system to a periodic stimulation by addition of noise at the input [R. Benzi, A. Sutera, and A. Vulpiani, *J. Phys*. **A14**, L453 (1981).]



Example Overdamped particle in a bistable potential well

Escape rate from the right or left potential well is given by the Kramers formula and is asymetrically modulated by the periodic signal

$$r_{K\pm}(t)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\Delta V\mp A\sin\omega_s t}{D}\right),$$

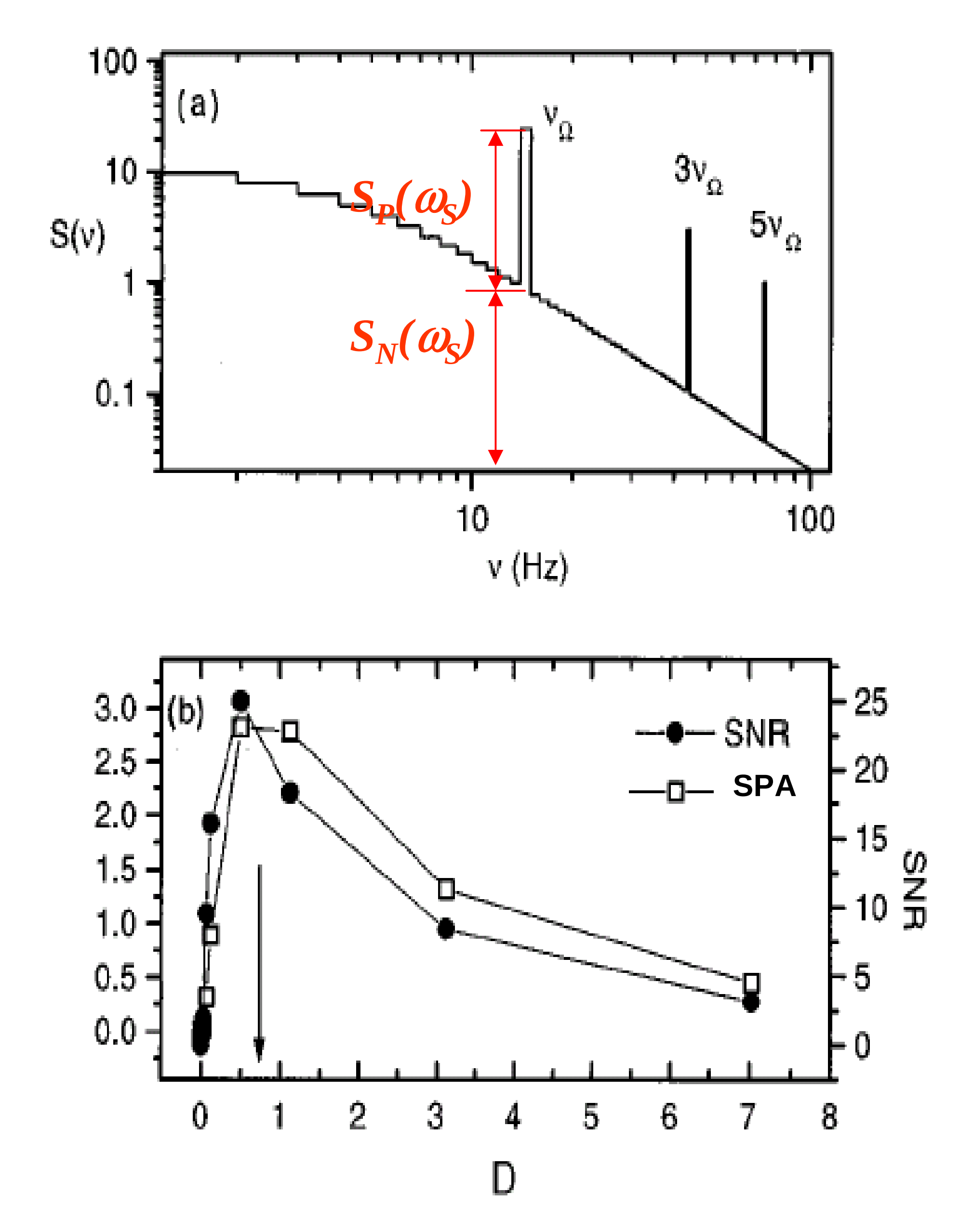


$$\frac{dx}{dt}=-\frac{dV}{dx}+A\sin\omega_s t+\xi(t)$$

$$V(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$$

$$\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$$

For optimum noise intensity D the position of the particle in the two potential wells is synchronized with the periodic signal



Characterization of stochastic resonance

Output signal

$$y(t) = \begin{cases} -1 \text{ dla } & x(t) < 0 \\ +1 \text{ dla } & x(t) > 0 \end{cases}$$

Measures:
- Signal-to-noise ratio (*SNR*)
- Spectral power amplification (*SPA*)

$$\mathrm{SNR} = 10 \log \frac{S_P(\omega_S)}{S_N(\omega_S)}$$

$$\mathrm{SPA} = \frac{S_P(\omega_S)}{A^2}$$

Typical power spectrum density of the output signal from a system exhibiting stochastic resonance (top) and dependence of the *SNR* and *SPA* on the noise intensity *D* (bottom).

Motivation for the study of stochastic resonance in systems with the structure of complex networks

- **Coupling bistable elements driven by a periodic signal and noise can lead to the enhancement of stochastic resonance (e.g., array enhanced stochastic resonance in spatially extended systems).**

- **Can suitably chosen structure of the coupling lead to qualitatively new phenomena in stochastic resonance?**

- **Example: structural stochastic multiresonance can occur in the Ising model on certain scale-free networks (the curves SPA vs. T show double maxima)** [A. Krawiecki, *European Phys. J.* **B69**, 81 (2009)].

- **Example: structural stochastic multiresonance can also occur in systems of coupled passive threshold elements on hierarchical networks, e.g., tree-like networks or Ravasz-Barabási networks** [M. Kaim and A. Krawiecki, *Phys. Lett.* **A374**, 4814 (2010)].

Stochastic multiresonance (concept): J.M.G. Vilar, J.M. Rubi, *Phys. Rev. Lett.* **78**, 2882 (1997); *Physica* **A264**, 1 (1999).

Stochastic resonance in the Ising model

- Periodic signal: oscillating magnetic field,
- Noise: thermal fluctuations (proportional to the temperature),
- Output signal: the time-dependent order parameter (e.g., magnetization).

The Ising model is treated as a complex system which consists of coupled bistable elements (spins), and its response to the periodic signal is studied as a function of the temperature and frequency of the magnetic field.

Exemplary results

- SR in the 1-dimensional Ising model (the paramagnetic phase) J.J. Brey and A. Prados, *Phys. Lett*. **A216**, 240 (1996); U. Siewert and L. Schimansky-Geier, *Phys. Rev*. **E58**, 2843 (1998);

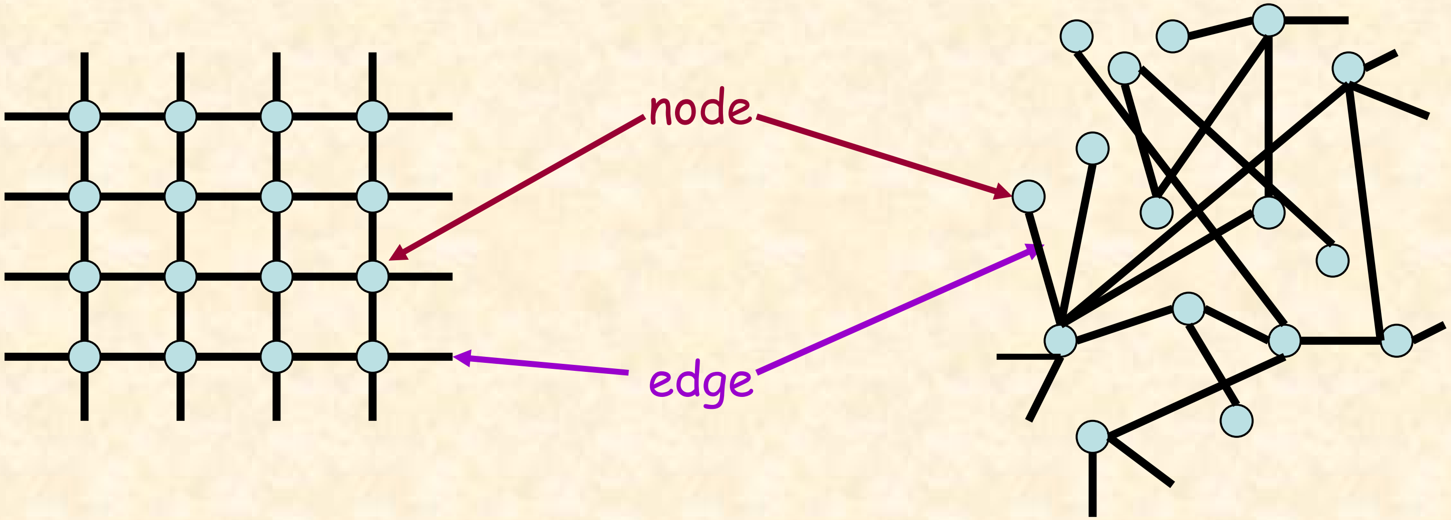
- SR in the 2- and 3-dimensional Ising model (Monte-Carlo simulations and theory in the mean-field approximation): Z. Neda, *Phys. Rev.* **E51**, 5315 (1995), K.-T. Leung and Z. Neda, *Phys. Lett*. **A246**, 505 (1998) et al.;

- Connection with dynamical phase transitions: B.J. Kim et al., *Europhys. Lett.* **56**, 333 (2001);

- SR in the Ising model on complex networks: H. Hong et al., *Phys. Rev.* **E66**, 011107 (2002) (Watts-Strogatz small-world networks), A. Krawiecki, *Int. J. Modern Phys.* **B18**, 1759 (2004) (Barabasi-Albert scale-free networks).

- Structural stochastic multiresonance in the Ising model on scale-free networks: A. Krawiecki, *Eur. Phys. J.* **B69**, 81 (2009).

Complex networks



Regular lattice

Complex network

$$p_k \propto \delta(z) = \delta(4)$$

$$p_k = \begin{cases} G_\sigma(k), \\ Ak^{-\gamma} \\ \dots \end{cases}$$

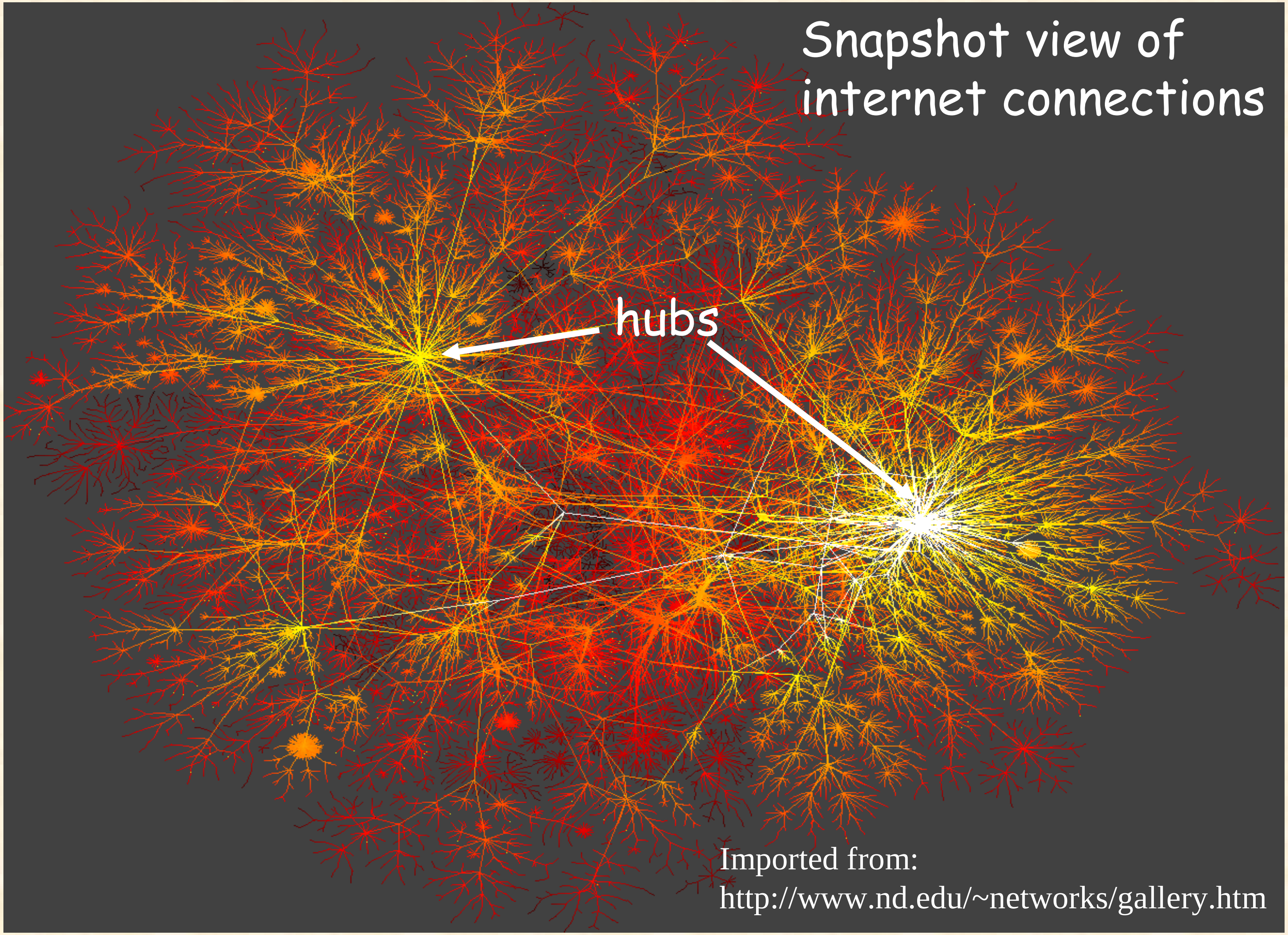
Connectivity of the node i: the number of edges attached to the node i

Quantity of interest: distribution of connectivity p_k (= probability distribution that a randomly selected node has connectivity k)

- **Networks with complex topology are ubiquitous in real world. An important class of complex networks are scale-free networks which look similar at any scale; e.g., the distribution of connectivity k obeys a power scaling law,** $p_k \propto k^{-\gamma}$.

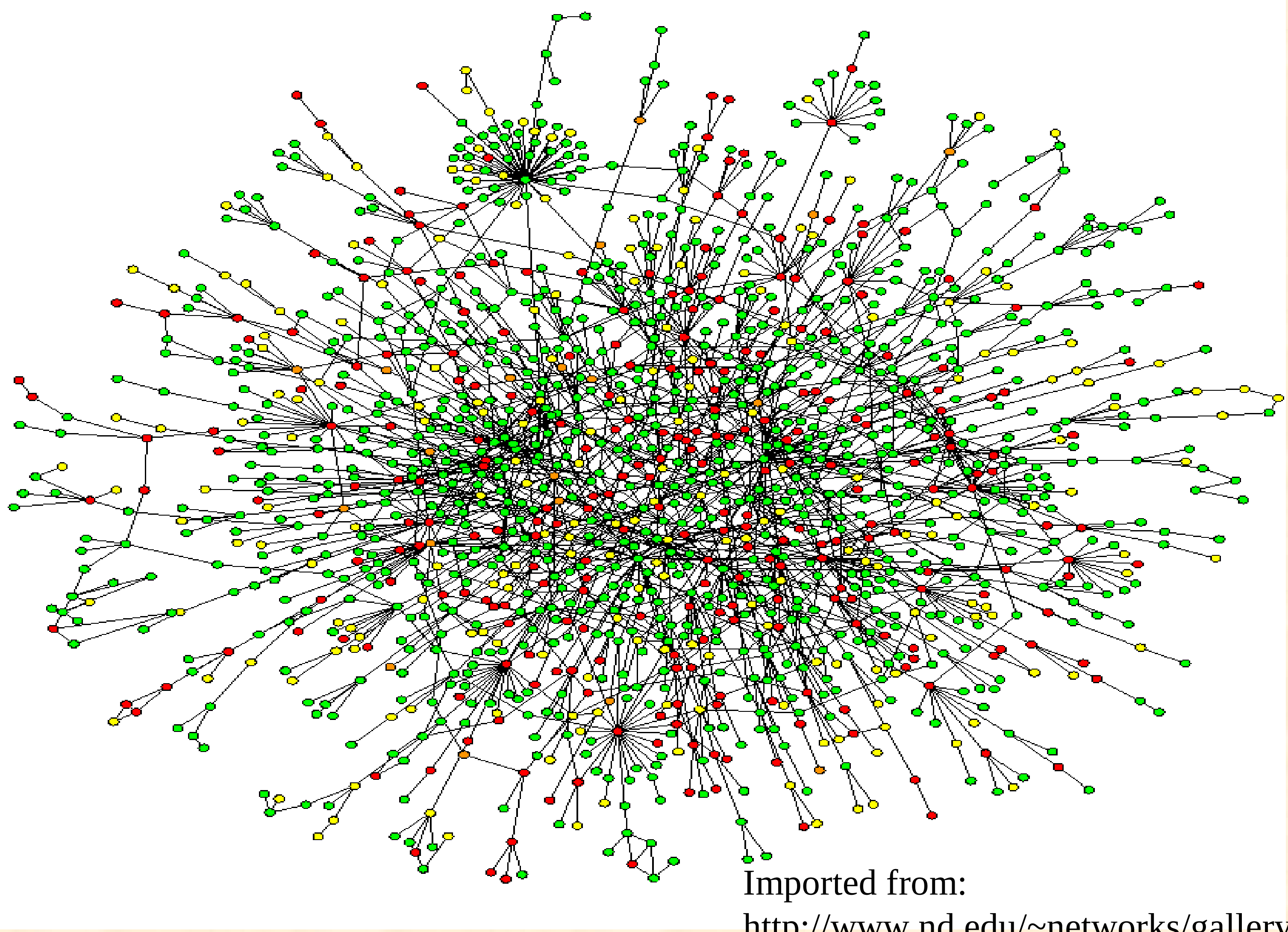
- Examples of scale-free networks comprise, e.g.,
- the internet activity,
- the www links,
- networks of cooperation (between scientists, actors, etc.),
- traffic networks (airplane & railway connections, city transport schemes),
- biological networks (sexual contacts, protein interactions, certain neural networks), etc.

Examples of scale-free networks: internet connections



Examples of scale-free networks: Map of protein-protein interactions.

The colour of a node signifies the phenotypic effect of removing the corresponding protein (red, lethal; green, non-lethal; orange, slow growth; yellow, unknown).



Imported from:
http://www.nd.edu/~networks/gallery.htm

The Ising model on complex networks

The spins s_i are located in the nodes i and edges between the nodes correspond to non-zero exchange interactions between the corresponding spins.

Hamiltonian

$$H = -\frac{1}{\langle k \rangle}\sum_{i,j} J_{ij} s_i s_j - h_0 \sin \omega_0 t \sum_i s_i$$

$J_{ij}=J$ if there is an edge between nodes i,j,
$J_{ij}=0$ otherwise

The local field acting on spin i

$$I_i(t) = \frac{1}{\langle k \rangle}\sum_j J_{ij} s_j + h_0 \sin \omega_0 t$$

The order parameter

$$S(t) = \frac{1}{N\langle k \rangle}\sum_i k_i s_i$$

Glauber dynamics (heat bath algorithm)

The transition rate between two spin configurations which differ by a single flip of one spin, e.g., that in node i

$$w_i(s_i) = \frac{1}{2}\left[1 - s_i \tanh\left(\frac{I_i(t)}{T}\right)\right]$$

Ferromagnetic phase transition

Critical temperature: maximum fluctuations of the order parameter

$$\delta S^2 = \langle S^2 \rangle - \langle |S| \rangle^2$$

Spectral Power Amplification (SPA)

$$SPA = \frac{|P_1|^2}{h_0^2} \qquad P_1 = \lim_{T\to\infty}\sum_{t=0}^{T-1} S(t)\exp(-i\omega_0 t)$$

Creating scale-free networks

[A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999) (for B=0, γ=3); S.N. Dorogovtsev, J.F.F. Mendes, *Adv. Phys.* **51**, 1079 (2002)]

- The algorithm starts with a small number m of fully connected nodes.
- Evolving network: New nodes are added step by step,
- Preferential attachment: From each new node m new links are created to the existing nodes, and the probability to create an edge to the node i is

$$p_i = (k_i + B) \Big/ \sum_i (k_i + B); \quad B > -m$$

where k_i is the actual number of edges attached to the node i.

- The network grows until a given number of nodes N is added.
- Evolving network + preferential attachment = scale-free network. The distribution of the degrees of nodes is

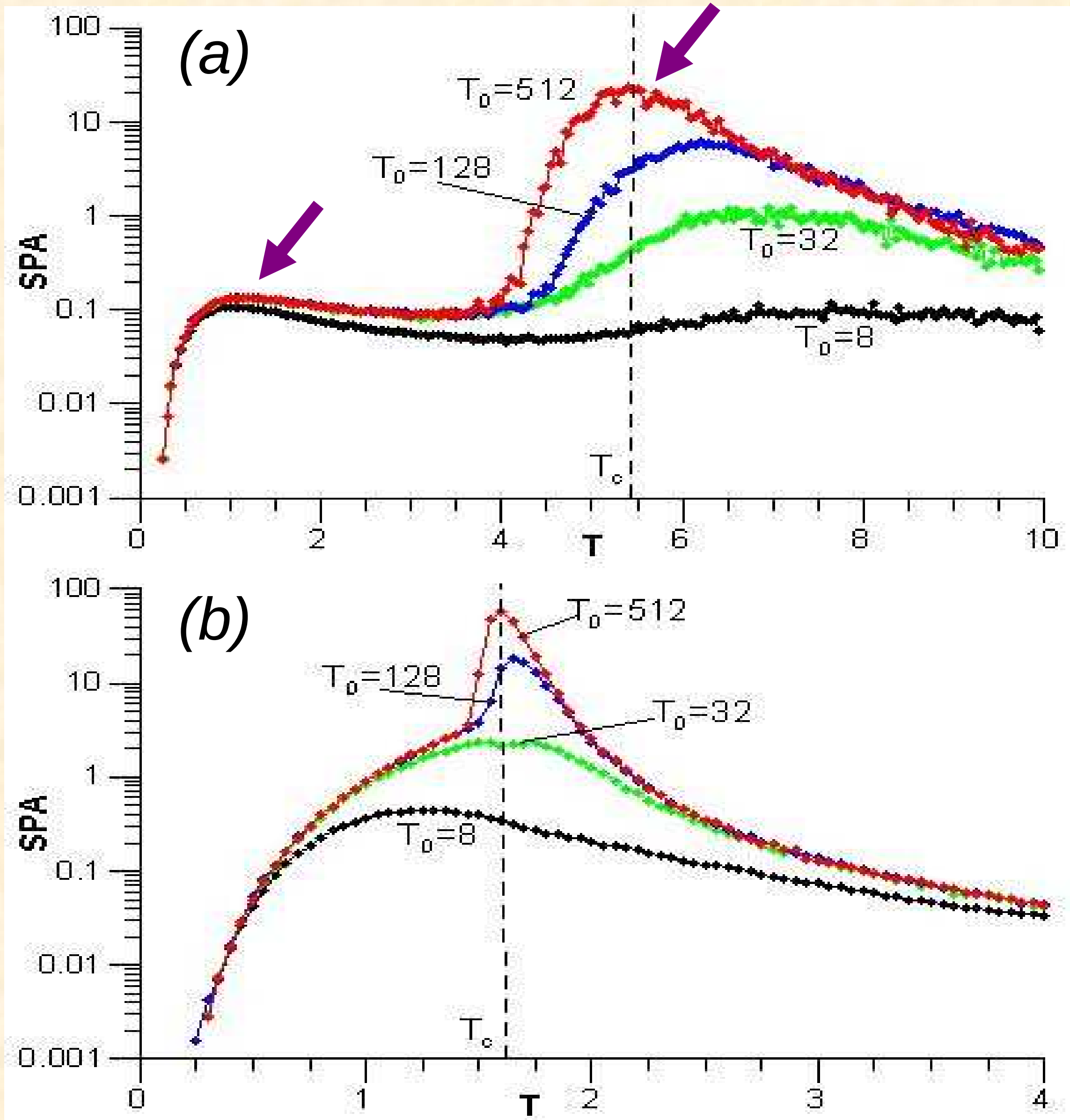
$$p_k \propto (k+B)^{-\gamma} \xrightarrow[k\to\infty]{} k^{-\gamma}, \quad \gamma = 3 + B/m$$

Structural stochastic multiresonance in the Ising model on scale-free networks

Multiresonance
(two maxima of the *SPA*)

- It can be seen that the occurrence of stochastic multiresonance depends on the structure of connections, thus in *(a)* structural stochastic multiresonance occurs,
- Structural stochastic multiresonance occurs for $2 < \gamma < 3$ provided that the power-law tails of the distribution of the degrees of nodes are fully developed, i.e., no artificial constraint is imposed on the maximum degree.

Signal: $S(t)$
$N = 10\ 000$,
$m = 5$,
$h_0 = 0.01$,
(a) $B = -4$ (γ=2.2),
(b) $B = 10$ (γ=5)



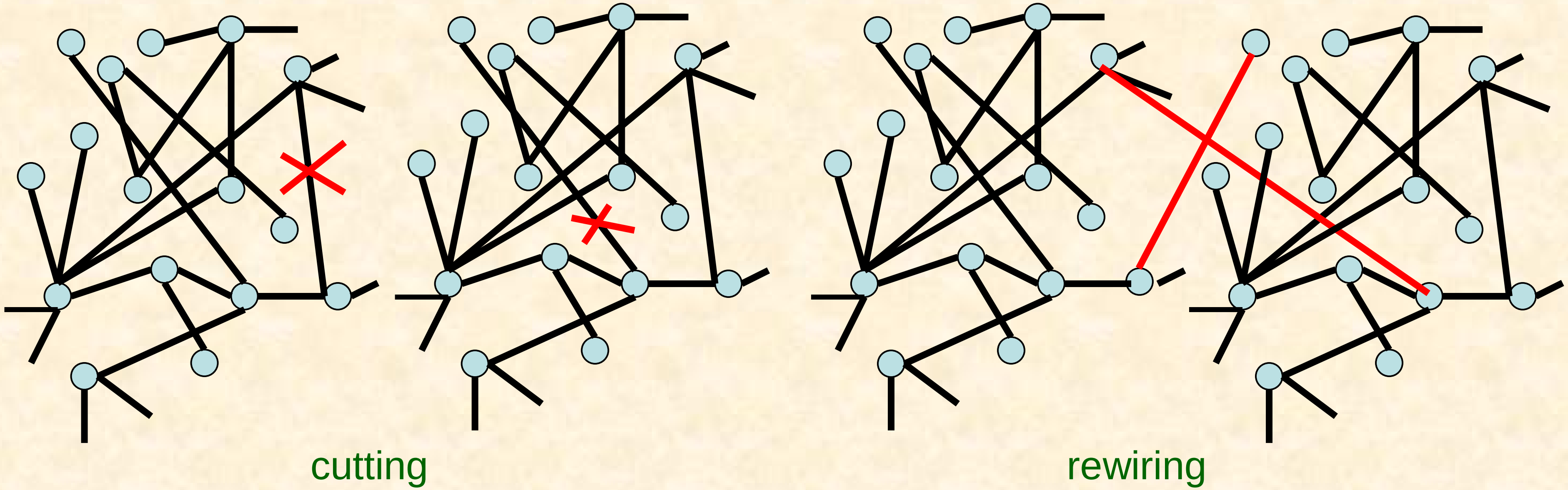
Creating coupled scale-free networks

Network A
N_A nodes,
exponent γ_A

Network B
N_B nodes,
exponent γ_B

<u>Subnetwork</u> A,
Rewiring probability $\pi^{(A)}$
N_A nodes, exponent γ_A

<u>Subnetwork</u> B,
Rewiring probability $\pi^{(B)}$
N_B nodes, exponent γ_B



Rewiring probability (in a given network) =
= number of rewired edges (in a given network) / number of edges (in a given network).

Cutting and rewiring do not change the distribution of degrees of nodes in the subnetworks.

For small rewiring probabilities the composite network consists of separate, densely connected subnetworks with few inter-subnetwork links, so it has modular structure;
For increasing rewiring probabilities the network structure becomes more and more uniform

The Ising model on coupled scale-free networks

The spins s_i are located in the nodes i and edges between the nodes correspond to non-zero exchange interactions between the corresponding spins. The sysyem is subject to the oscillating magnetic field

Order parameters

For the composite network

$$S(t) = \frac{1}{N\langle k \rangle} \sum_{i=1}^{N=N_A+N_B} k_i s_i$$

For the subnetworks

$$S_A(t) = \frac{1}{N_A \langle k \rangle_A} \sum_{i=1}^{N_A} k_i s_i, \quad S_B(t) = \frac{1}{N_B \langle k \rangle_B} \sum_{i=1}^{N_B} k_i s_i$$

Critical temperatures for the uncoupled subnetworks (ferromagnetic transition)

$$T_c^{(A,B)} = J \frac{\langle k^2 \rangle_{A,B}}{\langle k \rangle_{A,B}^2}$$

$$2 < \gamma_{A,B} < 3 \Rightarrow T_c^{(A,B)} = J \frac{(\gamma_{A,B} - 2)^2}{(\gamma_{A,B} - 1)(3 - \gamma_{A,B})} N_{A,B}^{\frac{3-\gamma_{A,B}}{\gamma_{A,B}-1}} \xrightarrow[N_{A,B} \to \infty]{} \infty$$

$$\gamma_{A,B} > 3 \Rightarrow T_c^{(A,B)} = J \frac{(\gamma_{A,B} - 2)^2}{(\gamma_{A,B} - 1)(3 - \gamma_{A,B})}$$

- Phase transitions in the Ising model on coupled scale-free networks (Barabási-Albert ones) in the absence of the magnetic field were considered by K. Suchecki, J.A. Hołyst, *Phys. Rev*. **E74**, 011122 (2006), Phys. Rev. **E80**, 031110 (2009).
- In general, there is ferromagnetic transition as the temperature decreases (spins in both subnetworks are aligned in parallel), although special preparation of the initial condition can lead to antiparallel alignement of spins in the two subnetworks.
- In the analysis based on the mean-field and linear response theory approximations it will be assumed that the ordered stationary state is the ferromagnetic one.

Mean field approximation

Equations of motion for the order parameter

• The Master equation for the probability that at time *t* the system is in the spin configuration (s_1, s_2, ..., s_n)

$$\frac{d}{dt}P(s_1,s_2,\dots,s_n;t)=-\sum_{j=1}^{N}w_j(s_j)P(s_1,s_2,\dots,s_j,\dots,s_n;t)+$$
$$+\sum_{j=1}^{N}w_j(-s_j)P(s_1,s_2,\dots,-s_j,\dots,s_n;t)$$

$$w_i(s_i)=\frac{1}{2}\left[1-s_i\tanh\left(\frac{I_i(t)}{T}\right)\right],\quad I_i(t)=\frac{1}{\langle k\rangle}\sum_j J_{ij}s_j+h_0\sin\omega_0 t$$

• Multiply both sides by s_i and perform an ensemble average (denoted by ⟨ ⟩), i.e., summation over all possible spin configurations of the composite network

$$\frac{d\langle s_i\rangle}{dt}=-\langle s_i\rangle+\left\langle\tanh\left(\frac{I_i(t)}{T}\right)\right\rangle,\ i=1,2,\dots,N=N_A+N_B$$

• The above *N* equations can be formally separated into two groups of N_A and N_B equations, for the mean values of spins belonging to the sunbentworks *A*, *B*, respectively

$$\frac{d\langle s_i^{(A)}\rangle}{dt}=-\langle s_i^{(A)}\rangle+\left\langle\tanh\left(\frac{I_i^{(A)}(t)}{T}\right)\right\rangle,\ i=1,2,\dots N_A;\quad \frac{d\langle s_i^{(B)}\rangle}{dt}=-\langle s_i^{(B)}\rangle+\left\langle\tanh\left(\frac{I_i^{(B)}(t)}{T}\right)\right\rangle,\ i=1,2,\dots N_B$$

• Make the following mean-field approximation

$$I_i^{(A,B)}(t)\approx\langle I_i^{(A,B)}(t)\rangle=\frac{J}{\langle k\rangle}\left(\sum_{j\in K_i^{(A)}}\langle s_j\rangle+\sum_{j\in K_i^{(B)}}\langle s_j\rangle\right)+h_0\sin\omega_0 t\Rightarrow\left\langle\tanh\left(\frac{I_i^{(A,B)}(t)}{T}\right)\right\rangle\approx\tanh\left(\frac{\langle I_i^{(A,B)}(t)\rangle}{T}\right)$$

• Divide the nodes of the subnetworks according to their degrees k and assume that the average values of spins located in the nodes belonging to the class with degree k are equal to $\langle s_k \rangle$
• Replace the sums over the nodes of the subnetworks with sums over the classes of nodes, e.g.

$$\left\langle S^{(A)}(t)\right\rangle = \sum_{k=m}^{k_{\max}^{(A)}} \frac{kp_k^{(A)}}{\langle k\rangle_A}\left\langle s_k^{(A)}(t)\right\rangle, \quad \left\langle S^{(B)}(t)\right\rangle = \sum_{k=m}^{k_{\max}^{(B)}} \frac{kp_k^{(B)}}{\langle k\rangle_B}\left\langle s_k^{(B)}(t)\right\rangle$$

$$\left\langle I_i^{(A)}(t)\right\rangle = \frac{Jk_i}{\langle k\rangle}\left[\left(1-\pi^{(A)}\right)\sum_{k=m}^{k_{\max}^{(A)}} \frac{kp_k^{(A)}}{\langle k\rangle_A}\left\langle s_k^{(A)}(t)\right\rangle + \pi^{(A)}\sum_{k=m}^{k_{\max}^{(B)}} \frac{kp_k^{(B)}}{\langle k\rangle_B}\left\langle s_k^{(B)}(t)\right\rangle\right] + h_0 \sin \omega_0 t =$$

$$= \frac{J_A k_i}{\langle k\rangle_A}\left[\left(1-\pi^{(A)}\right)\left\langle S^{(A)}(t)\right\rangle + \pi^{(A)}\left\langle S^{(B)}(t)\right\rangle\right] + h_0 \sin \omega_0 t = \frac{J_A k_i}{\langle k\rangle_A}\tilde{S}^{(A)}(t) + h_0 \sin \omega_0 t,$$

$$J_A \equiv J\frac{\langle k\rangle_A}{\langle k\rangle}, \quad \tilde{S}^{(A)}(t) \equiv \left(1-\pi^{(A)}\right)\left\langle S^{(A)}(t)\right\rangle + \pi^{(A)}\left\langle S^{(B)}(t)\right\rangle,$$

and similarly for the network *B*.
In the above equations it was taken into account that the probability that a node i with degree k_i belonging to the subnetwork *A* is connected to a node with degree k, belonging to

• subnetwork *A* is $\left(1-\pi^{(A)}\right)\frac{kp_k^{(A)}}{\langle k\rangle_A}$ • subnetwork *B* is $\pi^{(A)}\frac{kp_k^{(B)}}{\langle k\rangle_B}$

and similarily for a node i belonging to the subnetwork *B*.

• Multiply both sides of the equations of motion for $\langle s_i^{(A,B)}\rangle$ by k_i, perform the sum over all nodes of the respective subnetwork and replace it with a sum over the classes of nodes,

$$\frac{d\left\langle S^{(A)}\right\rangle}{dt} = -\left\langle S^{(A)}\right\rangle + \sum_{k=m}^{k_{\max}^{(A)}} \frac{p_k^{(A)} k}{\langle k\rangle_A} \tanh\left(\frac{J_A k}{\langle k\rangle_A T} \tilde{S}^{(A)}(t) + \frac{h_0}{T} \sin \omega_0 t \right),$$

$$\frac{d\left\langle S^{(B)}\right\rangle}{dt} = -\left\langle S^{(B)}\right\rangle + \sum_{k=m}^{k_{\max}^{(B)}} \frac{p_k^{(B)} k}{\langle k\rangle_B} \tanh\left(\frac{J_B k}{\langle k\rangle_B T} \tilde{S}^{(B)}(t) + \frac{h_0}{T} \sin \omega_0 t \right)$$

Stationary values of the order parameters and magnetizations for the subnetworks

$$h_0 = 0 \Rightarrow \begin{aligned} \left\langle S^{(A)}\right\rangle_0 &= \sum_{k=m}^{k_{\max}^{(A)}} \frac{p_k^{(A)} k}{\langle k\rangle_A} \tanh\left(\frac{J_A k}{\langle k\rangle_A T} \tilde{S}_0^{(A)}(t) \right), \\ \left\langle S^{(B)}\right\rangle_0 &= \sum_{k=m}^{k_{\max}^{(B)}} \frac{p_k^{(B)} k}{\langle k\rangle_B} \tanh\left(\frac{J_B k}{\langle k\rangle_B T} \tilde{S}_0^{(B)}(t) \right) \end{aligned}$$

The case of two coupled scale-free networks

$$p_k^{(A)} = Ak^{-\gamma_A}, \quad \gamma_A < 2, \quad k_{\max}^{(A)} = mN_A^{\frac{1}{\gamma_A - 1}}, \quad A = \frac{\gamma_A - 1}{m^{-\gamma_A+1} - \left(k_{\max}^{(A)}\right)^{-\gamma_A+1}}$$

$$p_k^{(B)} = Bk^{-\gamma_B}, \quad \gamma_A < 2, \quad k_{\max}^{(B)} = mN_B^{\frac{1}{\gamma_B - 1}}, \quad B = \frac{\gamma_B - 1}{m^{-\gamma_B+1} - \left(k_{\max}^{(B)}\right)^{-\gamma_B+1}}$$

Replacing summation with integration, one gets

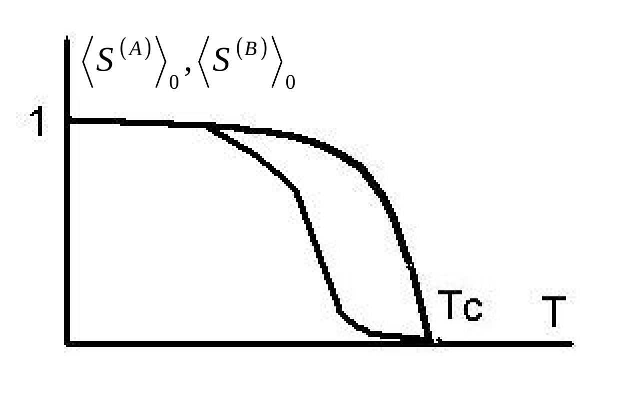
$$\left\langle S^{(A)} \right\rangle_0 = \int_m^{k_{\max}^{(A)}} \frac{Ak^{-\gamma_A+1}}{\left\langle k \right\rangle_A} \tanh\left(\frac{J_A k \tilde{S}_0^{(A)}}{\left\langle k \right\rangle_A T} \right) dk,$$

$$\left\langle S^{(B)} \right\rangle_0 = \int_m^{k_{\max}^{(B)}} \frac{Bk^{-\gamma_B+1}}{\left\langle k \right\rangle_B} \tanh\left(\frac{J_B k \tilde{S}_0^{(B)}}{\left\langle k \right\rangle_B T} \right) dk$$

$$\left\langle M^{(A)} \right\rangle_0 = \sum_{k=m}^{k_{\max}^{(A)}} p_k^{(A)} \tanh\left(\frac{J_A k \tilde{S}_0^{(A)}}{\left\langle k \right\rangle_A T} \right) \rightarrow$$

$$\rightarrow \left\langle M^{(A)} \right\rangle_0 = \int_m^{k_{\max}^{(A)}} Ak^{-\gamma_A} \tanh\left(\frac{J_A k \tilde{S}_0^{(A)}}{\left\langle k \right\rangle_A T} \right) dk$$

$$\left\langle M^{(B)} \right\rangle_0 = \int_m^{k_{\max}^{(B)}} Bk^{-\gamma_B} \tanh\left(\frac{J_B k \tilde{S}_0^{(B)}}{\left\langle k \right\rangle_B T} \right) dk$$



For non-zero rewiring probability there is a common critical temperature for both subnetworks T_c.
For $T<T_c$ the mean-field stationary values of the order parameters deviate from zero and there is a ferromagnetic transition, but the critical behaviour in the vicinity of the critical point for both subnetworks can be different.
Below the critical point the magnetizations in both subnetworks are parallel.

Linear response theory

In order to solve the equations of motion

$$\frac{d\left\langle S^{(A)}\right\rangle}{dt} = -\left\langle S^{(A)}\right\rangle + \sum_{k=m}^{k_{\max}^{(A)}} \frac{p_k^{(A)}k}{\left\langle k\right\rangle_A} \tanh\left(\frac{J_A k}{\left\langle k\right\rangle_A T}\tilde{S}^{(A)}(t) + \frac{h_0}{T}\sin\,\omega_0 t\right),$$

$$\frac{d\left\langle S^{(B)}\right\rangle}{dt} = -\left\langle S^{(B)}\right\rangle + \sum_{k=m}^{k_{\max}^{(B)}} \frac{p_k^{(B)}k}{\left\langle k\right\rangle_B} \tanh\left(\frac{J_B k}{\left\langle k\right\rangle_B T}\tilde{S}^{(B)}(t) + \frac{h_0}{T}\sin\,\omega_0 t\right)$$

let us assume that the external magnetic field forces the order parameters of the subnetworks to perform small oscillations around their stationary values

$$\begin{aligned}\left\langle S^{(A)}(t)\right\rangle &= \left\langle S^{(A)}\right\rangle_0 + \xi_A(t)\\ \left\langle S^{(B)}(t)\right\rangle &= \left\langle S^{(B)}\right\rangle_0 + \xi_B(t)\end{aligned} \Rightarrow \begin{aligned}\frac{d\xi_A}{dt} &= -\frac{\xi_A}{\tau_{AA}} - \frac{\xi_A}{\tau_{AB}} + \frac{Q_A}{T}h_0 \sin\,\omega_0 t\\ \frac{d\xi_B}{dt} &= -\frac{\xi_B}{\tau_{BB}} - \frac{\xi_A}{\tau_{BA}} + \frac{Q_B}{T}h_0 \sin\,\omega_0 t\end{aligned}$$

$$\tau_{AA} = \left[1-\left(1-\pi^{(A)}\right)\frac{J_A}{T\left\langle k\right\rangle_A^2}\sum_{k=m}^{k_{\max}^{(A)}} p_k^{(A)}k^2\cosh^{-2}\left(\frac{J_A k\tilde{S}_0^{(A)}}{\left\langle k\right\rangle_A T}\right)\right]^{-1}, \quad \tau_{AB} = \left[\frac{\pi^{(A)}}{1-\pi^{(A)}}\left(\tau_{AA}^{-1}-1\right)\right]^{-1},$$

$$\tau_{BB} = \left[1-\left(1-\pi^{(B)}\right)\frac{J_B}{T\left\langle k\right\rangle_B^2}\sum_{k=m}^{k_{\max}^{(B)}} p_k^{(B)}k^2\cosh^{-2}\left(\frac{J_B k\tilde{S}_0^{(B)}}{\left\langle k\right\rangle_B T}\right)\right]^{-1}, \quad \tau_{BA} = \left[\frac{\pi^{(B)}}{1-\pi^{(B)}}\left(\tau_{BB}^{-1}-1\right)\right]^{-1}$$

$$Q_A = \frac{1}{\left\langle k\right\rangle_A}\sum_{k=m}^{k_{\max}^{(A)}} p_k^{(A)}k\cosh^{-2}\left(\frac{J_A k\tilde{S}_0^{(A)}}{\left\langle k\right\rangle_A T}\right), \quad Q_B = \frac{1}{\left\langle k\right\rangle_B}\sum_{k=m}^{k_{\max}^{(B)}} p_k^{(B)}k\cosh^{-2}\left(\frac{J_B k\tilde{S}_0^{(B)}}{\left\langle k\right\rangle_B T}\right)$$

The case of two coupled scale-free networks

In the ordered state, for $T < T_c$

$$\tau_{AA} = \left\{ \frac{A\left(1-\pi^{(A)}\right)}{\langle k\rangle_A \langle S^{(A)}\rangle_0} \left[m^{-\gamma_A+2} \tanh\left(\frac{J_A m \tilde{S}_0^{(A)}}{\langle k\rangle_A T} \right) - \left(k_{\max}^{(A)}\right)^{-\gamma_A+2} \tanh\left(\frac{J_A k_{\max}^{(A)} \tilde{S}_0^{(A)}}{\langle k\rangle_A T} \right) \right] + \right.$$

$$\left. + 1 - \frac{\langle S^{(A)}\rangle_0}{\tilde{S}_0^{(A)}} \left(1-\pi^{(A)}\right)\left(\gamma_A - 2\right) \right\}^{-1}$$

$$Q_A = \frac{AT}{J_A \langle S^{(A)}\rangle_0} \left[\left(k_{\max}^{(A)}\right)^{-\gamma_A+1} \tanh\left(\frac{J_A k_{\max}^{(A)} \tilde{S}_0^{(A)}}{\langle k\rangle_A T} \right) - m^{-\gamma_A+1} \tanh\left(\frac{J_A m \tilde{S}_0^{(A)}}{\langle k\rangle_A T} \right) + \left(\gamma_A - 1\right) \frac{\langle M^{(A)}\rangle_0}{J_A \tilde{S}_0^{(A)}} \right]$$

In the disordered state, for $T > T_c$

$$\tau_{AA} = \left[1 - \left(1-\pi^{(A)}\right) \frac{\langle k\rangle_A}{\langle k\rangle} \frac{T_c^{(A)}}{T} \right]^{-1} \qquad T_c^{(A)} = J \frac{\langle k^2\rangle_A}{\langle k\rangle_A^2}$$

(critical temperatue for the subnetwork *A* in the case with no rewiring)

$$Q_A = 1$$

The parameters τ_{BB}, Q_B are obtained in a similar way.

Spectral power amplification *SPA* as a function of temperature

Solution of the equations of motion in the linear response approximation is

$$\xi_A = \alpha_1 \sin \omega_0 t + \alpha_2 \cos \omega_0 t$$
$$\xi_B = \beta_1 \sin \omega_0 t + \beta_2 \cos \omega_0 t$$

where the coefficients can be obtained from the system of linear equations

$$\begin{bmatrix} \tau_{AA}^{-1} & -\omega_0 & \tau_{AB}^{-1} & 0 \\ \omega_0 & \tau_{AA}^{-1} & 0 & \tau_{AB}^{-1} \\ \tau_{BA}^{-1} & 0 & \tau_{BB}^{-1} & -\omega_0 \\ 0 & \tau_{BA}^{-1} & \omega_0 & \tau_{BB}^{-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \frac{Q_A}{T} h_0 \\ 0 \\ \frac{Q_B}{T} h_0 \\ 0 \end{bmatrix}$$

Spectral power amplification for the subnetworks is

$$SPA_A = SPA_A(T) = \frac{\alpha_1^2 + \alpha_2^2}{4h_0^2}, \quad SPA_B = SPA_B(T) = \frac{\beta_1^2 + \beta_2^2}{4h_0^2}$$

and for the composite network

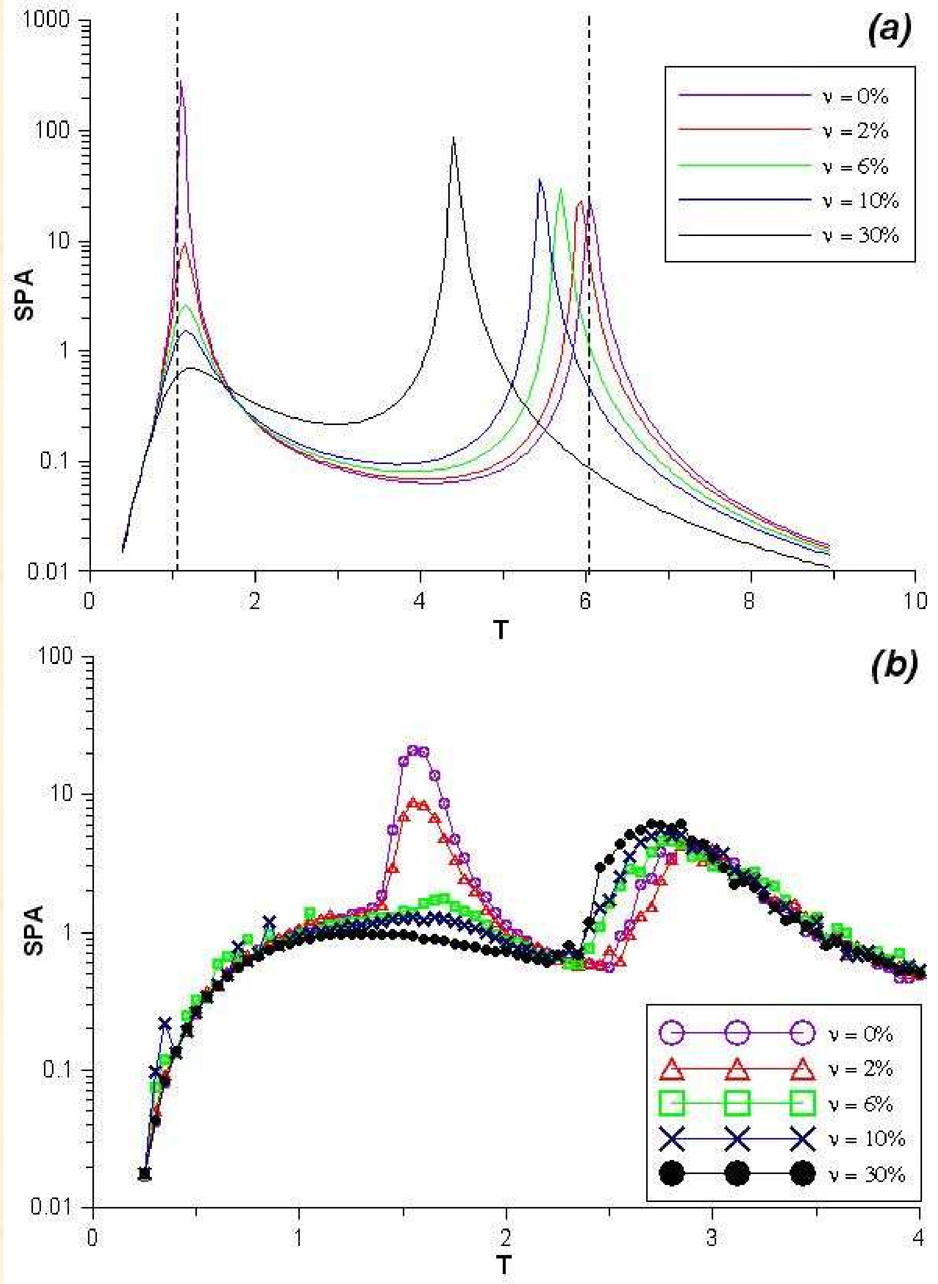
$$SPA(T) = \frac{N_A}{N} SPA_A(T) + \frac{N_B}{N} SPA_B(T)$$

Structural stochastic multiresonance in the Ising model on two coupled scale-free networks

Networks with the same number of nodes, $N_A = N_B$, and different exponents $\gamma_A \neq \gamma_B$.

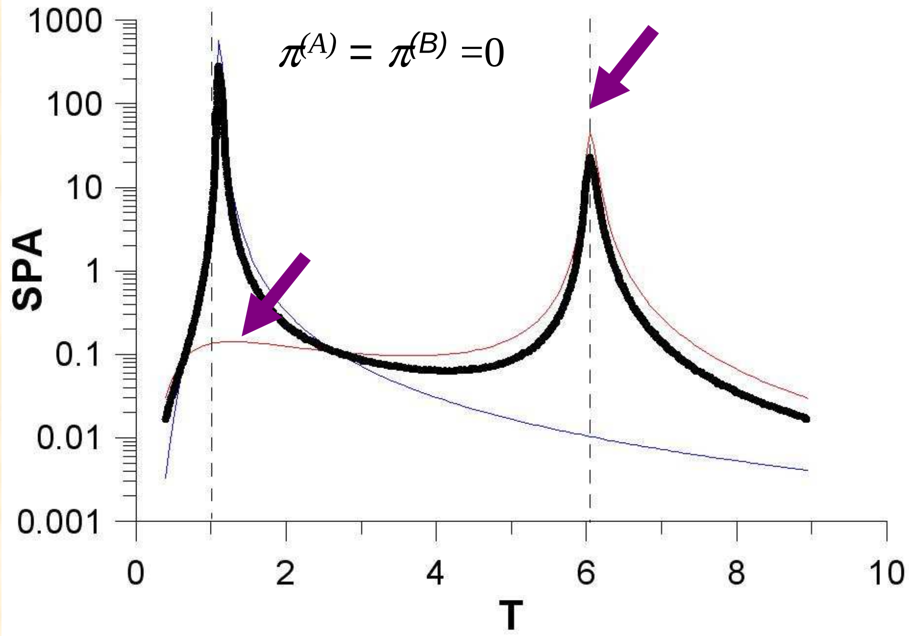
• Coupling networks with different critical temperatures results in the curves *SPA* vs. *T* with two distinct maxima.
• As the rewiring probability increases the maxima approcah each other and the maximum at lower temperature gradually disappears.

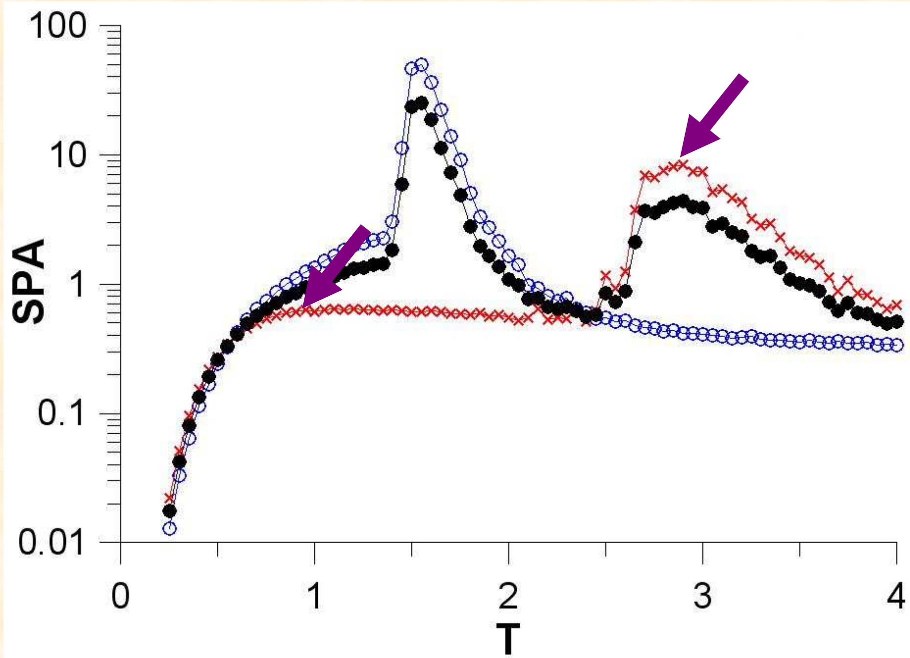
Signal: $S(t)$
$N_A = N_B = 5\,000$,
$m = 5$,
$\omega_s = 2\pi/512$, $h_0 = 0.01$,
for different probabilities of rewiring $\pi^{(A)} = \pi^{(B)} = \nu$ (see legends);
Subnetwork A: $B = 10$ ($\gamma_A = 5$),
Subnetwork B: $B = -3$ ($\gamma_B = 2.5$)
(a) Theoretical results in the mean-field and linear response theory approximations,
(b) Numerical results from Monte Carlo simulations,



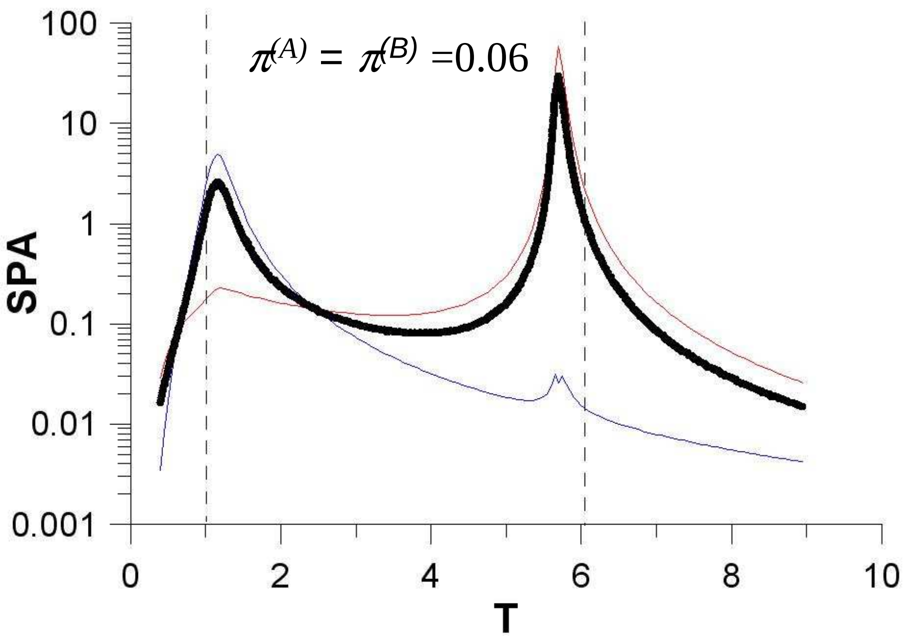
The same system as on the previous slide, but apart from *SPA* vs. *T* for the whole network also $SPA_{A,B}$ vs. *T* for the two subnetworks are shown.

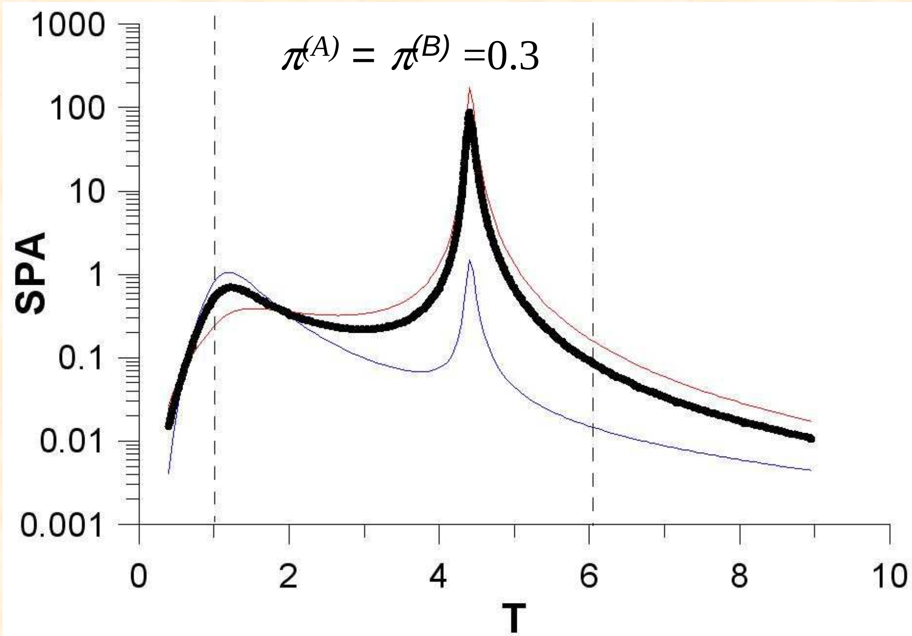
- For small rewiring probability the network has modular structure and the curves $SPA_{A,B}$ vs. *T* resemble those for uncoupled networks. Two maxima of SPA_B vs. *T* are observed for the network with $\gamma_B < 3$.
- As the rewiring probability increases and the density of inter-subnetworks links becomes higher the curves *SPA* vs. *T* for the two subnetworks approach each other and double maxima of the SPA_A appear also for the network with $\gamma_A > 3$,
- The curve SPA vs. T for the composite network resembles in general that for the subnetwork with higher critical temperature.

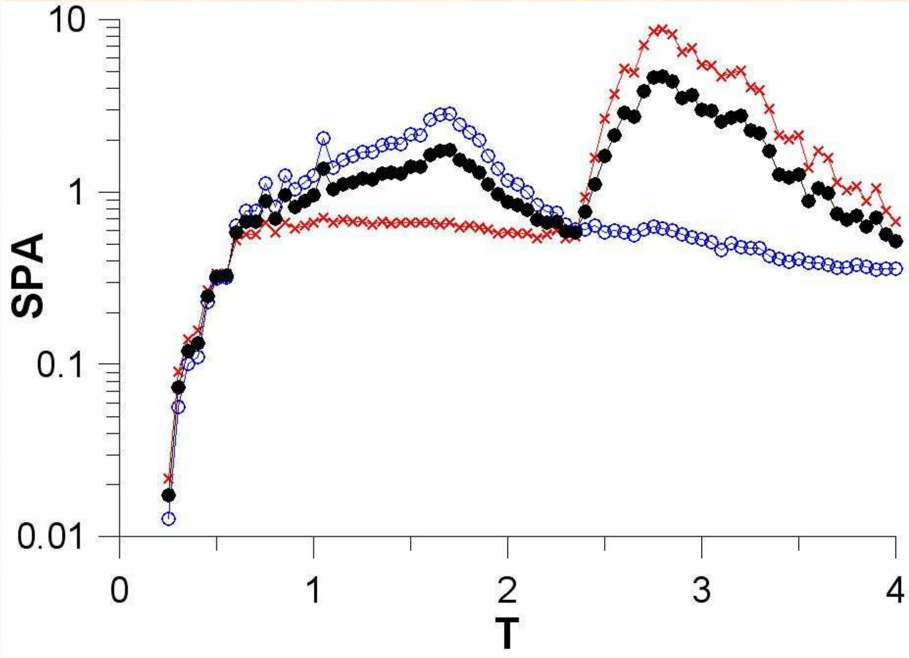


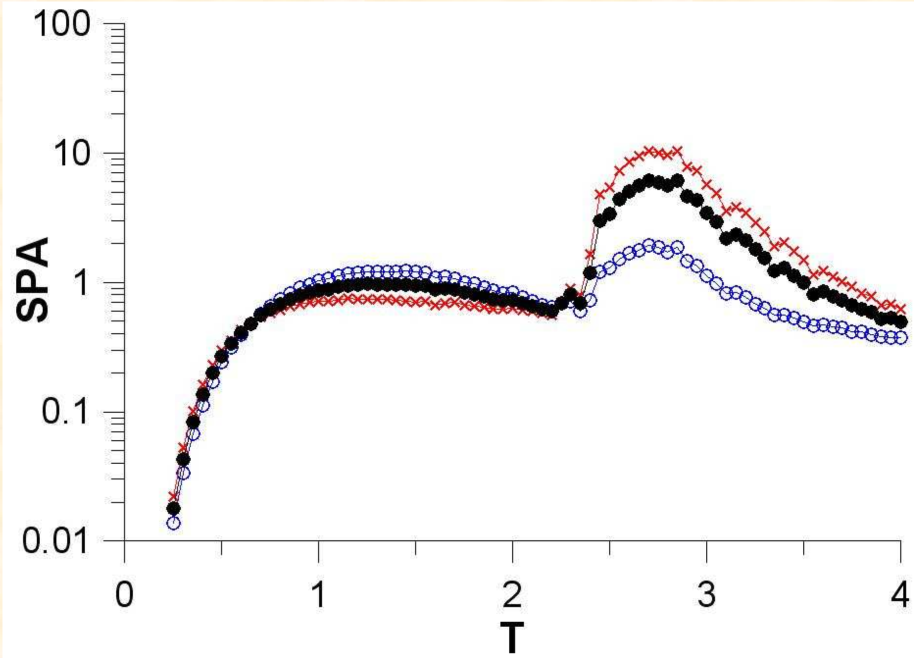


Black symbols/line – *SPA* vs. *T* for the whole network,
Blue symbols/line – SPA_A vs. *T*,
Red symbols/line – SPA_B vs. *T*,
Upper panels: Theoretical results in the mean-field and linear response theory approximations,
Lower panels: Numerical results from Monte Carlo simulations


$\pi^{(A)} = \pi^{(B)} = 0.06$
SPA
T
100
10
1
0.1
0.01
0.001
0
2
4
6
8
10

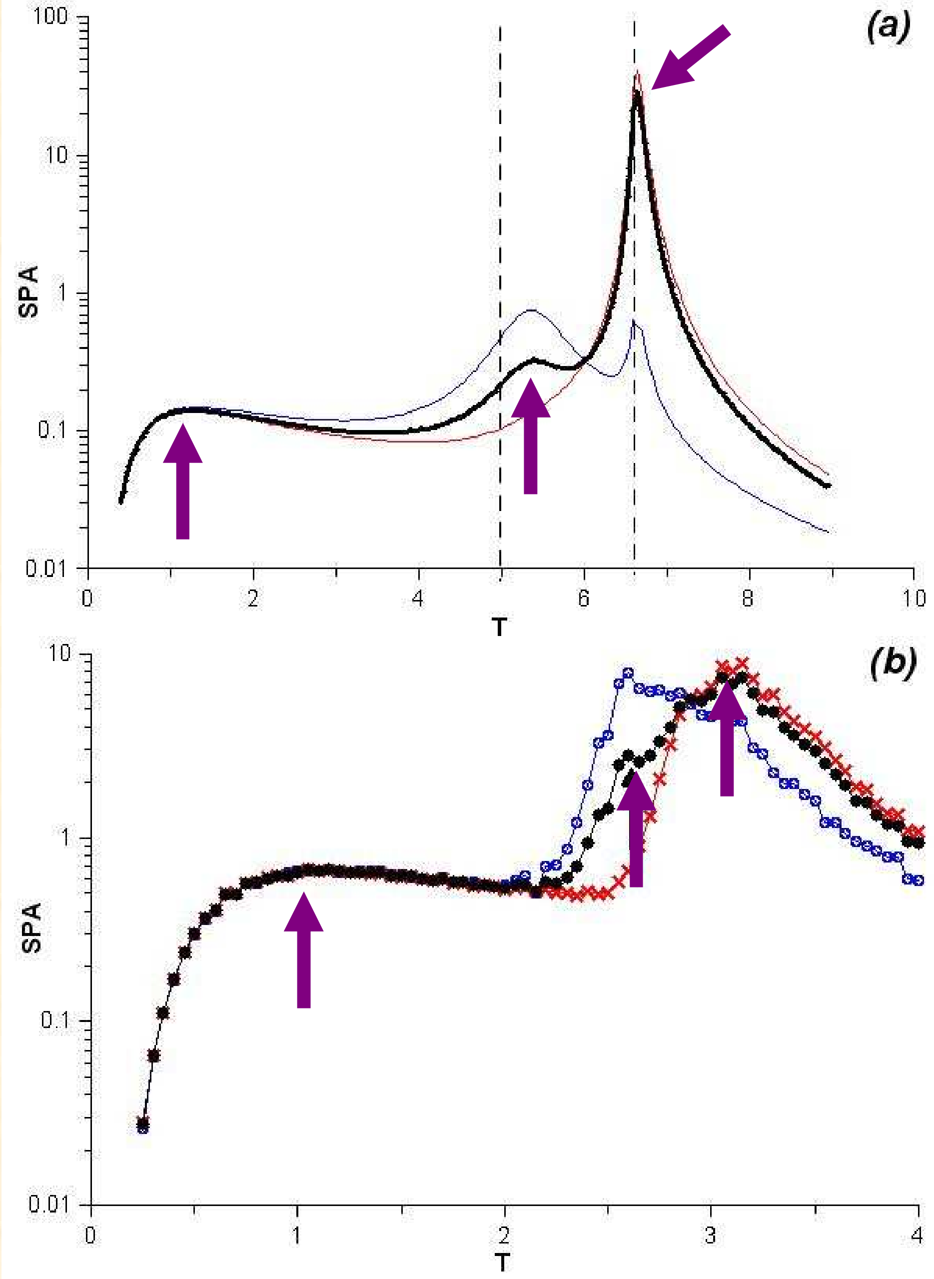

$\pi^{(A)} = \pi^{(B)} = 0.3$
SPA
T
1000
100
10
1
0.1
0.01
0.001
0
2
4
6
8
10


SPA
T
10
1
0.1
0.01
0
1
2
3
4


SPA
T
100
10
1
0.1
0.01
0
1
2
3
4

Networks with different number of nodes, $N_A \neq N_B$, and the same exponents $\gamma_A = \gamma_B$.

• Coupling networks with different critical temperatures can result in the curves *SPA* vs. *T* with two or even three distinct maxima for small to moderate rewiring probability,
• Two maxima are located close to the critical temperatures for the two uncoupled subnetworks, and one deep in the ferromagnetic phase,
• As the rewiring probability rises the two former maxima usually merge into one, located close to the higher of the two critical temperatures



N_A = 3000, N_B = 5 000, m = 5, $\omega_s = 2\pi/512$, h_0 = 0.01, $\pi^{(A)}$ = 0.02, $\pi^{(B)}$ = 0.012

B = -3 ($\gamma_A = \gamma_B$ =2.5)

(a) Theoretical results in the mean-field and linear response theory approximations,

(b) Numerical results from Monte Carlo simulations,

Black symbols/line – *SPA* vs. *T* for the whole network,
Blue symbols/line – SPA_A vs. T,
Red symbols/line – SPA_B vs. T.

Conclusions

• Stochastic multiresonance can be observed in the Ising model on scale-free networks with $2 < \gamma < 3$, for small and moderate frequencies of the oscillating magnetic field and for a large enough number of interacting spins N. One maximum of the curve SPA vs. T (a trivial one) appears at $T \approx T_c$ and the other one usually at $T << T_{c.}$.

• The necessary condition for the occurrence of stochastic multiresonance is the presence of the fully developed power-law tails in the distribution of the degrees of nodes p_k (structural stochastic multiresonance).

• Coupling scale-free networks with different critical temperatures (different number of nodes or exponents γ) also leads to the occurrence of structural stochastic multiresonance. Curves SPA vs. T exhibit two or three maxima.

• The maxima of the SPA vs. T for the composite network can be related to those for the uncoupled subnetworks if the rewiring probability is small to moderate and the composite network has modular structure.

• As the rewiring probability increases and the structure of the composite network becomes more uniform the curve SPA vs. T for the composite network becomes similar to that for the subnetwork with higher critical tempetrature.


Thank you for your attention