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Deterministic coherence resonance in systems with on-off intermittency and delayed feedback

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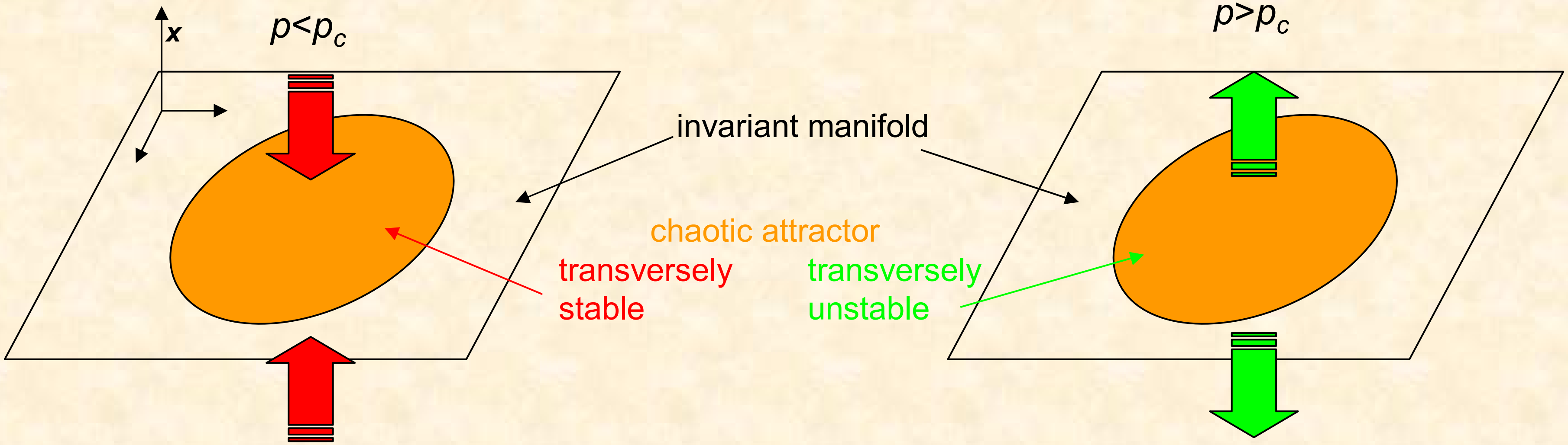
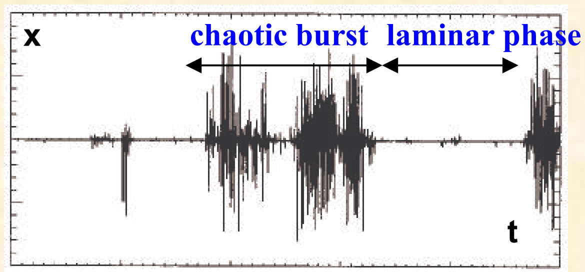
On-off intermittency is a sort of extreme bursting which occurs in systems posessing a chaotic attractor within an invariant manifold whose dimension is less than that of the phase space [e.g, N. Platt, E.A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. **E49**, 1140 (1994)]



As a control parameter p crosses a certain threshold p_c this attractor undergoes a **supercritical blowout bifurcation** and loses transverse stability, and a new attractor is formed which encompasses that contained within the invariant manifold [e.g., E. Ott and J. Sommerer, Phys. Lett. **A188**, 39 (1994)]



Attractor bubbling occurs if the invariant manifold is destabilized below the threshold for the blowout bifurcation, e.g., due to the action of noise in the direction transverse to the invariant manifold. Chaotic bursts occur already for $p<p_c$.

Example

Logistic map with time-dependent control parameter [J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. **E49**, 1140 (1994)]

$$y_{n+1} = a\zeta_n y_n \left(1 - y_n\right)$$

$\zeta_n \in (0,1)$ chaotic process

$y_n = 0$ invariant manifold

Characteristics of on-off intermittency are not sensitive to the details of the dynamics within the invariant manifold (ζ_n) provided that its correlation time is small; thus, ζ_n can be approximated by random process ξ_n with uniform distribution on the interval (0,1).

$$\zeta_n = \xi_n \in (0,1) \Rightarrow a_c = e = 2.71\ldots$$

Characteristics of on-off intermittency

Distribution of lengths of laminar phases

$$P(\tau) \propto \tau^{-3/2}$$

Mean duration of laminar phases

$$\langle \tau \rangle \propto \left| p - p_c \right|^{-1}$$

Example

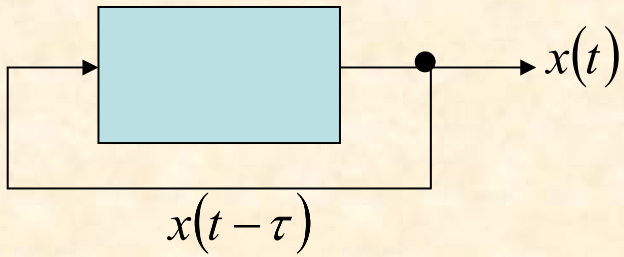
Diffusively coupled chaotic oscillators at the edge of identical synchronization

$$\dot{\vec{x}} = f(\vec{x}) + H(\vec{y} - \vec{x})$$
$$\dot{\vec{y}} = f(\vec{y}) + H(\vec{x} - \vec{y})$$

$H(\vec{z})$ linear coupling function

On-off intermittency occurs as the control parameter (e.g., the strength of the linear diffusive coupling) decreases below the threshold for identical synchronization. The trajectories of the two oscillators are synchronized during long time intervals (laminar phases) but occassionally deviate from each other (chaotic bursts).

In this work we are interested in the effect of delayed feedback (memory) on systems with on-off intermittency



The logistic map with time-dependent control parameter and delayed feedback

$$y_{n+1} = (1-K)a\zeta_n y_n (1-y_n) + K y_{n-k}$$

For $\zeta_n \equiv 1 = const$ this map is a model for chaos control [T. Buchner and J.J. Żebrowski, Phys. Rev. E63, 16210 (2001)]

For $y_n \approx 0$ the dynamics is well aproximated by the linearization $y_{n+1} \approx (1-K)a\zeta_n y_n + K y_{n-k}$

Introducing new variables $y_n^{(1)} = y_n,\, y_n^{(2)} = y_{n-k}, \ldots, y_n^{(j)} = y_{n-k+j-2}, \ldots, y_n^{(k+1)} = y_{n-1}$

the linearized map can be written in the form of a *k*+1 – dimensional map

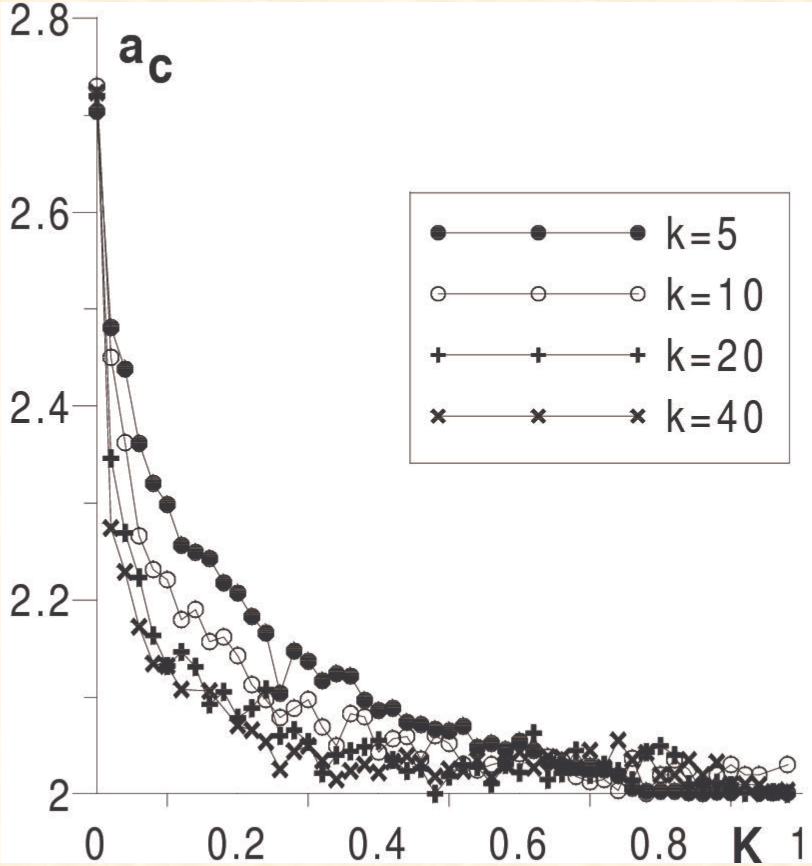
$$\vec{y}_{n+1} = \hat{M}_n \vec{y}_n, \quad \hat{M}_n \equiv \begin{pmatrix} (1-K)a\zeta_n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & & \cdots & 0 \end{pmatrix}, \quad \vec{y}_n \equiv \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(k+1)} \end{pmatrix}$$

with the invariant manifold $\vec{y}_n \equiv 0$

Transverse stability of the attractor within the invariant manifold is determined by the transverse Lyapunov exponent

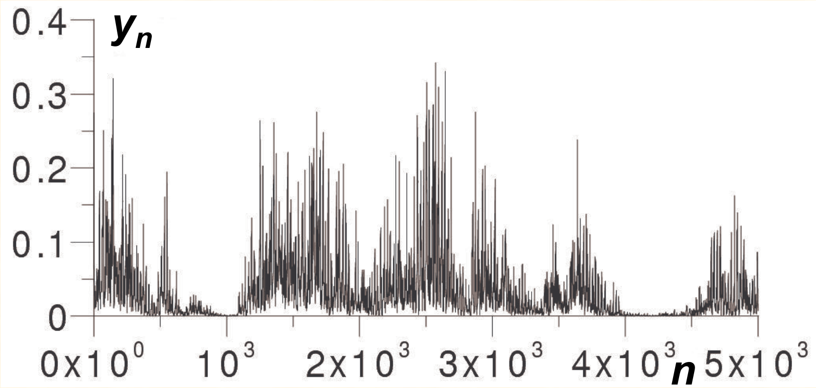
$$\lambda_T = \lim_{N\to\infty} \frac{1}{N} \ln \frac{\left\| \hat{M}_{N-1}\hat{M}_{N-2\ldots}\hat{M}_1 \vec{y}_0 \right\|}{\left\| \vec{y}_0 \right\|}, \; \vec{y}_0 \perp \{(\zeta_n, \vec{y}_n) : \vec{y}_n = 0\}$$

Intermittency threshold a_c is defined as the value of the control parameter a for which the attractor within the invariant manifold y_n = 0 loses transverse stability, i.e., the transverse Lyapunov exponent λ_T crosses zero.



Intermittency threshold a_c vs. K for various k (see legend) for the map with $\zeta_n = \xi_n$ (white noise on the unit interval)

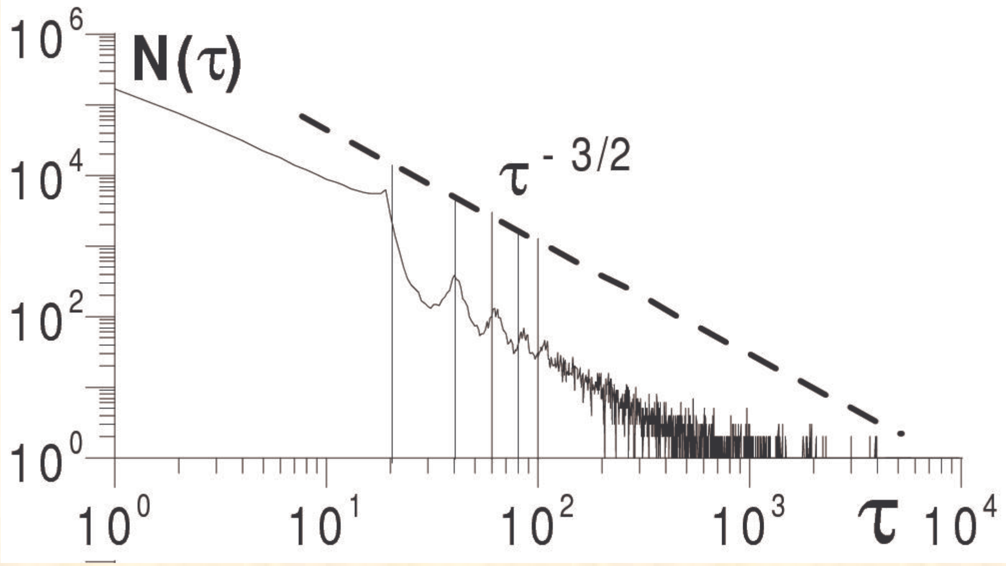
For $a > a_c$ the output signal y_n exhibits intermittent bursts separated by quiescent phases during which $y_n \approx 0$.



Time series y_n for k=20, K=0.2, a=2.2 (just above a_c)

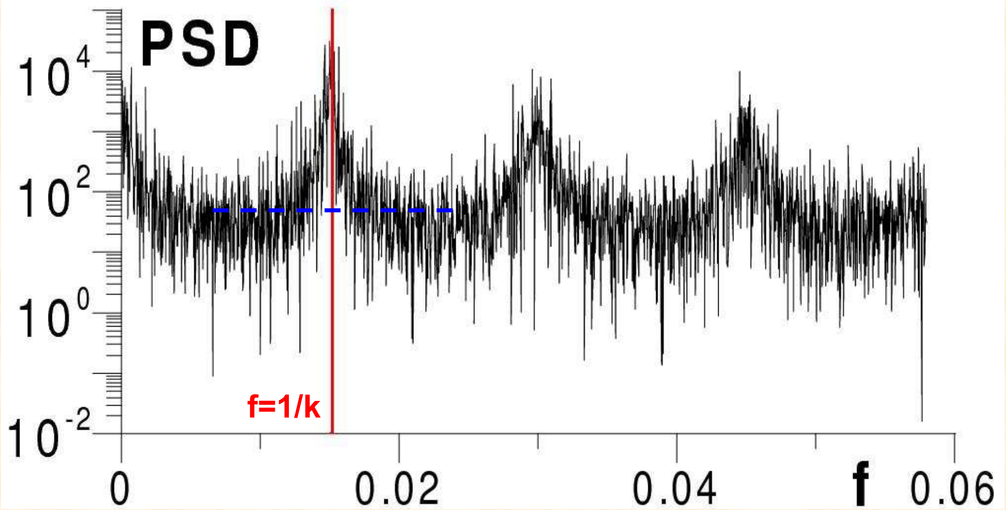
Output signal $$Z_n \equiv \begin{cases} 0 & y_n < y_{thr} = 0.01 \\ 1 & y_n > y_{thr} \end{cases}$$

For K>0 the distribution of laminar phase lengths $N(\tau)$ for a just above a_c exhibits a series of maxima at the values of τ equal to k and its multiples superimposed on a power-law trend typical of on-off intermittency, $N(\tau) \propto \tau^{-3/2}$.



Histogram of the number of laminar phases $N(\tau)$ of duration τ for k=20, K=0.3, a=2.1, vertical lines are drawn at multiples of k

A broad peak centered at the frequency 1/k appears in the power spectrum density (PSD) of Z_n; similar peaks occur at the multiples of 1/k



Power spectrum density from the time series Z_n for k=64, K=0.3, a=2.1

Deterministic coherence resonance in the logistic map with delayed feedback

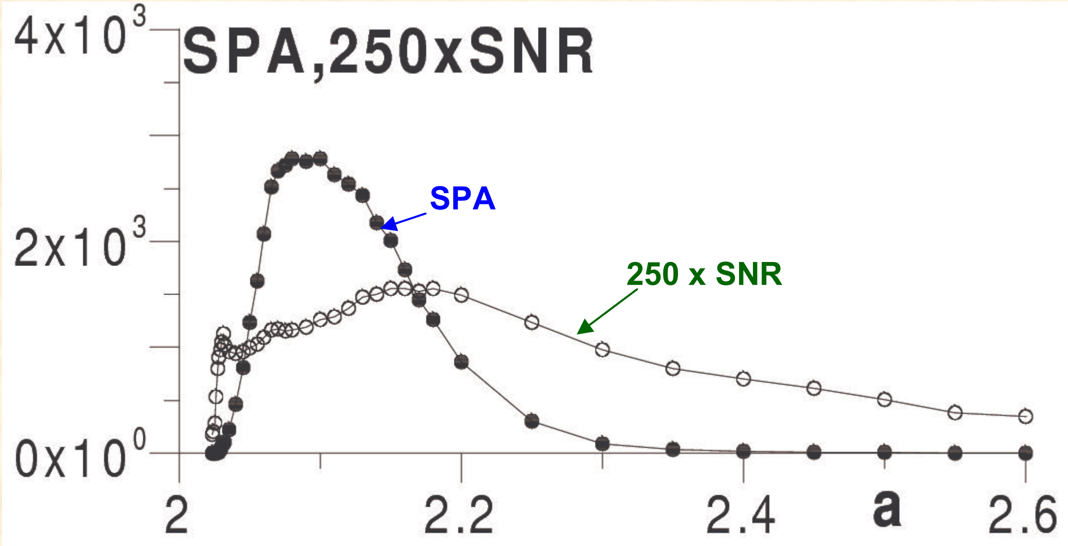
Coherence resonance manifests itself as a peak of regularity of the output signal of certain nonlinear stochastic systems for optimum intensity of the input noise and without any external periodic stimulation [e.g., H. Gang *et al*., Phys. Rev. Lett. 71, 807 (1993); W.-J. Rappel and S.H. Strogatz, Phys. Rev. E50, 3249 (1994); A. Longtin, Phys. Rev. E55, 868 (1997); A.S. Pikovsky and J. Kurths, Phys. Rev. Lett. 78, 775 (1997)].

Coherence resonance was observed in systems with delayed fedback, including

- bistable systems [e.g., L.S. Tsimring and A. Pikovsky, *Phys. Rev. Lett.* **87**, 250602 (2001); K. Panajotov *et al., Phys. Rev.* **A69**, 011801 (2004); M. Arizaleta Arteaga *et al., Phys. Rev. Lett*. **99**, 023903 (2007)]
- excitable systems [e.g., G.C. Sethia, J. Kurths, and A. Sen, *Phys. Lett*. **A364**, 227 (2007)]
- threshold crossing detectors [e.g., R. Morse and A. Longtin, *Phys. Lett*. **A359**, 640 (2006)].

SPA and *SNR* show maxima as functions of the control parameter *a*

SPA and *SNR* vs. *a* from the PSD of Z_n for *k*=64, *K*=0.3.
***SPA* is defined as the absolute height of the maximum of the PSD at *f=1/k*;**
***SNR* is defined as the relative height of the maximum of the PSD at *f=1/k* with respect to the mean value of the PSD on the interval** (*-1/2k, 3/2k*)



Here, maximum regularity of the output signal is observed as the dynamics within the invariant manifold, controlled by the parameter a, is changed, without external noise; so this phenomenon is an example of **deterministic coherence resonance** [e.g., J.F. Martinez Avila, H.L.D. de S. Cavalcante, and J.R. Rios Leite, *Phys. Rev. Lett*. **93**, 144101 (2004)]

Diffusively coupled Rössler oscillators with delayed feedback

$$\begin{aligned}
\dot{x}_1 &= -(y_1 + z_1) \\
\dot{y}_1 &= x_1 + a y_1 + \boxed{k(y_2 - y_1)} + \boxed{Ks(\tau)} \\
\dot{z}_1 &= b + z_1(x_1 - c) \\
\dot{x}_2 &= -(y_2 + z_2) \\
\dot{y}_2 &= x_2 + a y_2 + \boxed{k(y_1 - y_2)} - \boxed{Ks(\tau)} \\
\dot{z}_2 &= (b + \delta b) + z_2(x_2 - c)
\end{aligned}$$

(diffusive coupling: $k(y_2 - y_1)$, $k(y_1 - y_2)$; delayed feedback: $Ks(\tau)$)

$$a = 0.2,\ b = 0.2,\ c = 11$$

$$s(\tau) \equiv y_2(t-\tau) - y_1(t-\tau) = \Delta y(t-\tau)$$

$\delta b \neq 0$ models small parameter mismatch in experimental systems, which leads to the occurrence of attractor bubbling

- For K=0 and δb=0 the oscillators are identically synchronized if $k > k_c \approx 0.12\ldots$, and for $k < k_c$ intermittent bursts of desynchronization (on-off intermittency) occur
- For K=0 and δb>0 chaotic bursting occurs already for $k > k_c$ (due to attractor bubbling)

For K=0 and δb=0 and $k > k_c \approx 0.12\ldots$ chaotic attractor exists within the invariant synchronization manifold

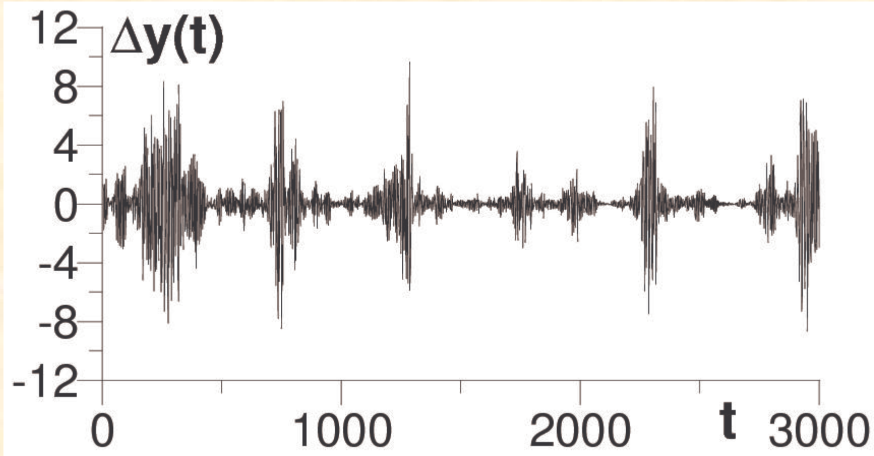
$$x_1 = x_2;\ y_1 = y_2;\ z_1 = z_2$$

which loses transverse stability for $k < k_c$

Output signal $$Z(t)=\begin{cases}0 & \Delta y(t)=y_1(t)-y_1(t)<0.1\\ 1 & \Delta y(t)=y_1(t)-y_1(t)>0.1\end{cases}$$

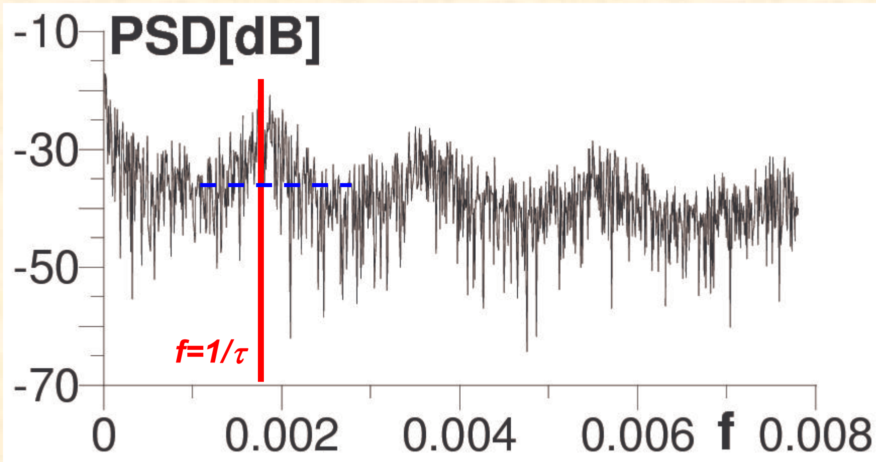
For $k < k_c$ (below the intermittency threshold, when the synchronization is lost) in the system of diffusively coupled oscillators and delayed feedback similiar phenomena are observed as in the logistic map with time-dependent control parameter.

For $k < k_c$ the output signal $\Delta y(t)$ exhibits intermittent bursts separated by quiescent phases during which $\Delta y(t) \approx 0$.



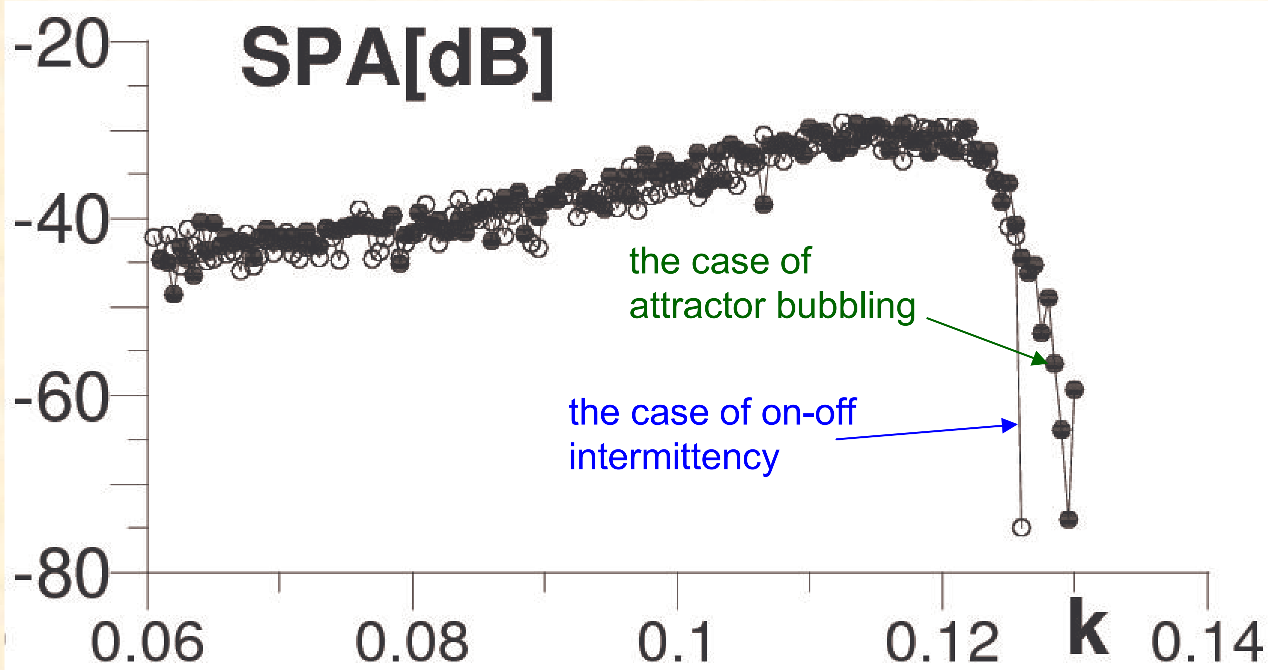
Time series $\Delta y(t)$ for τ=512, k=0.12, δb= 10^{-4}

A broad peak centered at the frequency $1/\tau$ appears in the power spectrum density (PSD) of $Z(t)$; similar peaks occur at the multiples of $1/\tau$



Power spectrum density from the time series $Z(t)$ for τ=512, k=0.12, δb= 10^{-4}

Deterministic coherence resonance in the system of diffusively coupled Rössler oscillators with delayed feedback



SPA vs. *k* for δb= 0 (circles, the case of on-off intermittency) and δb= 10^{-4} (dots, the case of attractor bubbling)

Deterministic coherence resonance occurs in the system of diffusively coupled Rössler oscillators and the output signal exhibits maximum regularity for optimum value of the parameter k which controls the internal dynamics within the invariant synchronization manifold. Hence, the results of numerical simulations suggest that deterministic coherence resonance can be observed experimantally in systems of coupled chaotic oscillators at the edge of identical synchronization.

Conclusions

• The influence of delayed feedback on systems with on-off intermittency was studied using generic maps with the time-dependent control parameter and synchronized chaotic oscillators.

• Delayed feedback can decrease the threshold for the blowout bifurcation, and its effect resembles that of noise applied transversely to the invariant manifold.

• **Deterministic coherence resonance** was observed in systems under consideration, characterized by the appearance of a series of maxima at the multiples of the delay time in the probability distribution of the lengths of laminar phases, superimposed on the power-law trend typical of on-off intermittency, and by the presence of a strong periodic component in the intermittent time series, with period equal to the delay time.

• The strength of the latter component exhibits maximum as the control parameter is varied, due to the changes of the internal dynamics of the system within the invariant manifold, in analogy with the case of coherence resonance in stochastic systems.


Thank you for your
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courtesy of Dr. P. Jóźwiak, Svalbard 2001