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Structural stochastic multiresonance in the Ising model on scale-free networks

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Motivation

- Coupling bistable elements driven by a periodic signal and noise can lead to the enhancement of stochastic resonance (*e.g.*, AESR in spatially extended systems).

- Can suitably chosen **structure of the coupling lead to qualitatively new phenomena in stochastic resonance**?

- Example: **structural stochastic multiresonance** can occur in the Ising model on certain scale-free networks (the curves SPA vs. T show double maxima).

Stochastic multiresonance (concept): J.M.G. Vilar, J.M. Rubi, *Phys. Rev. Lett.* **78**, 2882 (1997); *Physica* **A264**, 1 (1999).

Stochastic resonance in the Ising model

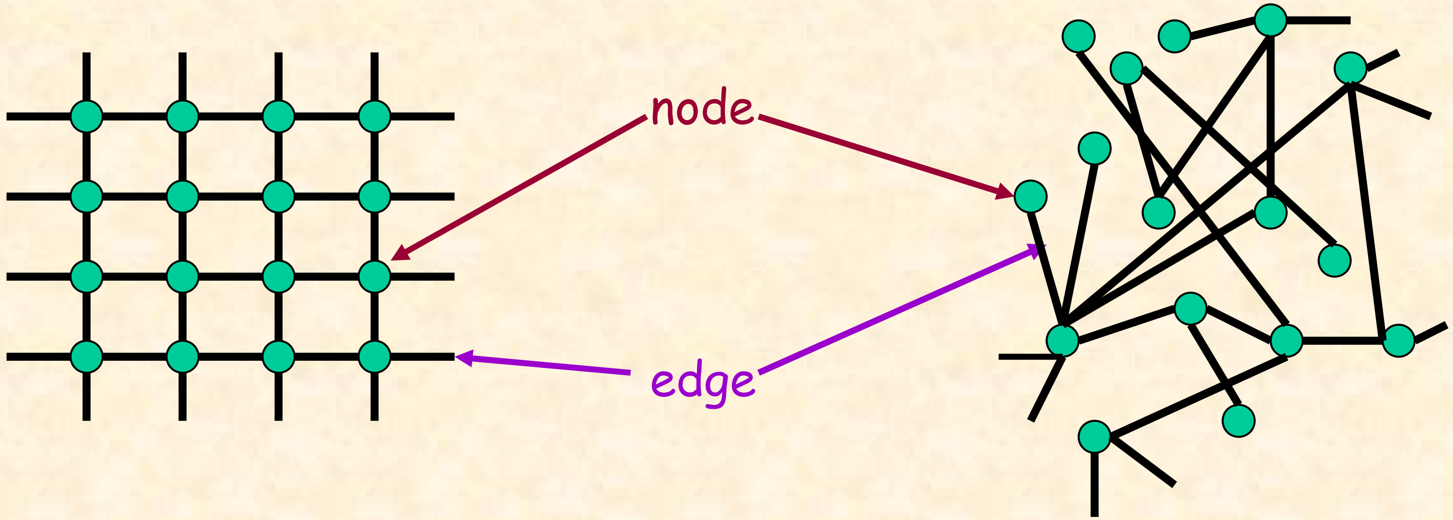
- Periodic signal: oscillating magnetic field,
- Noise: thermal fluctuations (proportional to the temperature),
- Output signal: the time-dependent order parameter (e.g., magnetization).

The Ising model is treated as a complex system which consists of coupled bistable elements (spins), and its response to the periodic signal is studied as a function of the temperature and frequency of the magnetic field.

Exemplary results

- SR in the 1-dimensional Ising model (the paramagnetic phase) J.J. Brey and A. Prados, *Phys. Lett.* **A216**, 240 (1996); U. Siewert and L. Schimansky-Geier, *Phys. Rev.* **E58**, 2843 (1998);
- SR in the 2- and 3-dimensional Ising model (Monte-Carlo simulations and theory in the mean-field approximation): Z. Neda, *Phys. Rev.* **E51**, 5315 (1995), K.-T. Leung and Z. Neda, *Phys. Lett.* **A246**, 505 (1998) et al.;
- Connection with dynamical phase transitions: B.J. Kim et al., *Europhys. Lett.* **56**, 333 (2001);
- SR in the Ising model on complex networks: H. Hong et al., *Phys. Rev.* **E66**, 011107 (2002) (Watts-Strogatz small-world networks), A. Krawiecki, *Int. J. Modern Phys.* **B18**, 1759 (2004) (Barabasi-Albert scale-free networks).

Complex networks



Regular lattice

$$p_k \propto \delta(z) = \delta(4)$$

Complex network

$$p_k = \begin{cases} G_\sigma(k), \\ Ak^{-\gamma} \\ \dots \end{cases}$$

Connectivity of the node i: the number of edges attached to the node i

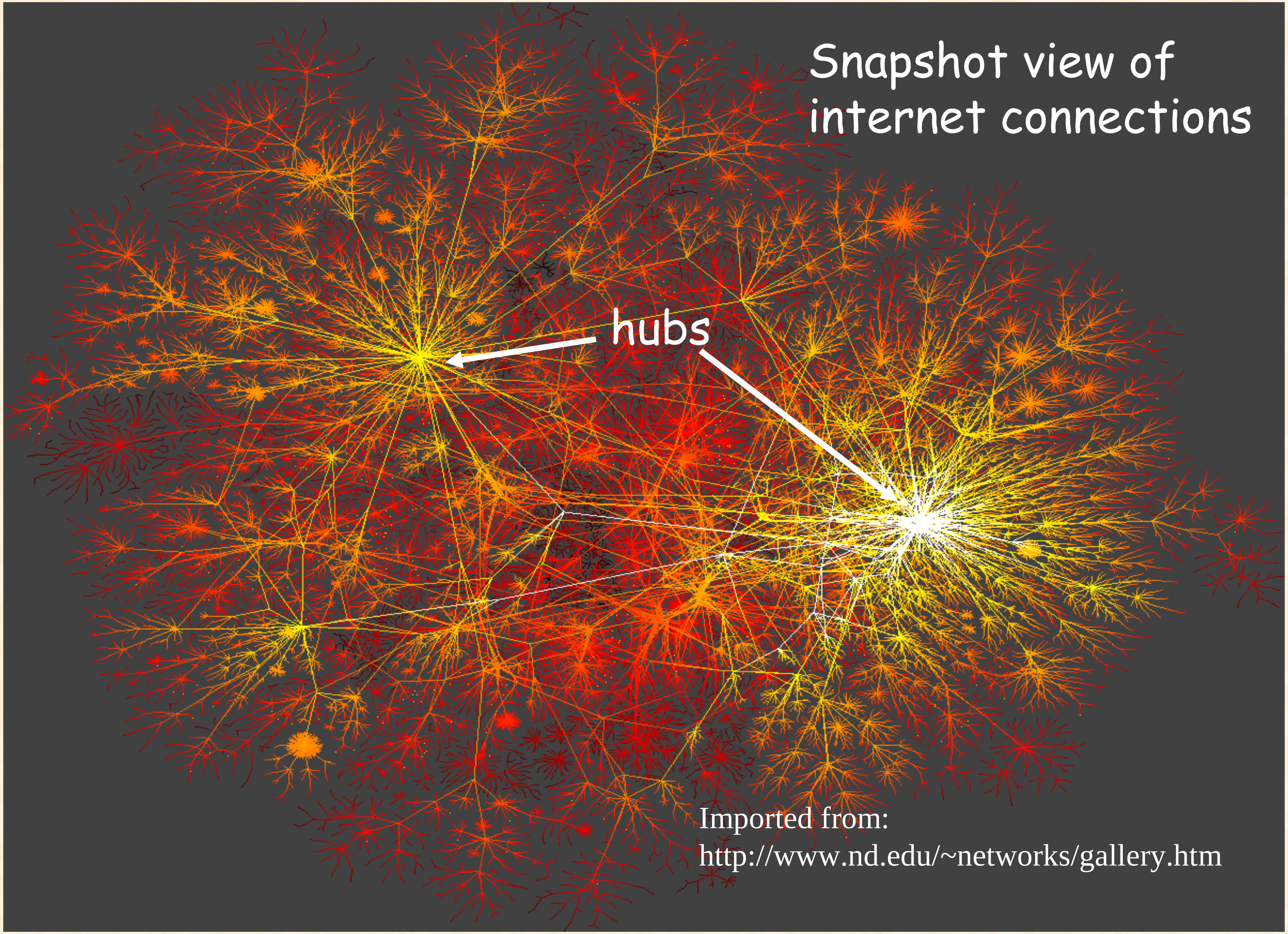
Quantity of interest: distribution of connectivity p_k (= probability distribution that a randomly selected node has connectivity k)

Scale-free networks

- Networks with complex topology are ubiquitous in real world. An important class of complex networks are scale-free networks which look similar at any scale; e.g., the distribution of connectivity k obeys a power scaling law, $p_k \propto k^{-\gamma}$.

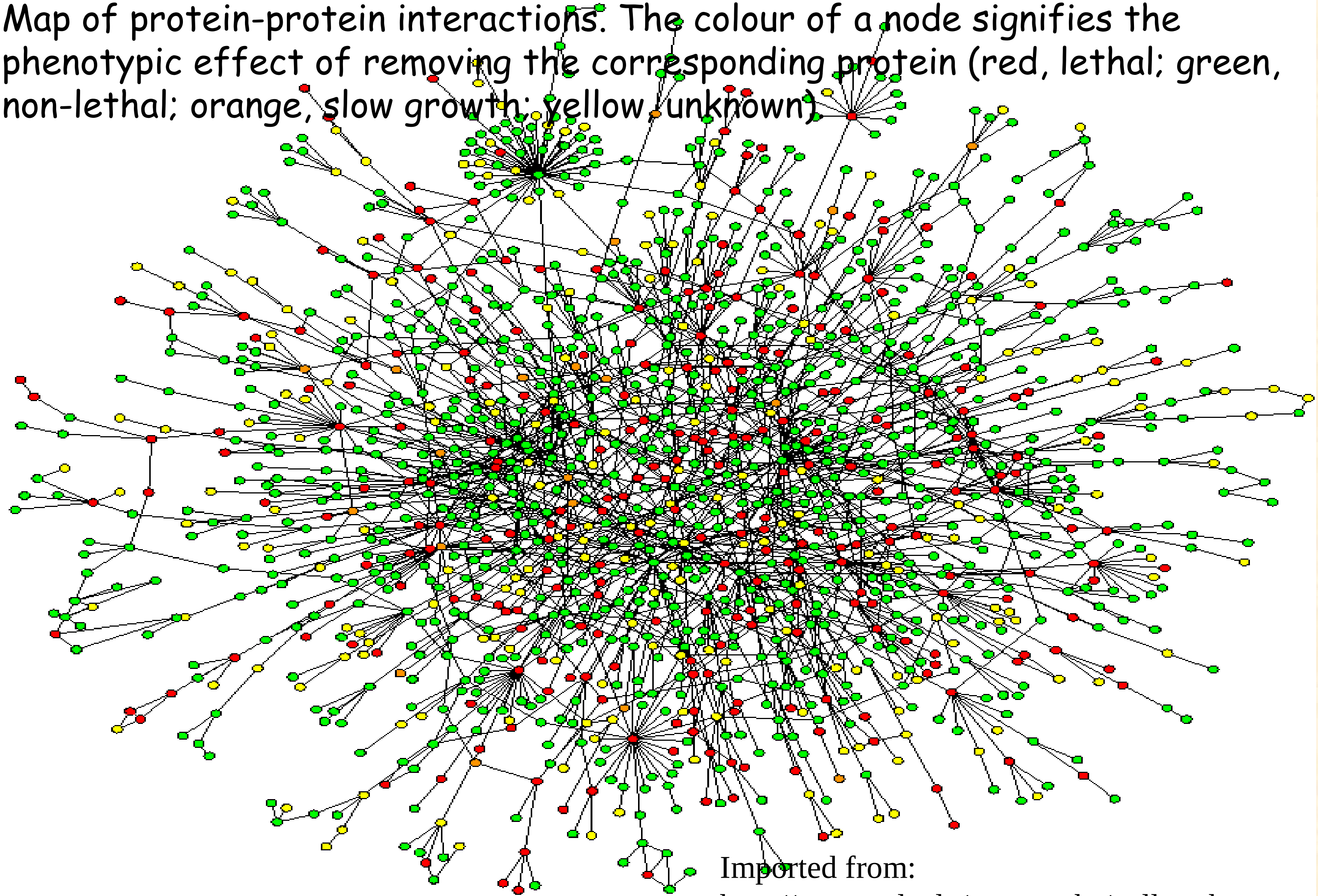
- Examples of scale-free networks comprise, e.g.,
- the internet activity,
- the www links,
- networks of cooperation (between scientists, actors, etc.),
- traffic networks (airplane & railway connections, city transport schemes),
- biological networks (sexual contacts, protein interactions, certain neural networks), etc.

Scale-free networks: examples



Scale-free networks: examples

Map of protein-protein interactions. The colour of a node signifies the phenotypic effect of removing the corresponding protein (red, lethal; green, non-lethal; orange, slow growth; yellow, unknown).



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http://www.nd.edu/~networks/gallery.htm

The Ising model on scale-free networks

The spins s_i are located in the nodes i and edges between the nodes correspond to non-zero exchange interactions between the corresponding spins.

Hamiltonian

$$H = -\frac{1}{\langle k \rangle}\sum_{i,j} J_{ij} s_i s_j - h_0 \sin \omega_0 t \sum_i s_i$$

$J_{ij}=J$ if there is an edge between nodes i,j,
$J_{ij}=0$ otherwise

The local field acting on spin i

$$I_i(t) = \frac{1}{\langle k \rangle}\sum_j J_{ij} s_j + h_0 \sin \omega_0 t$$

The order parameter

$$S(t) = \frac{1}{N\langle k \rangle}\sum_i k_i s_i$$

Glauber dynamics (heat bath algorithm)

The transition rate between two spin configurations which differ by a single flip of one spin, e.g., that in node i

$$w_i(s_i) = \frac{1}{2}\left[1 - s_i \tanh\left(\frac{I_i(t)}{T}\right)\right]$$

Ferromagnetic phase transition

Critical temperature: maximum fluctuations of the order parameter

$$\delta S^2 = \langle S^2 \rangle - \langle |S| \rangle^2$$

Spectral Power Amplification (SPA)

$$SPA = \frac{|P_1|^2}{h_0^2} \qquad P_1 = \lim_{T\to\infty} \sum_{t=0}^{T-1} S(t) \exp(-i\omega_0 t)$$

Mean field approximation

• The Master equation for the probability that at time t the system is in the spin configuration (s_1, s_2, ..., s_n)

$$\frac{d}{dt}P(s_1, s_2, \ldots, s_n; t) = -\sum_{j=1}^{N} w_j(s_j)\,P(s_1, s_2, \ldots, s_j, \ldots, s_n; t) +$$

$$+\sum_{j=1}^{N} w_j(-s_j)\,P(s_1, s_2, \ldots, -s_j, \ldots, s_n; t)$$

• Multiply both sides by s_i and perform an ensemble average

$$\frac{d\langle s_i\rangle}{dt} = -\langle s_i\rangle + \left\langle \tanh\left(\frac{I_i(t)}{T}\right)\right\rangle \qquad (⍂)$$

• Replace s_i ✹ ↑ s_i ⇧, I_i ✹ ↑ I_i ⇧

• Divide the nodes of the network according to their degrees k and assume that the average values of spins located in the nodes belonging to the class with degree k are equal to ↑ s_k ⇧

• Replace the sums over the nodes of the network with sums over the classes of nodes, e.g.

$$\langle S(t)\rangle = \sum_{k=m}^{k_{\max}} \frac{kp_k}{\langle k\rangle}\langle s_k(t)\rangle, \quad \langle I_i(t)\rangle = \frac{Jk_i}{\langle k\rangle}\sum_{k=m}^{k_{\max}} \frac{kp_k}{\langle k\rangle}\langle s_k(t)\rangle + h_i(t)$$

• Multiply both sides of (⍂) by k_i, perform the sum over all nodes of the network and replace it with a sum over the classes of nodes,.

$$\boxed{\frac{d\langle S\rangle}{dt} = -\langle S\rangle + \sum_{k=m}^{k_{\max}} \frac{p_k k}{\langle k\rangle}\tanh\left(\frac{Jk\langle S\rangle}{\langle k\rangle T} + \frac{h_0}{T}\sin\omega_0 t\right)}$$

Mean field approximation

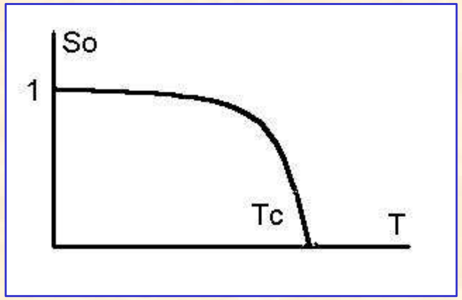
Stationary values of the order parameter and magnetization

$$h_0 = 0 \Rightarrow S_0 = \sum_{k=m}^{k_{max}} \frac{p_k k}{\langle k \rangle} \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) = \sum_{k=m}^{k_{max}} \frac{Ak^{-\gamma+1}}{\langle k \rangle} \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) \rightarrow$$

$$\boxed{S_0 = \int_m^{k_{max}} \frac{Ak^{-\gamma+1}}{\langle k \rangle} \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) dk}$$

$$M_0 = \sum_{k=m}^{k_{max}} p_k \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) = \sum_{k=m}^{k_{max}} Ak^{-\gamma} \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) \rightarrow$$

$$\boxed{M_0 = \int_m^{k_{max}} Ak^{-\gamma} \tanh\left(\frac{JkS_0}{\langle k \rangle T}\right) dk}$$



Critical (crossover) temperature for the ferromagnetic phase transition

$$\sum_{k=m}^{k_{max}} \frac{p_k k}{\langle k \rangle} \frac{Jk}{\langle k \rangle T_c} = 1 \Rightarrow \boxed{T_c = \frac{J}{\langle k \rangle^2} \sum_{k=m}^{k_{max}} k^2 p_k = \frac{J\langle k^2 \rangle}{\langle k \rangle^2}}$$

Mean field approximation

$$p_k = Ak^{-\gamma}, \quad \gamma < 2, \quad k_{\max} = mN^{\frac{1}{\gamma-1}}, \quad A = \frac{\gamma - 1}{m^{-\gamma+1} - k_{\max}^{-\gamma+1}}$$

$$\gamma = 3 \Rightarrow T_c = \frac{J}{4} \ln N$$

$$\gamma \neq 3 \Rightarrow \begin{array}{l} \langle k \rangle = \dfrac{A}{\gamma - 2}\left(m^{-\gamma+2} - k_{\max}^{-\gamma+2}\right) \\ \langle k^2 \rangle = \dfrac{A}{\gamma - 2}\left(m^{-\gamma+3} - k_{\max}^{-\gamma+3}\right) \end{array}$$

$$2 < \gamma < 3 \Rightarrow T_c = J\frac{(\gamma - 2)^2}{(\gamma - 1)(3 - \gamma)} N^{\frac{3-\gamma}{\gamma-1}} \xrightarrow[N \to \infty]{} \infty$$

$$\gamma > 3 \Rightarrow T_c = J\frac{(\gamma - 2)^2}{(\gamma - 1)(3 - \gamma)}$$

[A. Aleksiejuk, J.A. Hołyst, and D. Stauffer, *Physica* **A310**, 260 (2002); A. Aleksiejuk, *Int. J. Modern Phys.* **C13**, 1415 (2002); G. Bianconi, *Phys. Lett.* **A303**, 166 (2002); M. Leone, A. Vázguez, A. Vespignani, R. Zecchina, *Eur. Phys. J.* **B28**, 191 (2002); S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, *Phys. Rev.* **E66**, 016104 (2002); F. Iglói, L. Turban, *Phys. Rev.* **E66**, 036140 (2002); C.P. Herrero, *Phys. Rev.* **E69**, 067109 (2004)]

Linear Response Theory

$$\frac{d\langle S\rangle}{dt} = -\langle S\rangle + \sum_{k=m}^{k_{\max}} \frac{p_k k}{\langle k\rangle} \tanh\left(\frac{Jk}{\langle k\rangle T}\langle S\rangle + \frac{h_0 \sin \omega_0 t}{T}\right)$$

$$\langle S(t)\rangle = S_0 + \xi(t)$$

$$\frac{d\xi}{dt} = -\frac{\xi}{\tau_{MF}} + \frac{Q}{T} h_0 \sin \omega_0 t$$

$$-\frac{1}{\tau_{MF}} = -1 + \frac{J}{T\langle k\rangle^2} \sum_{k=m}^{k_{\max}} p_k k^2 \cosh^{-2}\left(\frac{JkS_0}{\langle k\rangle T}\right) =$$

$$= \frac{A}{\langle k\rangle S_0}\left[k_{\max}^{-\gamma+2} \tanh\left(\frac{Jk_{\max} S_0}{\langle k\rangle T}\right) - m^{-\gamma+2} \tanh\left(\frac{JmS_0}{\langle k\rangle T}\right)\right] - 3 + \gamma$$

$$Q = \frac{1}{\langle k\rangle} \sum_{k=m}^{k_{\max}} p_k k \cosh^{-2}\left(\frac{JkS_0}{\langle k\rangle T}\right) =$$

$$= \frac{A}{JS_0}\left[k_{\max}^{-\gamma+1} \tanh\left(\frac{Jk_{\max} S_0}{\langle k\rangle T}\right) - m^{-\gamma+1} \tanh\left(\frac{JmS_0}{\langle k\rangle T}\right) - (-\gamma+1)\frac{M_0}{A}\right]$$

Linear Response Theory

Linear response to the external periodic signal

$$\xi(t) = \xi_0 \sin(\omega_0 t - \theta)$$

$$\xi_0 = \frac{AQ}{T}\left(\frac{1}{\tau_{MF}^2} + \omega_0^2\right)^{-1/2} \qquad \theta = \operatorname{arctg}(\omega_0 \tau_{MF})$$

SPA in the Linear Response approximation

$$SPA = \frac{\xi_0^2}{4h_0^2}$$

In the paramagnetic phase, *T* > *Tc*

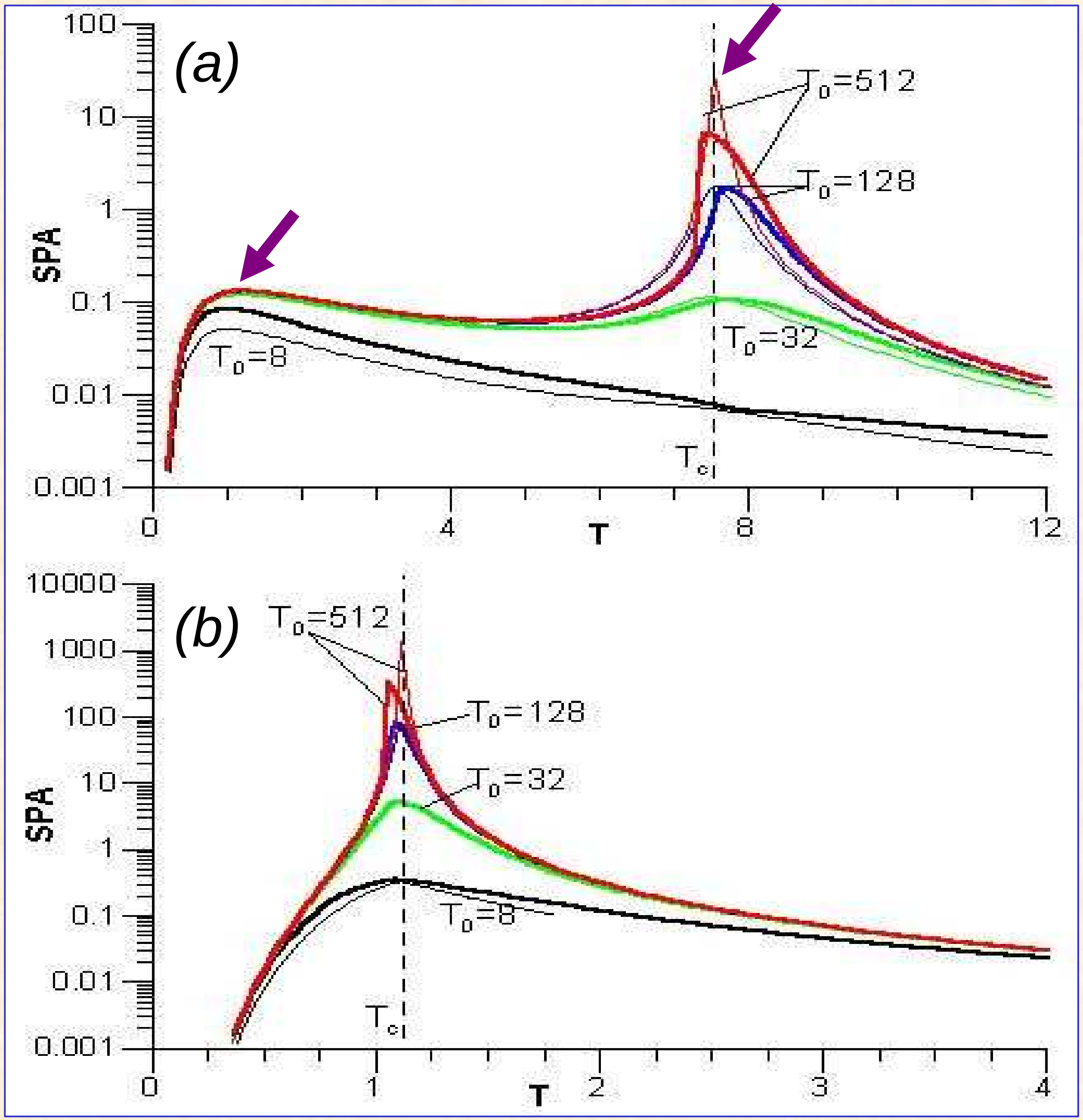
$$S_0 = 0 \Rightarrow Q = 1, \quad \tau_{MF} = \left(1 - \frac{T_c}{T}\right)^{-1}$$

$$SPA = SPA(T) = \frac{1}{4T^2}\left[\left(1 - \frac{T_c}{T}\right)^2 + \omega_0^2\right]^{-1}$$

$$T = T_c \Rightarrow SPA(T_c) = \frac{1}{4T_c^2\omega_0^2} \Rightarrow SPA(T_c) \underset{\gamma \to 2}{\longrightarrow} 0$$

Results: Mean field simulations and Linear Response Theory

Multiresonance (double maximum of the SPA)



Thick: MF simulation,
Thin: LRT

Signal: *S(t)*
N = 10 000,
h_0= 0.01,
(a) γ= 2.5,
(b) γ= 5

How to create scale-free networks: Preferential attachment

[A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999) (for B=0, γ=3);
S.N. Dorogovtsev, J.F.F. Mendes, *Adv. Phys.* **51**, 1079 (2002)]

• The algorithm starts with a small number m of fully connected nodes.

• Evolving network: New nodes are added step by step,

• Preferential attachment: From each new node m new links are created to the existing nodes, and the probability to create an edge to the node i is

$$\boxed{p_i = (k_i + B) \Big/ \sum_i (k_i + B); \quad B > -m}$$

where k_i is the actual number of edges attached to the node i.

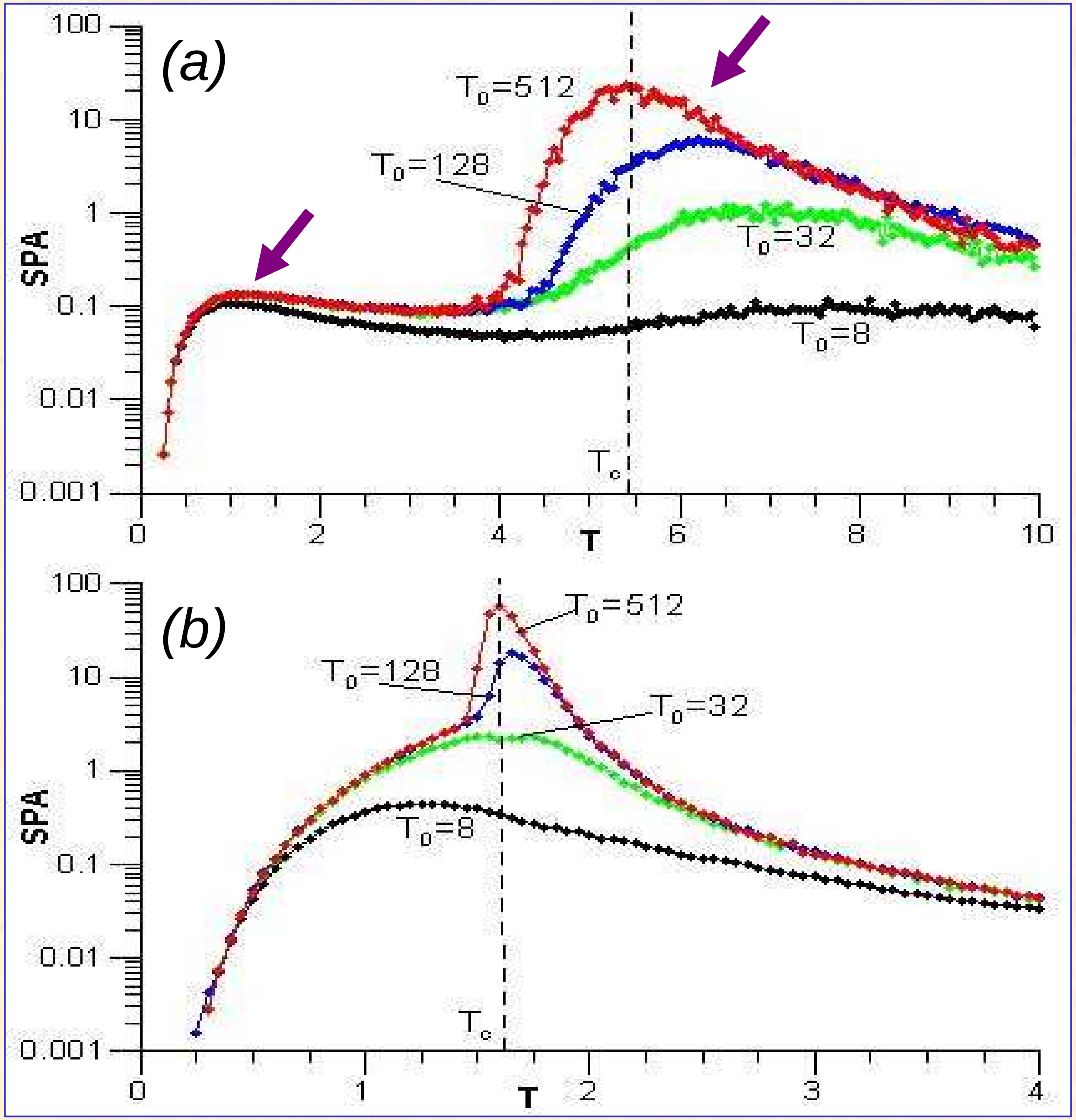
• The network grows until a given number of nodes N is added.

• Evolving network + preferential attachment = scale-free network.
The distribution of the degrees of nodes is

$$\boxed{p_k \propto (k+B)^{-\gamma} \xrightarrow[k\to\infty]{} k^{-\gamma}, \quad \gamma = 3 + B/m}$$

Results: Preferential attachment

Multiresonance (double maximum of the SPA)



Signal: $S(t)$

N = 10 000,

m = 5,

h_0= 0.01,

(a) B = -4 (γ=2.2),

(b) B = 10 (γ=5)

How to create scale-free networks: Configuration Model (CM)

[e.g., M.E.J. Newman, in *Handbook of Graphs and Networks: From the Genome to the Internet*, ed. S. Bornholdt and H. G. Schuster (Wiley - VCH, Berlin 2003), p. 35-68]

• The algorithm starts with assigning to each node i, in a set of N nodes, a random number k_i of „stubs" (ends of edges) drawn from a given probability distribution p_k, with $m \leq k_i < N$, (the maximum degree of node is N-1), with the condition that the sum $\sum_i k_i$ is even.

• The network is completed by connecting pairs of these stubs chosen uniformly at random to make complete edges, respecting the preassigned sequence k_i.

• Physical constraint: Multiple and self-connections are forbidden.

• For scale-free networks with $p_k \propto k^{-\gamma}$, $2 < \gamma < 3$, the latter constraint introduces correlations in the network, in the sense that, e.g., that the average degree of the nearest neighbours

$$\boxed{\bar{k}_{nn}(k) = \sum_{k'} k' P(k'|k)}$$

decreases with k (disassortative mixing - highly connected nodes are more probably linked to poorly connected ones)

Results: CM

Multiresonance (double maximum of the SPA)

Split of the main maximum of the SPA at T T_c (origin unknown)

Signal: $S(t)$

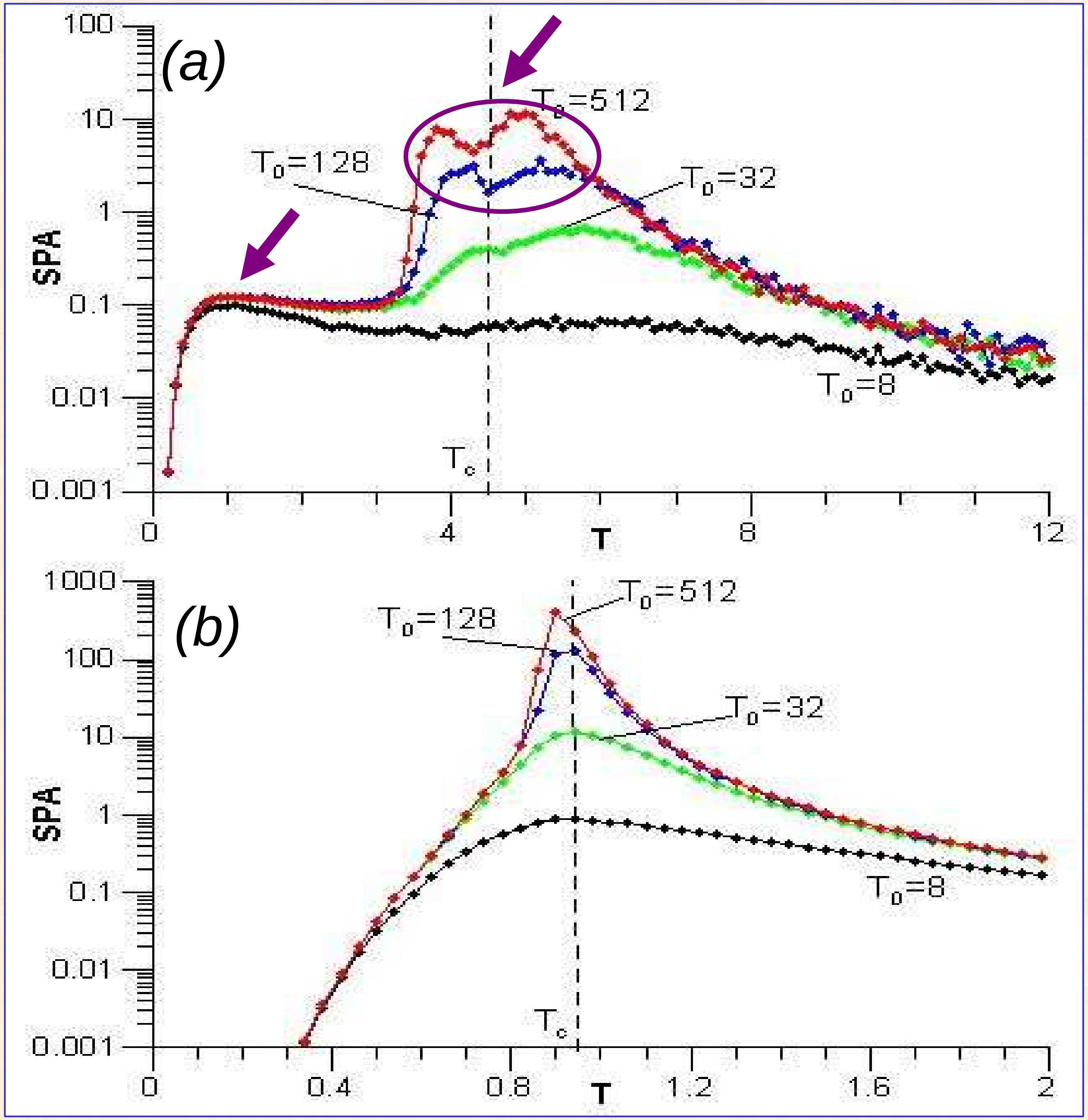
N = 10 000,

m = 5

h_0= 0.01,

(a) γ= 2.5,

(b) γ= 5



How to create scale-free networks: Uncorrelated CM (UCM)

[e.g., M. Catanzaro, M. Boguñá, and R. Pastor-Satorras, *Phys. Rev. E* **71**, 027103 (2005)]

• The algorithm starts with assigning to each node i, in a set of N nodes, a random number k_i of „stubs" (ends of edges) drawn from a given probability distribution p_k, with $m \le k_i < N^{1/2}$, (the maximum degree of node is $N^{1/2} - 1$), with the condition that the sum $\sum_i k_i$ is even.

• The network is completed by connecting pairs of these stubs chosen uniformly at random to make complete edges, respecting the preassigned sequence k_i.

• Physical constraint: Multiple and self-connections are forbidden.

• For scale-free networks with $p_k \propto k^{-\gamma}$ the resulting network is uncorrelated in the sense that, e.g., that the average degree of the nearest neighbours does not depend on k. However, for $2 < \gamma < 3$ the power-law tails in the distribution $p_k \propto k^{-\gamma}$ are not fully developed.

Results: UCM

Signal: *S(t)*

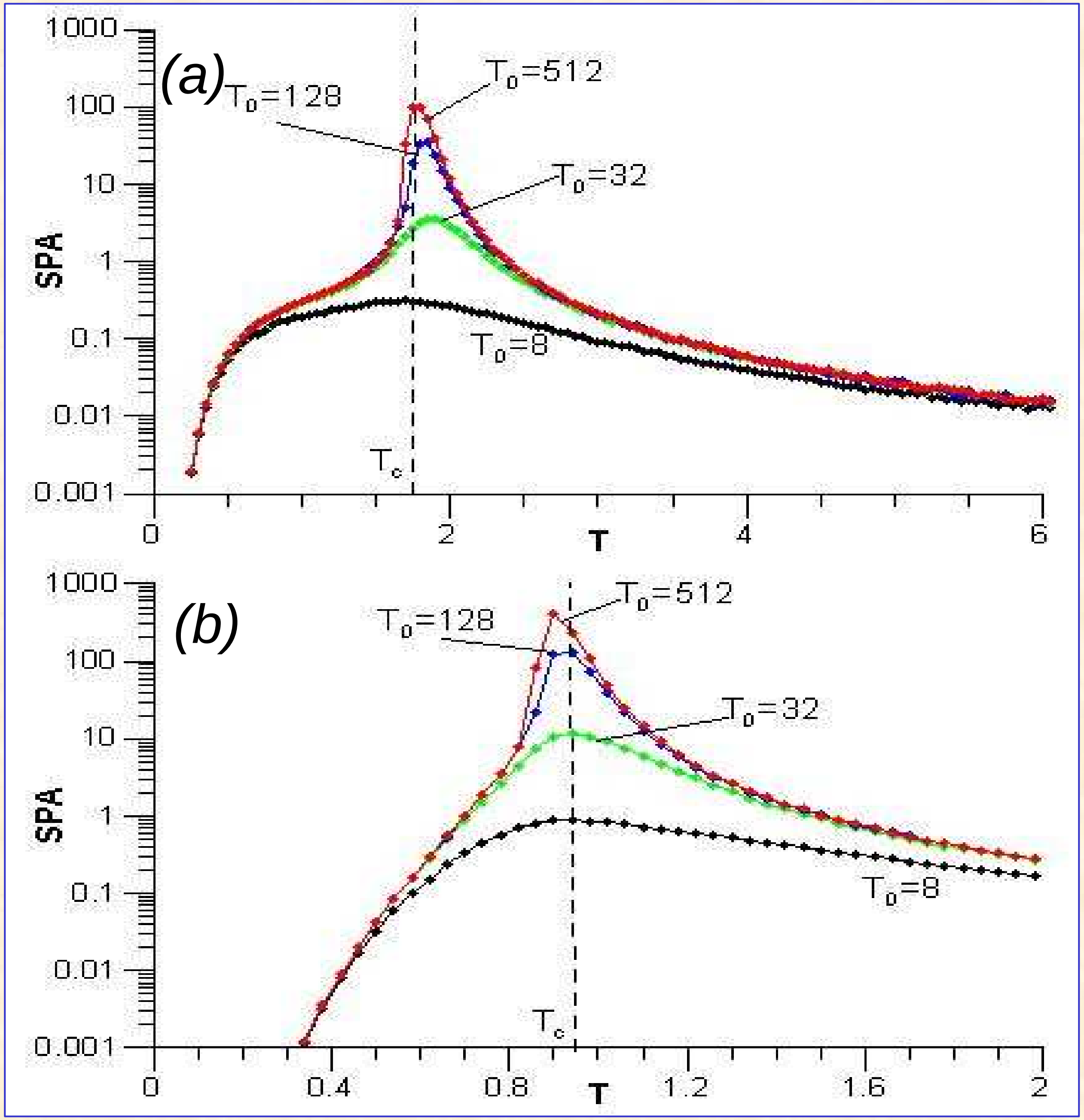
N = 10 000,

m = 5,

h_0= 0.01,

(a) γ= 2.5,

(b) γ= 5



Conclusions

• Stochastic multiresonance is observed in the Ising model on scale-free networks with $2 < \gamma < 3$, for small and moderate frequencies of the oscillating magnetic field and for a large enough number of interacting spins N. One maximum of the curve SPA vs. T (a trivial one) appears at T 🕐 T_c and the other one usually at $T << T_{c.}$.

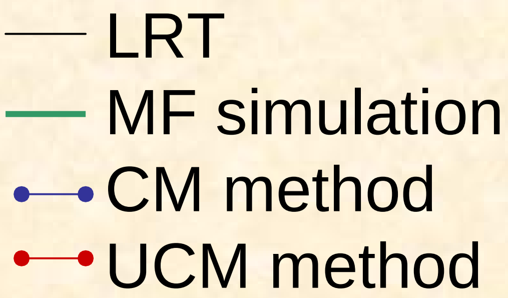
• The necessary condition for the occurrence of stochastic multiresonance is the presence of the fully developed power-law tails in the distribution of the degrees of nodes p_k (structural stochastic multiresonance).

• Open problem: the presence of double (or multiple) maxima of the curve SNR vs. T in the Ising model on scale-free networks.

• Open problem: the presence of structural stochastic multiresonance in other complex systems (coupled bistable stochastic oscillators, etc.) .

Thank you for your attention

Results: Comparison for different methods of creating SF networks



Signal: *S(t)*

N = 10 000,

h_0= 0.01,

T_0 = 128

(a) γ= 2.5,

(b) γ= 5