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Dynamics of a one-dimensional neural network with a "small world"—topology of synaptic connections

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Abstract

Dynamics of a one-dimensional neural network with external periodic stimulation are investigated numerically. Synaptic connections with constant and random values were assumed. Three ranges of network dynamics—periodic, intermediate and chaotic—were found, depending on the number of short-range synaptic connections k and the gain parameter g, which are the control parameters. The influence of the number of k and additional r long-range connections (shortcuts), typical for a "small-world" network, on the dynamics of the system is discussed. With an increase in the values of g and k, clusters of neurons which do not follow external stimulation appear in the network. For large enough values of g and k, a network has chaotic dynamics. The presence of shortcuts may have either a stabilizing or destabilizing effect on the dynamics of a network. In particular, the presence of shortcuts with certain locations may increase the susceptibility of network to external stimulation.

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1. Introduction

The dynamics of the cellular neural networks (CNNs) plays a fundamental role in applications like pattern recognition or complex system modeling [1]. Neurons in

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CNN, which form a regular lattice, are connected only with some neighbors of the first, second or third order, i.e., synaptic connections are short-range [1]. The dynamics of discrete-time CNNs had been investigated for a one-dimensional case [1–3] as well as for a two-dimensional case [4]. Recently, many authors investigated networks in which a small number of long-range connections (shortcuts) were added to standard short-range connections. The topology of such connections is called the topology of a “small world”. It can be found in different types of real networks, like computer networks, networks of social contacts, power networks and others (for a review, see e.g. papers [5–8]). Initially, most research investigated static properties of small-world networks, like the average shortest-path distance, the clustering coefficient, the correlation length or scaling properties [5–8]. Next, also dynamic models—of information propagation, properties of regular networks of coupled dynamic systems, spreading of infections in populations, etc.—based on small-world networks were extensively investigated [5,9–13].

Investigations of a one-dimensional CNN with a standard short-range connections and an external sinusoidal stimulation with the period T showed the network’s complex dynamics. Four types of attractors were found: they depend on the gain of the neuron g and the range of neighbors k [4]. In this paper, we present the results of investigations of the dynamic properties of a one-dimensional CNN, discuss the effect of the shortcuts and compare them with the case of a standard short-range connections.

2. The model

In our model, a one-dimensional CNN is represented by a chain of N neurons with periodic boundary conditions and discrete-time dynamics given by

$$S_i(t+1) = \tanh[g h_i(t) + I(t)] , \tag{1}$$

where the local field h_i depends on short-range J_{ij} and long-range $J_{iu}^{(l)}$ synaptic connections

$$h_i(t) = \sum_{j=i-k}^{i+k} J_{ij} S_j(t) + \sum_{u=1}^{N} c_{iu} J_{iu}^{(l)} S_u(t) . \tag{2}$$

External stimulation has the form

$$I(t) = A \sin\left(\frac{2\pi}{T} t\right) . \tag{3}$$

In those equations g is the gain parameter, $J_{ij} \in [-1, 1]$ is a short-range synaptic connection between the neurons S_i and S_j located in the neighborhood with the radius k. Moreover, in the network there are r additional long-range connections $J_{iu}^{(l)}$. Fig. 1 illustrates the structure of connections for a network with $N = 12$, $k = 2$ and $r = 2$. The shortcuts in the network occur with the probability p, i.e., the number of

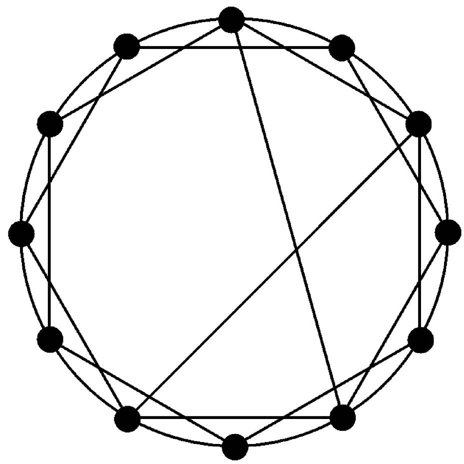


Fig. 1. The structure of the network with $N = 12$ neurons, $k = 2$ and the number of shortcuts $r = 2$.

shortcuts divided by the number of all possible connections:

$$p = \frac{r}{N(N-1)/2} . \tag{4}$$

The location of shortcuts is generated randomly. This means that the presence of a long-range synaptic connection $J_{iu}^{(l)}$ between the $S_i - S_u$ pair of neurons is a result of the value of the random variable c_{iu} ($c_{iu} = 1$ for an existing long-range connection, otherwise $c_{iu} = 0$).

Numerical simulations were used to investigate network dynamics. To observe the properties of local dynamics, profiles were calculated for the states of all neurons $S(i)$ (where $i = 1, 2, 3, \ldots, N$) for different times. Return maps $S_i(t + T)[S_i(t)]$, showing the character of the Poincaré sections, were observed, too. To observe the properties of global dynamics of the network, average deviation

$$\sigma(t) = \frac{1}{N} \sum_{i=1}^{N} [S_i(0) - S_i(t)]^2 \tag{5}$$

and the return map $\sigma(t + T)[\sigma(t)]$ were used. The Fourier transformation in the FFT form was used to calculate the characteristic frequencies of the system and to observe the ranges of the network's periodic, quasiperiodic and chaotic time evolutions.

Numerical experiments were made for $N = 100$ and 10000, the period of external stimulation $T = 24$, different values of the gain parameter g, the order of neighbors k (the number of short-range connections of each neuron equals $2k + 1$) and the amplitude A, using synchronous dynamics. Constant or random values of J_{ij}, random initial values of the neurons $S_i(0)$ for $i = 1, 2, 3, \ldots, N$ were assumed. All random numbers were generated from uniform distribution and were not correlated. In the computations, transient time did not exceed 1500 steps.

3. Dynamics of a network without shortcuts

When the synaptic connections J_{ij} are constant, the values of neurons $S_i(t)$ for $i = 1, 2, 3, \ldots, N$ and $t = T, 2T, 3T, \ldots,$ show uniform oscillations of all neurons with the period T of external stimulation. This is so because the local field h_i has the same value for all neurons (dashed curve in Fig. 2). Uniform oscillation appears irrespectively of the values of k and J_{ij}. A change in the values J_{ij} and k does not break the symmetry of the system and its dynamics do not change.

When the synaptic connections are random, the network's dynamics are more complex. For the smallest number of neighbors of neurons, $k = 1$, and small enough values of the gain parameter g, the values of the neurons $S_i(t)$ for $i = 1, 2, 3, \ldots, N$ and $t = T, 2T, 3T, \ldots$ (see Fig. 2) show oscillation with the period T. The network is spatially disordered (textiti.e. S_i have different values), which results from the random values of the synaptic connection. For g exceeding slightly $g = 0.5$, a cluster of neurons that oscillate with a period other than T appears in the network (solid line in Fig. 2). Depending on the current random distribution of synaptic connections, one or more clusters may be present in the network at the same time. The number of neurons m that form the clusters increases with g and reaches 90% for $A = 0$ (Fig. 3a). The amplitude of the external stimulation A influences the value of m, too. The larger the values of the amplitude, the smaller the number of neurons that form the clusters. This is so because the influence of external stimulation is greater (cf. curves in Fig. 3a).

For $k = 3$, the clusters spread to the whole network and m reaches 100% for large enough g. As in the case $k = 1$, greater values of the amplitude A cause a decrease in m (see Fig. 3b). A further increase in k does not cause qualitative changes on the $m(g)$ curves (see Fig. 3c). For small g the number $m = 0$, irrespectively of k, but this range with no clusters decreases with k. An increase in the network's size to over $N = 1000$

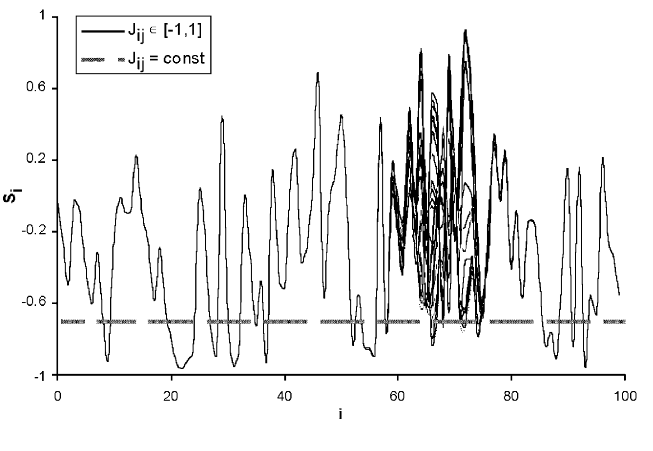


Fig. 2. The values of the neurons $S_i(t)$ for $i = 1, 2, 3, \ldots, N$ depicted for $t = T, 2T, 3T, \ldots,$ for $k = 1$, $g = 1$, the case of the constant values of synaptic connections (dashed line, $J_{ij} = 0.5$) and random values of synaptic connections (solid curve). A cluster of neurons is located between S_{60} and S_{80}.

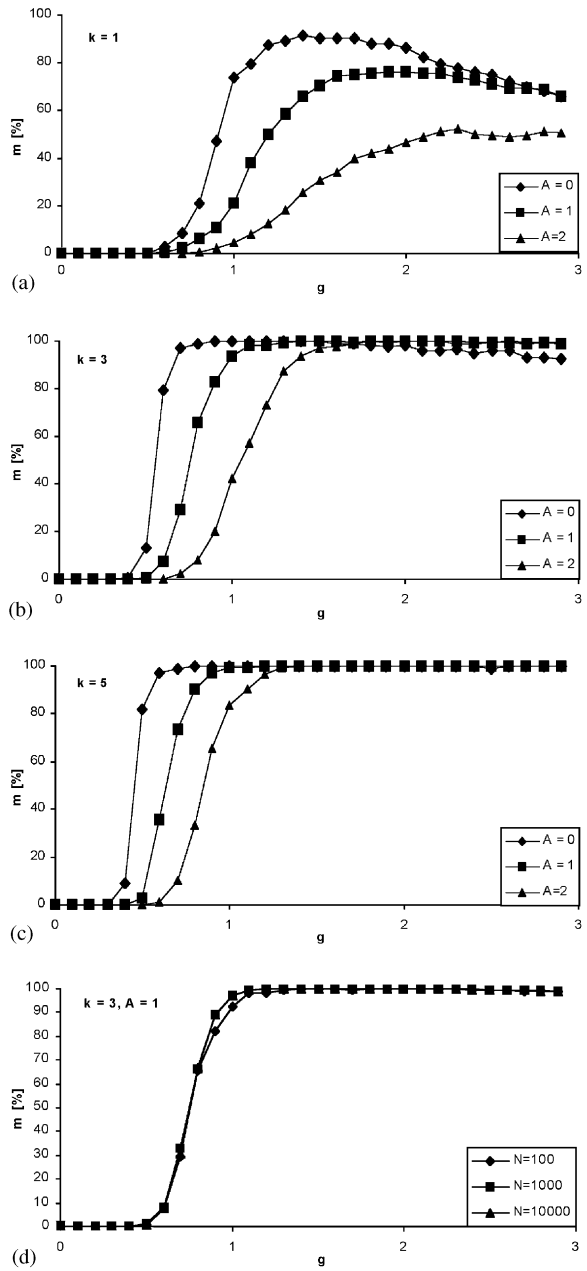


Fig. 3. The number of neurons m forming clusters as a function of the gain parameter g, for $N = 100$, averaged over 100 (a,b,c) and 10000 (d) distributions of synaptic connections and different values of the amplitude of external stimulation. a—for $k = 1$; b—for $k = 3$; c—for $k = 5$. (d) illustrates the relation $m(g)$, for $N = 100$, 1000 and 10000.

does not influence the number m of neurons that form the clusters (see Fig. 3d). Note that, there are some differences between results obtained for $N = 100$ and $N \geqslant 1000$ in specific range of g. The behavior of the curve $m(g)$ for $N = 100$ is the result of same finite size effect.

Investigations of local dynamics in clusters show quasiperiodic or periodic oscillations of the time evolution of neurons with the period $T_1 = 3T$. For quasiperiodic oscillations, the return map for a neuron located within a cluster has the form of a closed loop (Fig. 4a). When oscillations of a neuron with the period $T_1 = 3T$ are observed, there are three points on the return map (Fig. 4b). Sometimes the dynamics of the neurons in the clusters are chaotic (Fig. 4c). These types of the time evolutions of neurons in clusters may be treated as intermediate dynamics leading to a chaotic evolution of the whole network. According to the literature, quasiperiodic motion is a typical route to chaos [14]. According the Li–Yorke theorem [15], oscillations with the period of $3T$ may lead to chaos in certain dynamic systems, too.

Application of the Fourier Transformations in the FFT form [14] to the average deviation $\sigma(t)$ makes investigating the emergence of chaos in the network possible. Fig. 5 illustrates the results: the horizontal and vertical axis correspond to the frequency and gain g, respectively. For a definite value of g, black points denote characteristic frequencies (peaks in the Fourier spectrum). Fig. 5a, b and c correspond to $k = 1, 3, 5$, respectively. Figures in the left column are calculated for $A = 0$, whereas figures in the right column for $A = 1$. It can be seen that in the case of no external stimulation, for the smallest values of g, the network settles on the point attractor and no characteristic frequencies are visible (white area in the figures in the left column). On the other hand, for $A = 1$ and the same range of g, some points (black on white) are visible. These points correspond to the characteristic frequencies of the system and are connected with the frequency of external stimulation or its harmonics. A comparison of Fig. 5a, b and c, reveals that for large enough values of g, there is a wide-band noise related to the emerging chaos in the network. With an increase in A, there is an increase in the value of g at which there is wide-band noise. With an increase in k, the lower limit of the chaotic evolution is observed for smaller values of g (cf. Fig. 5a, b, and c). It should be noted that for the smallest value of $k = 1$ (Fig. 5a) the area of the wide-band noise is less visible. This results from the presence of windows of periodic (quasiperiodic) oscillations observed in the network and will be explained further in the text.

It is interesting to investigate the internal structure of the chaotic region using the relation between the number of peaks n in the Fourier spectrum of the network and gain g (Fig. 6). As we see, peaks of characteristic frequencies occur in the network starting from certain critical value of g_c. Their number increases as g increases. However, for certain values of $g > g_c$, narrow minima. They correspond to a sudden decrease in the number of characteristic frequencies. These minima may be treated as a periodic (quasiperiodic) windows in the chaotic region of the network's dynamics. In particular, such periodic windows are observed for a two-neuron network, where an analytical approach is possible [16]. The number and the width of those windows increase with an increasing amplitude of the external stimulation and with a decreasing k (cf. Fig. 5a, for $k = 1$). Fig. 7 shows a bifurcation diagram for $k = 1$ and $A = 1$, where

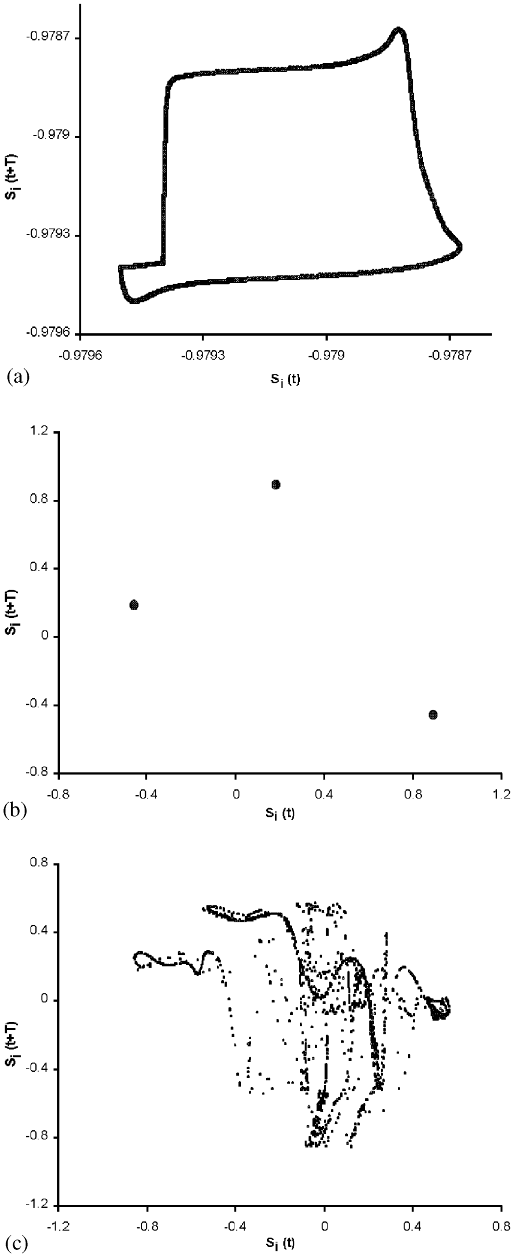


Fig. 4. Return maps for quasiperiodic oscillations (a), oscillations with the period $T_1 = 3T$ (b) and chaotic evolution (c) of a neuron located in a cluster.

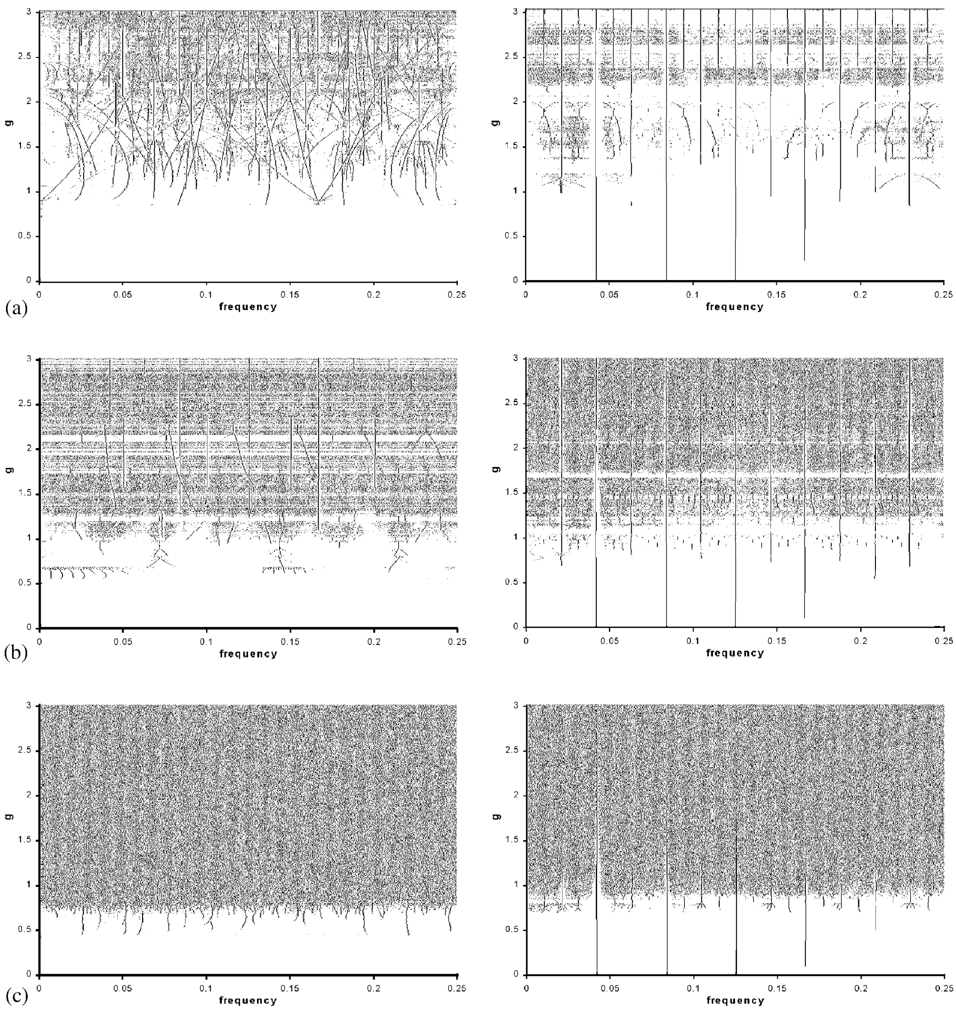


Fig. 5. The Fourier spectrum of the averaged deviation $\sigma(t)$ shows the characteristic frequencies for the different values of the amplitude of external stimulation: $A = 0$ (left column), $A = 1$ (right column). (a), (b) and (c) correspond to $k = 1, 3$ and 5, respectively. Characteristic frequencies are shown as a black points on a white background.

the location and width of periodic windows with the period of oscillations equal to $3T$ are visible.

Fig. 8 illustrates the ranges of the three types of network's dynamics found in the observations of the quantifiers of global network dynamics. As we see, there are periodic attractors for the smallest values of k and g. The upper limit of this region (squares in the Fig. 8) may be approximated with the relation $g(k) = 1.35k^{-0.46}$ (solid line in Fig. 8). Above this region there is an intermediate, quasiperiodic type of dynamics. For

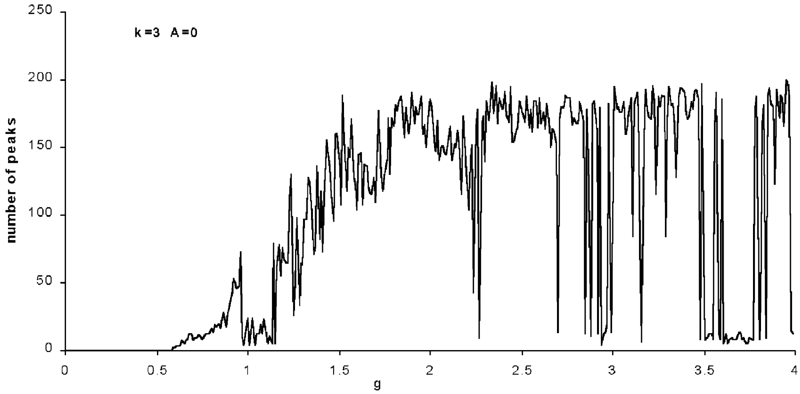


Fig. 6. Number of peaks in the Fourier spectrum as a function of g, for $k=3$ and $A=0$. Narrow minima appearing for $g > g_c \approx 0.6$ show periodic or quasiperiodic windows.

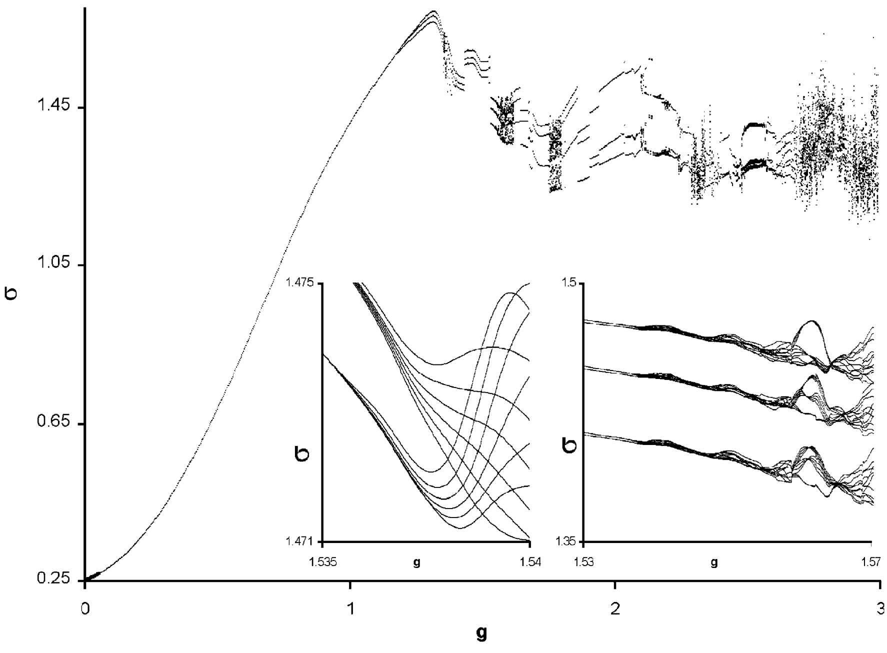


Fig. 7. The Bifurcation diagram $\sigma(g)$ shows the locations of periodic (quasiperiodic) windows. Onset show the details of the internal structure of the diagram.

large enough values of k and g, there is a chaotic region with a lower limit (marked with triangles). This limit can be approximated with the relation $g(k) = 1.5k^{-0.4}$ (solid line in Fig. 8). The route to chaos in the network under investigation is mixed. It occurs via a quasiperiodic evolution or an evolution with the period $T_1 = 3T$ appearing

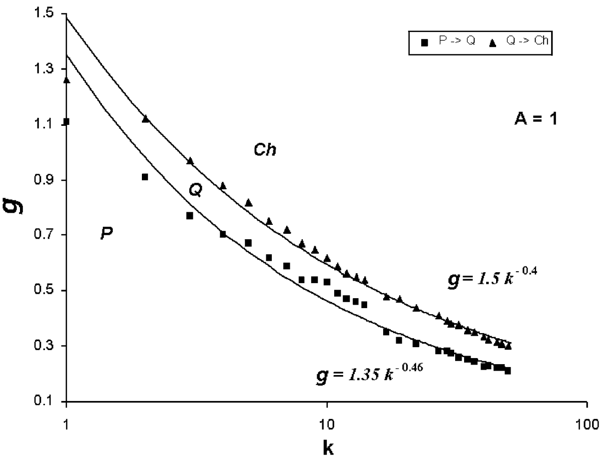


Fig. 8. The ranges of the types of dynamics of a network: P—periodic with $m = 0\%$, Q—intermediate with $m > 0\%$, Ch—chaotic with $m \approx 100\%$; limits of regions (solid lines) may be approximated by the relations shown in the figure. Note that $k = 50$ means full connected network ($N = 100$). The results were averaged over 100 different distributions of synaptic connections.

in the clusters of neurons. At the same time—outside the clusters—oscillations with the period of external stimulation are visible (cf. Fig. 2). As k increases, the number m of neurons that form the clusters increases, too. For large enough k, clusters spread to the whole network.

4. Influence of shortcuts on the dynamics of the network

When there are r shortcuts in the network, as r increases, the average shortest-path distance L decreases. The relation $L(p)$ obtained in our calculations as well as the relation between the clustering coefficient C and p (Fig. 9) are similar to the results obtained in other papers [5,9]. For constant values of synaptic connections J_{ij}, the influence of shortcuts on the dynamics of the neurons is observed only for the neurons located at the ends of shortcuts. Let us consider the case of one shortcut ($r = 1$) connecting neurons $i = 10$ and $j = 60$ (solid line in Fig. 10a). The values of the neurons in the neighborhood of S_{10} and S_{60} have different values than the neurons distant from the ends of shortcut. This change results from the different values of the local field h_i acting on the neurons near the ends of the shortcut. For two or more shortcuts, the behavior of the network is similar (dashed line in Fig. 10a, for $r = 2$).

For random values of synaptic connections the dynamics of the network in the presence of shortcuts is similar to the case without shortcuts (Fig. 10b). Because of the random distribution of the values of short-range synaptic connections (and the values of the local fields of neurons), the influence of shortcuts is manifested for their certain location only. In those cases, which we describe further in the text, this influence is significant.

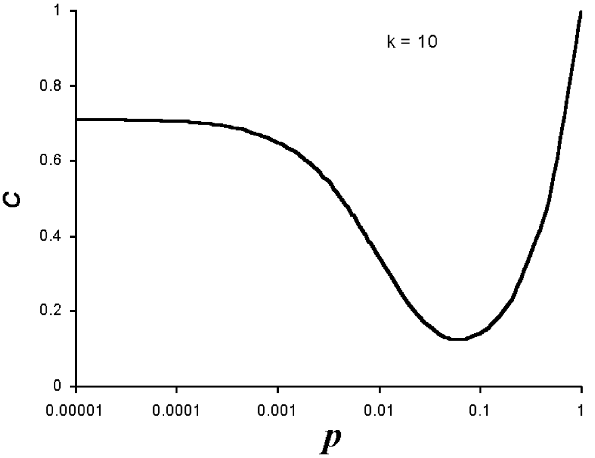


Fig. 9. The relation between clustering coefficient C and the probability of presence of shortcuts p for $N=1000$ and $k=10$. Initially the value of C decreases, because with an increase in the number of shortcuts the regularity of the network disappears. For large values of p the situation is opposite, the clustering coefficient increases. This is so because for a large number of shortcuts the structure of network become similar to a full connected graph. The results were averaged over 100 different location of shortcuts.

In our calculations we assume values of shortcuts $J_{iu}^{(l)}=1$; other values of $J_{iu}^{(l)}$ do not qualitatively change the dynamics of the network. A typical influence of the shortcuts on the relation $m(g)$ is shown in Fig. 11, for a different number of shortcuts r and a chosen distribution of short-range synaptic connections J_{ij}.

In most cases shortcuts increase the number m, which means that a smaller number of neurons oscillates periodically with the period of external stimulation T (Fig. 11a). However, for $k>2$ and large enough g, shortcuts have an opposite influence: the number m decreases (Fig. 11b). With an increasing number of short-range connections (given by k), the influence of shortcuts is smaller. This is so because the additional term connected related to the shortcuts has a smaller influence on the local field h_i (Eq. (2)) when the number of terms in sum over k is greater (cf. Fig. 11a and b).

In certain cases, the presence of shortcuts may have a greater influence on network dynamics and may significantly change the number of neurons that form clusters. Fig. 12a illustrates the influence of the location of long-range connections, for g=0.876. The horizontal axis has the label of the network with selected, random location of shortcuts. The vertical axis has the number m. For the majority of labels, m increases; however, for certain labels, m abruptly decreases. This means that the presence of shortcuts reduces the number of neurons that form clusters. Thus, in some cases, shortcuts may have stabilizing effect on the oscillations in the network and may increase the susceptibility of the network to external stimulation. This effect is better visible for the case of a network with the values of g greater than 8.76, which causes the increase in the value of m (Fig. 12b). In this case the decreasing in m is more significant than for networks with a smaller value of g (cf. Fig. 12a and b). For constant values of

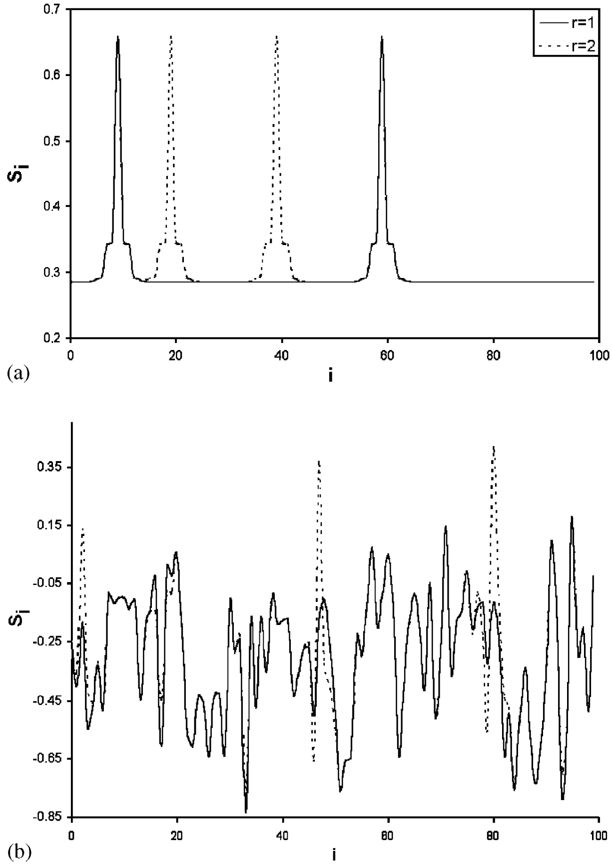


Fig. 10. The values of the neurons $S_i(t)$ for $i = 1, 2, 3, \ldots, N$ depicted for $t = T, 2T, 3T, \ldots,$. Case (a): constant values of synaptic connections $J_{ij} = 0.5$. One shortcut present in the network ($r = 1$), connecting neurons S_{10} and S_{60}-solid line; two shortcuts ($r = 2$) and neurons $S_{10} - S_{60}$ and $S_{20} - S_{40}$ are connected—dashed line. Case (b): random values of synaptic connections for $r = 0$ (solid line) and for $r = 3$ (dashed line). Values of other parameters for both figures are $k = 2$, $A = 1$ and $g = 0.5$.

synaptic connections—and no shortcuts in the network—the influence of neurons in the neighborhood in the radius k is the same for all neurons. Fig. 2 indicates that there are no clusters in the network and $m = 0$. Therefore, the presence of the long-range connections does not influence on the number m, which equals zero (see Fig. 10a).

5. Conclusions

In our calculation the cases of constant and random distribution of values of synaptic connections J_{ij}, were studied. Most interesting is dynamics of our system for the case of a random distribution of J_{ij}. Then, there are three ranges of the dynamics of the

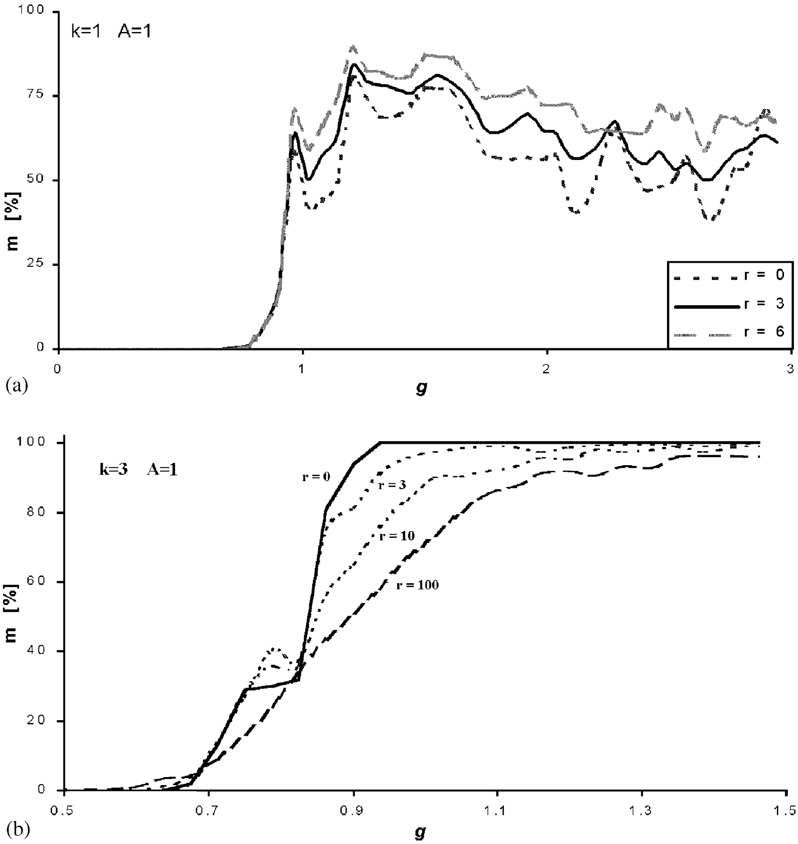
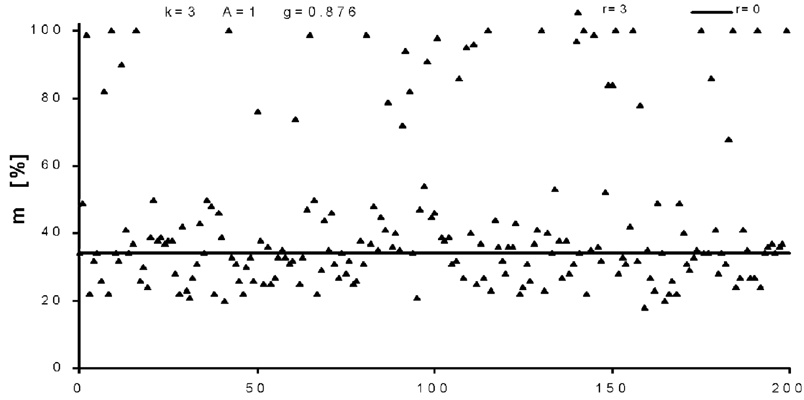


Fig. 11. The number of neurons m forming clusters as a function of the gain parameter g, for random values of synaptic connections and different number of shortcuts r. The results were averaged over 100 different location of shortcuts.

network: periodic, intermediate and chaotic. They depend on the number of neighboring neurons, given by k, gain of the neuron g and the amplitude A of an external, periodic stimulation. In the periodic region, all neurons follow the external stimulation. In the intermediate region, there are clusters of neurons that do not follow external stimulation. The dynamics of the neurons in clusters is quasiperiodic, periodic with the period $3T$ or, sometimes, chaotic. The number of neurons m that form the clusters increases with an increase in the value of k. For large enough k and g, there are chaotic dynamics in the whole network. Correlated random values of synaptic connections, the specific distribution of J_{ij}, influence the number m and location of clusters, for both cases $r = 0$ and $r > 0$.

The presence of additional long-range connections (shortcuts) influences the dynamics of neurons located mainly near the ends of shortcuts. For the case of constant synaptic connections, there is only a phase shift in the oscillations of those neurons



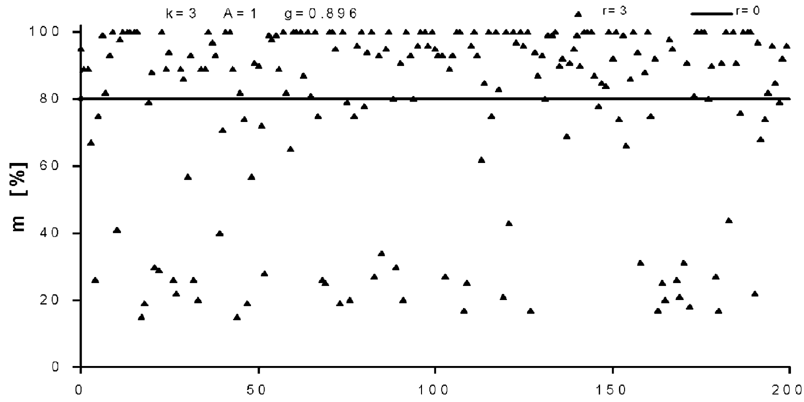


Fig. 12. The number m of neurons forming the clusters for the network with random values of synaptic connections, a different number of shortcuts $r = 0$ (solid line) and $r = 3$ (triangles). Different locations of shortcuts are marked as a label of the network in the horizontal axis. All results were obtained for the same values of the short-range synaptic connections. For certain locations of shortcuts m abruptly decreases, which means that the presence of shortcuts reduces the number of neurons forming the clusters.

(cf. Fig. 10a). For the case of the random synaptic connections, the presence of shortcuts, in most cases causes a moderate increase in the number m of neurons that form clusters. For some special locations of shortcuts, however, there is a sudden decrease or increase of m is observed. This means that shortcuts have either a stabilizing or destabilizing effect on the dynamics of the network. In particular, the presence of shortcuts with certain locations may significantly increase or decrease the susceptibility of the network to external stimulation. Investigations of local dynamics show, that there is a stabilizing effect of shortcuts when one of the periodically oscillating neurons is connected with a neuron located in the middle of a cluster with the non-periodic dynamics.

The relation between the average shortest-path distance L, the clustering coefficient C and the probability of the presence of shortcuts p found in our computations are typical for a small-world systems and have been obtained by other authors [5,9].

References

[1] L.O. Chua, CNN: A Paradigm for Complexity, World Scientific, Singapore, 1998.
[2] R. Dogaru, A.T. Murgan, Proceedings of the Third IEEE Workshop on Cellular Neural Networks, Rome, 1994, pp. 303–308.
[3] R.A. Kosiski, A. Jaroszewicz, Chaos, Solitons and Fractals 9 (1998) 1481.
[4] R.A. Kosiski, A. Krawiecki, A. Sukiennicki, Acta Phys. Pol. A 100 (2001) 89.
[5] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440.
[6] A.L. Barabashi, R. Albert, Science 286 (1999) 509.
[7] S.H. Strogatz, Nature 410 (2001) 268.
[8] R. Albert, A.-L. Barabasi, Rev. Mod. Phys. 74 (2002) 47.
[9] M.E.J. Newman, D.J. Watts, Phys. Rev. E 60 (1999) 7332.
[10] B.A. Huberman, L.A. Adamic, Nature 401 (1999) 131.
[11] C.F. Moukarzel, Phys. Rev. E 60 (1999) R6263.
[12] C. Moore, M.E.J. Newman, Phys. Rev. E 61 (2000) 5678.
[13] J. Lathinen, J. Kertesz, K. Kaski, Physica A 311 (2002) 571.
[14] J. Argyris, G. Faust, M. Haase, An Exploration of Chaos, North-Holland, Amsterdam, 1994.
[15] T.Y. Li, J.A. Yorke, Am. Math. Mon. 82 (1975) 985.
[16] S. Wang, Complex Systems 5 (1991) 425.