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Generalizations of the concept of marginal synchronization of chaos

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Abstract

Generalizations of the concept of marginal synchronization between chaotic systems, i.e. synchronization with zero largest conditional Lyapunov exponent, are considered. Generalized marginal synchronization in drive–response systems is defined, for which the function between points of attractors of different systems is given up to a constant. Auxiliary system approach is shown to be able to detect this synchronization. Marginal synchronization in mutually coupled systems which can be viewed as drive–response systems with the response system influencing the drive system dynamics is also considered, and an example from solid-state physics is analyzed. Stability of these kinds of synchronization against changes of system parameters and noise is investigated. In drive–response systems generalized marginal synchronization is shown to be rather sensitive to the changes of parameters and may disappear either due to the loss of stability of the response system, or as a result of the blowout bifurcation. Nonlinear coupling of the drive system to the response system can stabilize marginal synchronization. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Recently there has been growing interest in the investigation of various kinds of synchronization in chaotic oscillators [1–7] (for review see [8,9]). This interest is spurred by the possible applications of synchronous chaos in private communication [10–16]. Among various forms of synchronization the simplest and most often studied one is identical synchronization (IS). It can be observed both in the systems of diffusively coupled chaotic oscillators [1,17–19] and in drive–response systems [2,20–26]. In the former case two (or more) identical chaotic oscillators with vectors of variables $\mathbf{x}(t)$, $\mathbf{x}'(t)$ are coupled via a function of the difference between the variables, $\mathbf{x} - \mathbf{x}'$. If this coupling is strong enough the phase trajectories of the two oscillators, while remaining chaotic and lying on the same attractor as for a single system, approach each other and after a sufficiently long transient the equality $\mathbf{x}(t) = \mathbf{x}'(t)$ holds. In the drive–response method of synchronization, variables of the chaotic system described by an ordinary differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$ may be divided into two parts, $\mathbf{x} = (\mathbf{u}, \mathbf{v})$, so that the whole system is composed of two subsystems, $\dot{\mathbf{u}} = \mathbf{f}_{\mathbf{u}}(\mathbf{u}, \mathbf{v})$, $\dot{\mathbf{v}} = \mathbf{f}_{\mathbf{v}}(\mathbf{u}, \mathbf{v})$, where $\mathbf{f} = (\mathbf{f}_{\mathbf{u}}, \mathbf{f}_{\mathbf{v}})$. This system is called the drive system. Then, the variables $\mathbf{u}$ are used to drive another chaotic system, called the response system, which is identical or at least very similar to the $\mathbf{v}$ subsystem. In the simplest version of this method, called the replacement method, a replica $\mathbf{v}'$ of the $\mathbf{v}$ subsystem is considered which evolves according to the equation in which the $\mathbf{u}$ variable is taken directly from the drive system, $\dot{\mathbf{v}}' = \mathbf{f}_{\mathbf{v}}(\mathbf{u}, \mathbf{v}')$. If the two subsystems are chosen appropriately

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synchronization between the drive and response systems is achieved in the sense that $\mathbf{v}(t) = \mathbf{v}'(t)$ after all transients die out. A linearized equation for the vector $\Delta\mathbf{v}(t) = \mathbf{v}(t) - \mathbf{v}'(t)$ is $\mathrm{d}\Delta\mathbf{v}/\mathrm{d}t = \hat{D}_\mathbf{v}\mathbf{f}_\mathbf{v}(\mathbf{u}, \mathbf{v})$, where $\hat{D}_\mathbf{v}\mathbf{f}_\mathbf{v}$ is the Jacobian of $\mathbf{f}_\mathbf{v}$ with respect to $\mathbf{v}$, and the corresponding Lyapunov exponents are called conditional Lyapunov exponents (CLE) [2,20]; the necesssary condition for synchronization is that all CLE were negative.

In this paper we deal with the problem of marginal synchronization of chaos (MS) [3,27–29]. In the replacement method this kind of synchronization occurs when the largest CLE is 0 or at least there is a pair of complex conjugate eigenvalues of the matrix $\hat{D}_\mathbf{v}\mathbf{f}_\mathbf{v}$ with zero real part. Another important ingredient of MS is symmetry. For example if the $\mathbf{v}$, $\mathbf{v}'$ subsystems are invariant under the scaling transformation $\mathbf{v} \to A\mathbf{v}$, $A =$ const, then the response system attractor will be a sized copy of the projection of the drive system attractor on the $\mathbf{v}$ subspace, i.e. $\mathbf{v}'(t) = A(\mathbf{x}_0, \mathbf{v}'_0)\mathbf{v}(t)$ (sized MS). The scaling constant $A(\mathbf{x}_0, \mathbf{v}'_0)$ may depend on the initial conditions of the drive and response systems, $\mathbf{x}_0$, $\mathbf{v}'_0$. Similarly if the $\mathbf{v}$, $\mathbf{v}'$ subsystems are invariant under the shift of variables in the direction of a vector $\mathbf{e}$ of unit length, $\mathbf{v} \to \mathbf{v} + A\mathbf{e}$, $A =$const, then the response system attractor will be a shifted copy of the projection of the drive system attractor on the $\mathbf{v}$ subspace, i.e. $\mathbf{v}'(t) = \mathbf{v}(t) + A(\mathbf{x}_0, \mathbf{v}'_0)\mathbf{e}$ (constant MS). The above-mentioned situations occur when the largest CLE is 0 either on average (sized MS) or there is always a zero eigenvalue of the $\hat{D}_\mathbf{v}\mathbf{f}_\mathbf{v}$ matrix (constant MS). In the case of a pair of complex eigenvalues of $\hat{D}_\mathbf{v}\mathbf{f}_\mathbf{v}$ with zero real part more complicated time-dependent relationship between the trajectories of the drive and response systems may occur, e.g. so-called oscillatory MS [27,28], but in this paper we constraint our attention to the case of sized and constant MS.

A more general form of synchronization is generalized synchronization (GS) [4,30–35]. Let us consider two different unidirectionally coupled chaotic systems, the drive system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and the response system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{h}(\mathbf{x}), \mathbf{y})$. Here, $\mathbf{h}(\mathbf{x})$ is a suitably chosen function of $\mathbf{x}$ by which the response system is coupled to the drive system. The unidirectional coupling is not constrained only to the possibilities offered by the replacement method, but it is still sometimes possible to divide the drive system variables into two groups, $\mathbf{x} = (\mathbf{u}, \mathbf{v})$ of which only the $\mathbf{u}$ variables take part in the coupling. The systems $\mathbf{x}$ and $\mathbf{y}$ show GS if there is a function $\mathbf{F}$ which transforms points on the drive system attractor into points on the response system attractor, $\mathbf{y} = \mathbf{F}(\mathbf{x})$. This function depends neither on initial conditions of the drive and response systems, provided that all of them belong to the basins of attraction of the same attractors [4], nor on time. In the case of IS in the replacement method $\mathbf{F}$ is identity between the $\mathbf{v}$ subsystem of $\mathbf{x}$ and the $\mathbf{y} = \mathbf{v}'$ system. As this function cannot depend on the initial conditions $\mathbf{x}_0$, $\mathbf{y}_0$, MS is not a special case of GS, though a linear transformation between $\mathbf{v}$ and $\mathbf{v}'$ exists [30]. Moreover, $\mathbf{F}$ need not be smooth: this is the case of weak synchronization, while if $\mathbf{F}$ is smooth the synchronization is called strong [32,34]. It was shown that GS can occur with arbitrary drive and response systems (e.g. the Lorenz system driven by the Rössler system, etc.) provided that the response system is stable, and that GS is robust against the changes of parameters of the drive and response systems within reasonable limits [30,31].

A method of detecting GS in drive–response systems is often based on the so-called auxiliary system approach [30,31]. It works well both in the case of weak and strong synchronization [32,34]. The idea of the auxiliary system approach is as follows. It may be shown that GS occurs if and only if the response system is asymptotically stable, i.e. if we consider the drive system $\mathbf{x}$ with a given initial condition $\mathbf{x}_0$ and two identical response systems (the second one called the auxiliary system) $\mathbf{y}_1$, $\mathbf{y}_2$, with different initial conditions $\mathbf{y}_{10}$, $\mathbf{y}_{20}$, the phase trajectories of the response systems approach each other as time goes to infinity, $\lim_{t\to\infty} ||\mathbf{y}_1(t) - \mathbf{y}_2(t)|| = 0$. In other words, the two response systems $\mathbf{y}_1$, $\mathbf{y}_2$ show GS with the drive system $\mathbf{x}$ only if they show IS with each other when driven by the same signal $\mathbf{h}(\mathbf{x})$. Clearly this is not true if the response systems show only MS with the drive system. This confirms that MS is not a particular case of GS.

In this paper we generalize the notion of MS. First, we introduce the concept of generalized marginal synchronization (GMS) which concerns unidirectionally coupled drive–response systems. Second, we investigate MS in mutually coupled systems. Concerning the first problem, as mentioned above, MS has been investigated so far within the framework of the replacement method, in which the response system $\mathbf{v}'$ is a replica of the part of the drive system $\mathbf{v}$. In this paper we let the drive system be different from the response system. By analogy with GS we say that the drive and response systems show GMS when there exists a function which transforms points on the drive system attractor on the points on the response system attractor. This function does not depend on time; but, in contrast with GS it is known up to a certain constant which depends on the initial conditions of the drive and response systems; it can be, e.g. the multiplicative scaling constant or a constant vector by which the response system attractor may be shifted. When the auxiliary system approach is applied to systems with GMS the two response systems show MS. The problem of stability of GMS against the changes of parameters of the drive and response systems is also addressed. This problem is more complicated than in the case of GS. It may happen that GMS is robust against such changes, but in other cases they can lead even to the disappearance of the chaotic attractor of the response system, although certain properties of GMS between the drive and response systems are preserved. This is either due to the loss of stability of the response system, or due to the occurrence of the blowout bifurcation [36–40] in the system. Concerning the second problem, we define and consider MS in systems which may be treated as modifications of drive–response systems in which the drive system is influenced by the response system. It is sometimes possible to divide such systems into groups of subsystems so that the subsystems belonging to one group are marginally synchronized with one another. We investigate the stability of MS in such systems against changes of parameters on a suitably chosen example.

The rest of this paper is organized as follows. Definition of GMS and the application of the auxiliary system approach to the detection of GMS in unidirectionally coupled drive–response systems are given in Section 2.1. In Section 2.2 a class of mutually coupled systems is defined in which MS is then investigated. Sections 3.1 and 3.2 are devoted to the analysis of several examples of MS and GMS in unidirectionally coupled drive–response systems. In Section 3.1 examples of constant GMS between different systems are considered. In Section 3.2 sized GMS between Lorenz systems with slightly different parameters is investigated. In Section 3.3 an example of MS in a mutually coupled system is given, taken from solid-state physics. All equations of motion are solved using the fourth- and fifth-order Runge–Kutta method with permanent error control. Finally, in Section 4 summary and conclusions are given.

2. Generalizations of marginal synchronization: definitions and methods of analysis

2.1. Generalized marginal synchronization: the auxiliary system approach

Let us consider two unidirectionally coupled chaotic oscillators, the drive system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, and the response system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{h}(\mathbf{x}), \mathbf{y})$, $\mathbf{y} \in \mathbb{R}^m$. In order to introduce the concept of GMS the definition of GS from Ref. [30] is extended as follows. We say that $\mathbf{x}$ and $\mathbf{y}$ show GMS if there exists a family of functions $\tilde{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^m$, a family of manifolds $\tilde{M} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \ \mathbf{y} = \tilde{\mathbf{F}}(\mathbf{x})\}$, and a subset $B = B_x \times B_y \subseteq \mathbb{R}^n \times \mathbb{R}^m$ such that all trajectories $(\mathbf{x}(t), \mathbf{y}(t))$ with initial conditions $(\mathbf{x}_0, \mathbf{y}_0) \in B$ approach a manifold M belonging to the family $\tilde{M}$ as time goes to infinity. The family $\tilde{\mathbf{F}}$, and, equivalently, the family $\tilde{M}$ are in turn defined as classes of the following two equivalence relations: (i) Two functions $\mathbf{F}_1$, $\mathbf{F}_2$ belong to the same family $\tilde{\mathbf{F}}$ if there exists a constant A such that, for every $\mathbf{x}$, $\mathbf{F}_2(\mathbf{x}) = A\mathbf{F}_1(\mathbf{x})$ (sized GMS), or (ii) if there exists a unit vector $\mathbf{e}$ and a constant A such that, for every $\mathbf{x}$, $\mathbf{F}_2(\mathbf{x}) = \mathbf{F}_1(\mathbf{x}) + A\mathbf{e}$ (constant GMS); in both cases the constant A may depend on the initial conditions, $A = A(\mathbf{x}_0, \mathbf{y}_0)$.

If $A = 1$ in the case (i) or $A = 0$ in the case (ii) independently of the initial conditions, we obtain the definition of GS from Ref. [30], but in this paper we are interested in more general cases. To be exact, such a general definition of GMS does not imply synchronization in the common sense, as the transformation between the drive and response systems is known only up to a constant. Thus in contrast with GS, this definition does not imply predictability, which is an important ingredient of the common understanding of synchronization [30]. Nevertheless, the attractors for $\mathbf{x}$ and $\mathbf{y}$ are tightly connected, as for every initial condition $(\mathbf{x}_0, \mathbf{y}_0)$ there is a transformation relating points on the two attractors, and the only effect of the change of the initial conditions is a shift or scaling of the values of this transformation by a constant, without changing its general form. From a different point of view one can say that the function $\mathbf{F}$ assigns straight lines in $\mathbb{R}^m$ to the points on the attractor in $\mathbb{R}^n$.

In analogy with GS one can show that GMS occurs between the systems $\mathbf{x}$ and $\mathbf{y}$ if and only if there exists a constant A and a vector $\mathbf{e}$ such that for two different initial conditions $(\mathbf{x}_0, \mathbf{y}_{10})$ and $(\mathbf{x}_0, \mathbf{y}_{20})$ we have $\lim_{t\to\infty} ||\mathbf{y}_1(t) - A\mathbf{y}_2(t)|| = 0$ in the case of sized GMS, or $\lim_{t\to\infty} ||\mathbf{y}_1(t) - \mathbf{y}_2(t) - A\mathbf{e}|| = 0$ in the case of constant GMS. This is the auxiliary system approach applied to the GMS case. In other words, the two systems show GMS if two response systems show MS (of course, the reverse theorem is trivial). The proof is a slight modification of that in Ref. [30]. For example let us consider the case of constant GMS. Let us denote the flows connected with the systems $\mathbf{x}$, $\mathbf{y}$ as ϕ_x^t, ϕ_y^t, respectively, so that $\mathbf{x}(t) = \phi_x^t(\mathbf{x}_0)$, etc. For a given point $\mathbf{x}_A$ belonging to the attractor of the system $\mathbf{x}$ let us consider a full system $\mathbf{x}, \mathbf{y}$ with two initial conditions $\left(\phi_x^{-t}(\mathbf{x}_A), \mathbf{y}_{10}\right)$, $\left(\phi_x^{-t}(\mathbf{x}_A), \mathbf{y}_{20}\right)$, with t long enough so that all transients die out. Let us define the function $\mathbf{F}_1$ by $\mathbf{F}_1(\mathbf{x}_A) = \phi_y^t(\mathbf{y}_{10})$. On the other hand, we could define the function $\mathbf{F}_2$ by $\mathbf{F}_2(\mathbf{x}_A) = \phi_y^t(\mathbf{y}_{20})$. Using the properties of the auxiliary system we obtain that for sufficiently long t there are such A and $\mathbf{e}$ that $\|\phi_y^t(\mathbf{y}_{20}) - \phi_y^t(\mathbf{y}_{10}) - A\left(\phi_x^{-t}(\mathbf{x}_A), \mathbf{y}_{10}, \mathbf{y}_{20}\right)\mathbf{e}\| \to 0$. So if we start with different initial conditions in the $\mathbf{y}$ variables the function $\mathbf{F}$ between $\mathbf{x}$ and $\mathbf{y}$ is defined up to a shift, $\mathbf{F}_1(\mathbf{x}_A) = \mathbf{F}_2(\mathbf{x}_A) + A\left(\phi_x^{-t}(\mathbf{x}_A), \mathbf{y}_{10}, \mathbf{y}_{20}\right)\mathbf{e}$. Only the shift depends on the initial conditions of the drive and both response systems. Similarly we can start with two different initial conditions both in the $\mathbf{x}$ and $\mathbf{y}$ variables, $\left(\phi_x^{-t}(\mathbf{x}_A), \mathbf{y}_{10}\right)$, $\left(\phi_x^{-t}(\mathbf{x}_B), \mathbf{y}_{20}\right)$. As both $\mathbf{x}_A$, $\mathbf{x}_B$ lie on one attractor there exists such t_{BA} that $\mathbf{x}_A = \phi_x^{t_{BA}}(\mathbf{x}_B)$. After t_{BA} the phase trajectory of the system starting from the second initial condition will reach the point $(\phi_x^{-t}(\mathbf{x}_A), \phi_y^{t_{BA}}(\mathbf{y}_{20}))$. Then, as shown above, we can define the function $\mathbf{F}$ either as $\mathbf{F}_1(\mathbf{x}_A) = \phi_y^t(\mathbf{y}_{10})$ or $\mathbf{F}_2(\mathbf{x}_A) = \phi_y^t(\phi_y^{t_{BA}}(\mathbf{y}_{20}))$, up to a constant shift. Thus starting with different initial conditions in the $\mathbf{x}$ and $\mathbf{y}$ variables does not change the overall form of the function $\mathbf{F}$ and modifies only the constant shift.

2.2. Marginal synchronization in mutually coupled systems

MS and GMS are particularly easy to understand in unidirectionally coupled drive–response systems. However, there are many physical examples of chaotic systems which can be viewed as mutually coupled systems (e.g. systems of interacting modes in plasma physics, lasers and solid-state physics). Moreover, as it will be shown in Sections 3.2 and 3.3, MS in such systems may be more robust against changes of system parameters. Thus in this paper also such a case is considered.

We deal with MS in mutually interacting systems which may be viewed as modifications of the drive–response systems. The idea is to take a given drive–response system of which it is known that it shows MS or GMS and to include the terms which describe the influence of the response system in the equations for the drive system. Then, many copies of the response system may be added to the system under study, and their influence on the drive system is included in the same way. If the dynamics of the drive system is not modified qualitatively in such a way that the property of MS or GMS between the drive and response systems is lost, it is a simple consequence of the auxiliary system approach that the response systems will show MS with one another. So a group of subsystems may be found within the whole system which exhibit MS. There can be, of course, many such groups if different response systems which show MS or GMS with the drive system are added in a systematic way. It should be noted that this is a typical situation in Hamiltonian systems in which the interactions among various subsystems (e.g. modes) are symmetric. One such example is considered in Section 3.3.

Formally the above-mentioned concept may be written as follows. The equations for the drive system are $\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \tilde{\mathbf{h}}_\mathbf{y}(\mathbf{y}_1), \tilde{\mathbf{h}}_\mathbf{y}(\mathbf{y}_2), \ldots\right)$, and the equations for the response systems are $\dot{\mathbf{y}}_i = \mathbf{g}_\mathbf{y}\left(\mathbf{y}_i, \mathbf{h}_\mathbf{y}(\mathbf{x})\right)$, $i = 1, 2, \ldots$ All response systems $\mathbf{y}_i$ are identical and driven in the same way via the function $\mathbf{h}_\mathbf{y}(\mathbf{x})$. They, in turn, influence the drive system via the function $\tilde{\mathbf{h}}_\mathbf{y}(\mathbf{y}_i)$ which is also identical for all i. A group of N response systems show MS if the relationships characteristic of MS hold for every pair of i, j, so, e.g. in the case of sized MS for every i, j a constant A_{ij} exists such that $\mathbf{y}_i(t) = A_{ij}\mathbf{y}_j(t)$ after all transients die out. This constant may depend on the initial conditions of the drive system $\mathbf{x}_0$ and all response systems $\mathbf{y}_{10}, \mathbf{y}_{20}, \ldots$ It should be pointed out that the existence of GMS between $\mathbf{x}$ and all response systems $\mathbf{y}_i$ in the case of unidirectional coupling, i.e. when $\tilde{\mathbf{h}}_\mathbf{y}(\mathbf{y}_i) \equiv 0$ for all i, does not guarantee the existence of MS among response systems in the case of mutual coupling. The whole system $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \ldots$ is thus divided into two subsystems: the drive system $\mathbf{x}$ and a group of marginally synchronized response systems $\mathbf{y}_i$. The choice of the drive and response systems may be somewhat arbitrary, i.e. different subsystems may be sometimes treated as the drive and response systems; we will always explain which subsystem we treat as a drive system, and which as a response system. Next, another group of response systems $\mathbf{z}_i$, $i = 1, 2, \ldots$, which are different from the $\mathbf{y}_i$ systems may be added so that the equation for the drive system becomes $\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \tilde{\mathbf{h}}_\mathbf{y}(\mathbf{y}_1), \ldots, \tilde{\mathbf{h}}_\mathbf{z}(\mathbf{z}_1), \ldots\right)$, and the equations for the response systems are as previously, with the change of $\mathbf{y}$ into $\mathbf{z}$ in the case of the systems from the second group. This procedure may be repeated, and the whole system may be divided into groups of subsystems showing MS.

By analogy with the auxiliary system approach we can say that also in the case of mutual coupling between the drive and response systems the drive system $\mathbf{x}$ and the response system $\mathbf{y}_i$ show GMS (as a replica $\mathbf{y}_j$ of the response system shows MS with $\mathbf{y}_i$). However, as now, e.g. the proportionality constant between $\mathbf{y}_i$, $\mathbf{y}_j$ showing sized MS depends on the initial conditions of the drive system and all response systems $\mathbf{y}_i$, $i = 1, 2, \ldots$, the concept of GMS in mutually coupled systems is not as straightforward as in the unidirectionally coupled ones.

In the above definition we assumed that there is no direct interaction between response systems, neither within nor among groups. They interact only indirectly, via the drive system. This allows the distinction between the drive and response systems. The case when there is direct interaction among all subsystems deserves separate study.

3. Results and discussion

3.1. Constant generalized marginal synchronization: the analysis using auxiliary system

In this section several examples of constant GMS between different chaotic oscillators are considered. In the first example the Chua system is used:

$$\dot{x} = \alpha[z - x - G(x)], \quad \dot{y} = -\beta z, \quad \dot{z} = x - z + y, \tag{1}$$

where $G(x)$ is a function given by $G(x) = bx + a - b$ if $x \geqslant 1$, $G(x) = ax$ if $|x| < 1$ and $G(x) = bx - a + b$ if $x \leqslant -1$. In Eq. (1) the names of variables x, y were assumed as in Ref. [3] and interchanged in comparison with the original Chua system (cf. [8]). Using the replacement method for Eq. (1) the replica $\mathbf{v}' = (x', y')$ of the $\mathbf{v} = (x, y)$ subsystem can always be marginally synchronized with $\mathbf{v}$ if $\mathbf{u} = z$ is used as a drive [3,29]. This is an example of constant MS which appears because the equation for $\Delta y = y - y'$ is $\Delta\dot{y} = 0$ and thus the matrix $\hat{D}_{\mathbf{v}} f_{\mathbf{v}}$ has always one zero eigenvalue. González-Miranda [3,29] observed that for the parameters $\alpha = 9$, $\beta = 14\ (2/7)$, $a = -8/7$ and $b = -5/7$ the attractor for the $\mathbf{v}'$ subsystem was an exact copy of the attractor for $\mathbf{v}$ subsystem, only shifted in the y direction by a vector whose length depended on the initial conditions of the (x, y) and (x', y') subsystems, but not of the z subsystem. This means that for these parameters the second CLE, which is connected with the equation for $\Delta x = x - x'$, was always negative. In fact, he observed fast convergence of $x(t)$ and $x'(t)$ to a straight line $x = x'$.

Here, we consider GMS in a drive–response system composed of two unidirectionally coupled Chua circuits with significantly different parameters. Moreover, in the drive system a linear transformation of variables is performed (such transformations of variables were used in Ref. [4]), $\bar{x} = x$, $\bar{y} = y + z$, $\bar{z} = z$. As the coupling variable $\bar{z}$ is again used. Dropping bars over the drive system variables for simplicity, the system under study may be written as

$$\dot{x} = \alpha[z - x - G(x)], \quad \dot{y} = -(\beta + 2)z + x + y, \quad \dot{z} = x - 2z + y, \tag{2}$$

$$\dot{x}_1 = \alpha_1[z - x_1 - G(x_1)], \quad \dot{y}_1 = -\beta_1 z. \tag{3}$$

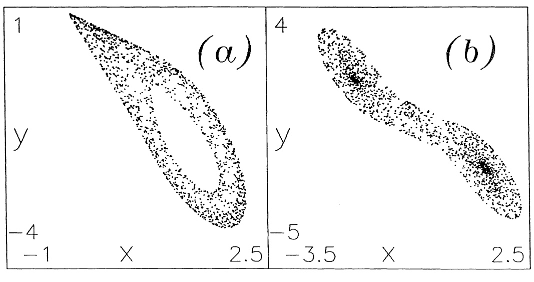
Here, the index 1 labels the response system; the auxiliary system, identical to (3), will be denoted as 2, with $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$. In both drive and response systems we used $G(x)$ as defined below Eq. (1), with $a = -8/7$, $b = -5/7$. Since the equations for y and y_1 are different, the systems (x, y), (x_1, y_1) do not show MS even in the case of identical drive and response system parameters – in this case, only the x, x_1 variables show IS. Hence we have two truly different systems which, as we show below, can exhibit GMS.

The analytic test for GMS consists in evaluating CLE between the response and auxiliary systems, using linearized equations of motion for $\Delta x = x_1 - x_2$, $\Delta y = y_2 - y_2$. If the largest exponent is zero then these two systems can show MS and thus GMS between the drive and response systems can occur. In the case of the z-driven system (3) the linearized equations will be identical to the ones obtained when MS in the system (1) is studied, and, moreover, this will be the case independently of the signal z. Therefore there will be always one zero and one negative CLE, which indicates the possibility to obtain GMS for the z-driven subsystem (x, y) of the Chua system for any z. As we show below, this is not a sufficient condition for GMS.

We solved numerically Eqs. (2) and (3) with the following parameters: $\alpha = 9$, $\beta = 15\ (2/7)$, $\alpha_1 = 10$, $\beta_1 = 14\ (2/7)$. The drive system (2) has then two symmetric separate attractors; one of them is shown in Fig. 1(a). The original Chua system (1) with parameters α_1, β_1 has in turn a "double scroll" attractor (Fig. 1(b)). But if the response system (3) is driven by the z variable from the drive system (2), only one loop of this attractor is visited, depending on the initial conditions. To check for GMS between the systems (2) and (3) we added the auxiliary system (x_2, y_2) and observed the plots of x_1 vs. x, x_2 vs. x_1 and y_2 vs. y_1 (Figs. 2 and 3). Due to the existence of two separate attractors for the systems (2), (3) we were also able to check the applicability of the auxiliary system method for the detection of GMS when the initial conditions were varied, as in Ref. [21].

Figs. 2 and 3 were obtained with different initial conditions for the response system and the same initial conditions for the drive and auxiliary systems. In Figs. 2(a) and 3(a) one can see that there is no evident functional dependence between the drive and response systems. However, as results from Fig. 2(b) and (c) the drive and response systems show GMS. If the initial conditions for the response and auxiliary system are chosen appropriately, the attractors of these systems are identical except that they are shifted with respect to each other in the y-direction. This shift depends linearly on the changes of the initial conditions of both the response and auxiliary system, but it is independent of such changes in the drive system. Thus the systems (x_1, y_1) and (x_2, y_2) show MS and, according to the theorem of Section 2.1, they both show GMS with the drive system. We checked that addition of zero mean Gaussian noise to the response and auxiliary system variables did not cause any noticeable loss of MS between these systems if the noise variance was below ca. 10^{-4}; for stronger noise, the straight lines in Fig. 2(b) and (c) became thicker. GMS in this case seems robust against changes of the drive and response system parameters, too.

On the other hand, if the initial conditions of the response system are chosen so that the common z drive directs the phase trajectories of the response and auxiliary systems to different accessible attractors, no GMS between the drive and response systems can be detected with the help of the auxiliary system. This can be seen in Fig. 3(b), where there is no linear dependence between x_1 and x_2. Although y_2 is still a shifted copy of y_1 (Fig. 3(c); this must be so in any case because of the zero CLE connected with the y variable), the response and auxiliary systems, treated as a whole, do not show MS. This, however, does not mean that there is no GMS between the drive and response system. Starting the auxiliary system with different initial conditions could lead it to MS with the response system and hence to the detection of GMS. The situation here is similar to that with IS in multi-attractor systems [21] and imposes some constraints on the applicability of the auxiliary system approach to the detection of not only GMS, but also GS. Such a constraint is not in contradiction with the theorems of Section 2.1, as there can be many subsets B in the full phase space $\mathbb{R}^n \times \mathbb{R}^m$ such that starting with initial conditions $\mathbf{x}_0$, $\mathbf{y}_0$ in various B will lead the phase trajectory to manifolds M belonging to various families $\tilde{M}$.



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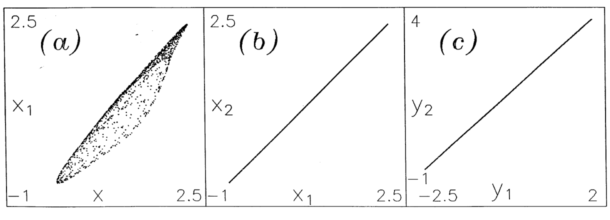

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Fig. 1. (a) One of the two chaotic attractors of the modified Chua system (2), with $\alpha = 9$, $\beta = 15\ (2/7)$; (b) "double scroll" attractor of the Chua system (1) with $\alpha = 10$, $\beta = 14\ (2/7)$.

Fig. 2. Constant GMS of two Chua systems with different parameters (2), (3): (a) chaotic attractor x_1 vs. x; (b) IS between the x variables of the response and auxiliary systems, $x_2 = x_1$; (c) MS between the y variables of the response and auxiliary systems, $y_2 = y_1 + 2$.

At this point it should be noted that multi-attractor chaotic systems can show IS independently of the choice of initial conditions, if a proper coupling is used [41]. The connection of this problem with the detection of GMS with the help of the auxiliary system is still to be clarified.

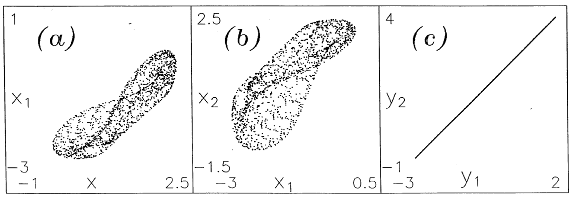
Our next example is with the Chua (x_1, y_1) subsystem driven by the y variable of the Lorenz system:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = (r - z)x - y, \quad \dot{z} = xy - bz, \tag{4}$$

$$\dot{x}_1 = \alpha_1[y - x_1 - G(x_1)], \quad \dot{y}_1 = -\beta_1 y, \tag{5}$$

with parameters $\sigma = 10$, $r = 28$, $b = 2\ (2/3)$, $\alpha_1 = 9$, $\beta_1 = 14\ (2/7)$. Both systems are well inside the chaotic regime. As explained above, on the basis of zero CLE connected with y_1, GMS between the Lorenz and Chua systems is expected although there is no evident functional dependence between the drive and response system variables (Fig. 4(a) and (b)). Using a copy of the (x_1, y_1) system as the auxiliary system we observed that the variables x_1 and x_2 show IS, $x_1(t) = x_2(t)$, and the variables y_1, y_2 lie on a straight line $y_2(t) = y_1(t) + A$, where A depends on the initial conditions of the response and auxiliary system. So there is MS between the response and auxiliary systems and thus GMS between the drive and response systems (4), (5). This GMS was also stable when the system parameters were changed. However, this case is not simple to analyze. It is important to note that the y variable of the Lorenz system (4) has zero mean, thus the right-hand side of the equation for y_1 in Eq. (5) averaged over time is 0. Nevertheless, the absolute value of y_1 can assume large values and be very different from what can be expected for the Chua system. The second point is that the averaged frequency of chaotic oscillations of the Lorenz system (4) is much higher than that of the Chua system (5). Both these factors cause that in the y_1 vs. y plot (Fig. 4(b)) one cannot observe in fact any attractor, but only chaotic wandering of the y_1 variable, probably without any constraint, although with zero mean. In the case when the phase trajectory for the response system does not converge to a constrained subset of the phase space it is rather impossible to say that there is any time-independent function relating points on the drive and response system attractors, and GMS between the systems (4) and (5) is doubtful.

The third example is Eq. (5) driven by the z variable of the Lorenz system (4) instead of y. The mean value of z is positive and thus the variable y_1 on average diverges to infinity. When the auxiliary system is added, the variables x_1, x_2 show IS and they are both constrained; this means that the CLE connected with $\Delta x = x_1 - x_2$ is negative. Also there is still a linear dependence $y_2(t) = y_1(t) + A$, with A dependent on the initial conditions, but both y_1 and y_2 diverge to infinity. Hence though there are some indications of GMS in the auxiliary system approach, in fact there is no GMS between the Lorenz and z-driven Chua system, as there is no chaotic attractor in the latter. However, the largest CLE, connected with $\Delta y = y_1 - y_2$, again vanishes. The present case is an example of the loss of GMS due to the loss of stability of the response system. An interesting conclusion from this example is that, in contrast with IS or GS, addition of random noise with non-zero mean to the driving variable (and thus the change of its mean value) may strongly affect the stability of MS and GMS. This example also shows that the existence of the zero largest CLE between the response and auxiliary system does not guarantee GMS between the drive and response systems, though it guarantees that certain features of MS between the response and auxiliary system are preserved.



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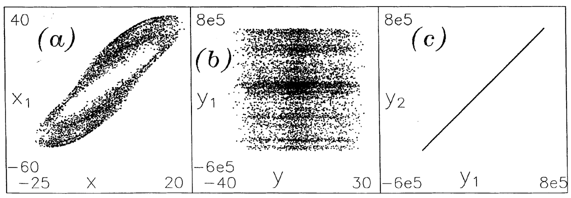

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Fig. 3. Apparent lack of GMS between two Chua systems with different parameters (2), (3) due to poor choice of initial conditions of the response and auxiliary system: (a) chaotic attractor x_1 vs. x; (b) chaotic attractor x_2 vs. x_1 between the x variables of the response and auxiliary systems; (c) MS between the y variables of the response and auxiliary systems, $y_2 = y_1 + 2$.

Fig. 4. Constant GMS between the Lorenz system (4) and the Chua system (5): (a) chaotic attractor x_1 vs. x, (b) chaotic "attractor" y_1 vs. y, unbounded wandering of y_1 can be seen; (c) MS between the y variables of the response and auxiliary systems, $y_2 = y_1 + 9$.

3.2. Sized generalized marginal synchronization: loss of stability via the blowout bifurcation

As a next example the (x_1, y_1) subsystem of the Lorenz system (4) will be studied, driven using the replacement method by the z variable of another Lorenz system with slightly different parameters

$$\dot{x} = \sigma(y - x), \quad \dot{y} = (r - z)x - y, \quad \dot{z} = xy - bz, \tag{6}$$

$$\dot{x}_1 = \sigma_1(y_1 - x_1), \quad \dot{y}_1 = (r_1 - z)x_1 - y_1. \tag{7}$$

In the case $\sigma_1 = \sigma = 10$, $r_1 = r = 28$, $b = 2\ (2/3)$ these two systems show sized MS [3,27–29] with $(x(t), y(t)) = (Ax_1(t), Ay_1(t))$ and the constant A which depends on the initial conditions for all equations. MS occurs because the largest CLE between the drive and response system is 0 on average (i.e. it converges to 0 when evaluated numerically). Some insight into the CLE spectrum may be obtained from the analysis of the time-dependent eigenvalues of $\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}$ (where $\mathbf{v} = (x, y)$), which are [28]:

$$\lambda_\pm = \frac{1}{2}\left\{ -(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma[z(t) - (r - 1)]} \right\}. \tag{8}$$

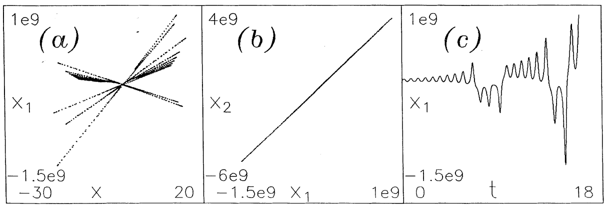
Approximating roughly the time average $\langle z \rangle \approx r - 1$, where $r - 1$ is the z coordinate of the two unstable fixed points of the Lorenz system we obtain a crude estimate of the largest CLE as $\lambda_{\max} \approx \langle \lambda_+ \rangle = 0$. This suggests that any small change of r changes $\langle z \rangle$ and destroys MS. Thus MS in the systems (4), (6) is sensitive to small changes of system parameters.

In the following we consider GMS between the systems (6), (7) with $r_1 \neq r$. To this purpose the auxiliary system (x_2, y_2) identical with the response system (7) is added to the above equations. It can be easily noted that the response system (7) is linear with respect to x_1, y_1. Hence the the largest CLE between the drive and auxiliary system is again roughly given by $\langle \lambda_+ \rangle$ from Eq. (8). If $r \neq r_1$ one can expect this average being either negative (then there is IS between the response and auxiliary system and GS between the drive and response system) or positive (no GMS at all). Another interesting observation is that the system (7) has a trivial fixed point $x_1 = y_1 = 0$ whose stability is again determined by the same eigenvalues (8). Thus we come to a conclusion that the occurrence of GS between the two different Lorenz systems will also lead to the convergence of the response system to the zero fixed point. The rate of this convergence and that of the appearance of IS between the response and auxiliary systems are identical: simply, the response and auxiliary systems synchronize after reaching the trivial fixed point. This form of synchronization between the chaotic drive system and the response systems possessing the fixed point attractor is, of course, some form of GS. On the other hand, if the largest CLE between the response and auxiliary systems is positive, the response system trajectories diverge from the fixed point and go to infinity. Thus the lack of GS and GMS between the drive and response systems leads to the disappearance of the response system attractor. MS reported in Refs. [3,27–29] for identical drive and response system parameters is observed just at the point where the fixed point $x_1 = y_1 = 0$ loses stability and a bifurcation which causes the destruction of the response system attractor occurs.

Despite the approximate character of the above analysis, the numerical simulations of Eqs. (6) and (7) with $\sigma = \sigma_1 = 10$, $b = 2\ (2/3)$, $r_1 = 28$ and r varied in the neighborhood of r_1 confirm the above conclusions (Fig. 5). If $r > r_1$ the response system attractor converges to the fixed point. If $r < r_1$ the response system attractor "swells" and diverges to infinity (Fig. 5). In both cases there is no apparent functional dependence between the drive and response system variables (Fig. 5(a)). In particular in the case with $r < r_1$ there is neither GS nor GMS between the drive and response systems since there is no response system attractor at all. However, due to the invariance of Eq. (7) against the scaling transformation $(x_1, y_1) \to (Ax_1, Ay_1)$ (Section 1) the variables of the response and auxiliary system show certain indications of MS, namely, they lie on a straight line $x_2 = Ax_1$, $y_2 = Ay_1$ (Fig. 5(b)). This happens both for $r > r_1$ and $r < r_1$. The constant A depends linearly on the changes of the response and auxiliary system initial conditions, and varies smoothly with the initial conditions of the drive system. The linear dependence between (x_1, y_1) and (x_2, y_2) retains in the presence of zero mean Gaussian noise with variance up to ca. 10^{-5}, added to the response system variables; for higher noise the constant A changes slowly in time. Here, the situation resembles that from the third example of Section 3.1 where GMS disappeared due to the loss of stability of the response system, but some features of MS between the drive and response systems were still observed.

The mechanism for the loss of stability of the response system may be readily identified as the blowout bifurcation [36–40]. This bifurcation occurs in dynamical systems which possess chaotic attractors inside an invariant manifold whose dimension is less than the dimension of the phase space. The blowout bifurcation occurs when this attractor loses transverse stability. This stability is controlled by the sign of the time-averaged largest transverse Lyapunov exponent which may be obtained from the linearization of the system equations around the invariant manifold. In our case, the systems (6), (7) may be viewed as a five-dimensional system with a chaotic attractor of the Lorenz system (6) contained inside the invariant manifold $x_1 = y_1 = 0$. Eq. (7) is already linear in the neighborhood of this manifold, so the largest transverse Lyapunov exponent is just the largest CLE between the response and auxiliary system. We calculated this CLE numerically and observed that its value crossed zero when $r = r_1$. When $r < r_1$ the mean value of this exponent is positive and the phase trajectory can depart from the invariant manifold. Usually, after the blowout bifurcation the phase trajectory is either captured by another attractor outside the invariant manifold (hysteretic bifurcation) or returns to the invariant manifold after a burst (non-hysteretic bifurcation) [36]. In the former case riddled basins of attraction [36,37,42–45]. can be observed before the blowout, in the latter case on–off intermittency appears [36,37,46,47]. But for all these phenomena to occur some kind of nonlinearity with respect to x_1, y_1 in Eq. (7) or any other equivalent mechanism would be necessary [40]. In the system under study they cannot be observed. However, in Eqs. (6) and (7) certain properties of a locally riddled basin of attraction for the attractor inside the $x_1 = y_1 = 0$ manifold can be seen. The basin is locally riddled [37,45] if in any neighborhood of a point belonging to this basin there is a positive measure set of points whose orbits depart from the attractor beyond a given distance. Then, e.g. under the influence of additive noise bursts are observed which drive the phase trajectory far from the invariant manifold even if the system is before blowout (attractor bubbling [37]). Such bursts of the x_1, y_1 variables in Eqs. (6), (7) are observed for $r > r_1$ if noise is added to the response system variables, in fact for any level of noise (Fig. 6).

It is known that in the drive–response method of IS it is possible to build a response system with parameters different from these of the drive system, and then to modify them with time so that they approach the parameters of the drive system. These modifications are based on the observation of the difference between the variables of the two systems. In this way IS between the two systems is obtained. This method is called adaptive synchronization [26]. It seems that the loss of stability of the response attractor if the parameters of the drive and response systems are slightly different makes it impossible to use the adaptive synchronization techniques to build systems which would show sized MS with a given drive system. The same is true if one wants to study sized MS in systems with parameters slowly varying in time [16].



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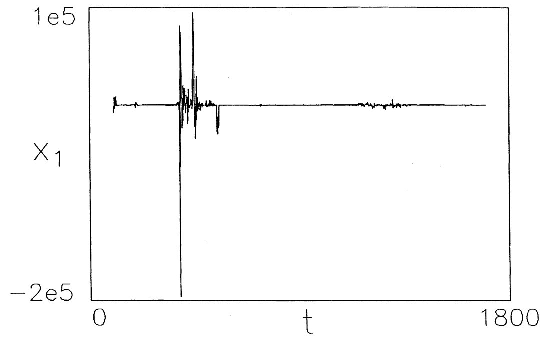

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Fig. 5. Sized GMS between two Lorenz systems with slightly different parameters (6), (7), with $r = 27.9$, $r_1 = 28$: (a) chaotic "attractor" x_1 vs. x, divergence of x_1 to infinity can be seen; (b) remnant of MS between the x variables of the response and auxiliary systems, $x_2 = 3.81x_1$, both variables diverge to infinity; (c) time series of x_1, divergence can be seen explicitly.

Fig. 6. Bursts of the response system variable in (6), (7) under the influence of additive noise for $r = 28.01$, $r_1 = 28$.

3.3. Marginal synchronization in mutually coupled systems: an example from solid-state physics

In this section, MS in a system of mutually coupled chaotic oscillators is investigated on the basis of a model taken from solid-state physics. The structure of mutual coupling is as explained in Section 2.2. The origin of the model is as follows. In ferromagnetic resonance the uniform mode with frequency ω_0 is excited by the rf field with frequency ω, perpendicular to the dc field direction. If both the dc field and the rf field frequency are chosen appropriately, and if the rf field amplitude exceeds a certain critical value, the uniform mode can decay into pairs of spin waves with opposite wave vectors and frequencies close to $\omega/2$ in the process called first-order Suhl instability [48]. This critical value is different for different pairs and depends on the wave vector length and propagation direction of spin waves. The threshold rf field amplitude for the occurrence of instability is equal to the critical amplitude minimized over all spin-wave pairs with frequencies close to $\omega/2$. Usually only one spin-wave pair is excited just above the threshold and it is called the critical pair. However, in the presence of symmetry the critical amplitude for many pairs may be equal and all of them can be excited. For example if the system has rotational symmetry with respect to the dc field direction, the critical rf field for all spin waves with frequency $\omega/2$, propagating at a given angle to this direction, is identical. The instability threshold is particularly low in the so-called coincidence regime, when $\omega \approx \omega_0$. For rf field amplitudes much above the threshold more pairs of different spin waves can be excited which can, e.g. lead to the appearance of multi-stability and chaos in the time dependence of absorption in the sample [49].

The model we consider below was shown to describe quite well certain phenomena which occur in chaotic spin-wave dynamics in coincidence regime (for details see Ref. [50]). Slowly varying in time parts of the complex amplitudes of uniform mode and spin-wave pairs are denoted as a_0, $a_{i,j}$, respectively. Here, i indexes the groups of spin-wave pairs with identical critical rf field amplitude, and j – spin-wave pairs within the groups. Excitation of spin waves slightly detuned from half of the pumping frequency is allowed. All spin-wave pairs belonging to one group have identical phenomenological damping η_i, detuning $\Delta\omega_i = \omega_i - \omega/2$ and coefficient of nonlinear interaction with the uniform mode V_{0i}; this yields identical critical rf field amplitude. The equations of motion for the amplitudes are

$$\begin{aligned} \dot{a}_0 &= |\eta_0 + i\Delta\omega_0||\eta_1 + i\Delta\omega_1|\varepsilon - (\eta_0 + i\Delta\omega_0)a_0 - i\textstyle\sum_{i,j} V_{0i}a_{i,j}^2, \\ \dot{a}_{i,j} &= -(\eta_i + i\Delta\omega_i)a_{i,j} - iV_{0i}^*a_{i,j}^*a_0. \end{aligned} \tag{9}$$

We considered Eq. (9) with $i = 1,2$, $j = 1,2$ (i.e. with two groups of two identical spin-wave pairs) and parameters (cf. [50]) $\eta_0 = 1.25$, $\Delta\omega_0 = -1.5$, $\eta_1 = 1.0$, $\Delta\omega_1 = 3.0$, $\eta_2 = 0.8$, $\Delta\omega_2 = 2.62$, $V_{01} = 1.0$, $V_{02} = 0.754$ and ε being the control parameter (rf field amplitude normalized to the instability threshold). Spin-wave pairs from the group $i = 2$ have higher critical value of the rf field amplitude and may be called "weak" pairs. It was shown in the model with $i = 1,2$, $j = 1$ [50] that for $\varepsilon < \varepsilon_c = 3.167\ldots$ the weak pair amplitude shows on–off intermittency: in its time series a sequence of laminar phases during which the amplitude is practically 0, and chaotic bursts is observed which is a trademark of on–off intermittency [46,47]. For $\varepsilon > \varepsilon_c$ the weak pair amplitude decays to 0. The transition at $\varepsilon = \varepsilon_c$ is, of course, blowout bifurcation.

With the parameters as given above, solutions of Eq. (9) are chaotic even if only one group of spin-wave pairs (that with $i = 1$) is present. We treat the system of equations for a_0, $a_{1,j}$ as the drive system and add the response system which consists of equations for $a_{2,j}$. This response system influences the drive system via the terms $V_{02}a_{2,j}^2$ in the first equation of Eq. (9). Therefore the mutual coupling structure between the drive and response systems is as described in Section 2.2. The results of the investigation of Eq. (9) for $\varepsilon = 3.0$ are summarized in Fig. 7, but they are similar for a wide range of $\varepsilon < \varepsilon_c$. A typical attractor is shown in Fig. 7(a). For $\varepsilon < \varepsilon_c$ we again observed on–off intermittency in the time series of all $a_{2,j}$ (cf. Fig. 7(b); the condensation of points along the $n_1 = |a_{2,1}| = 0$ axis betokens the occurrence of long laminar phases with $n_1 \approx 0$). We also observed that the system (9) could be subdivided into groups of subsystems (i.e. groups of spin-wave pairs with the same i) which have the property that there is no apparent functional dependence among the variables of subsystems belonging to different groups (Fig. 7(b)), while subsystems within groups show sized MS (Fig. 7(c)). In Fig. 7(c) MS between spin-wave pairs from

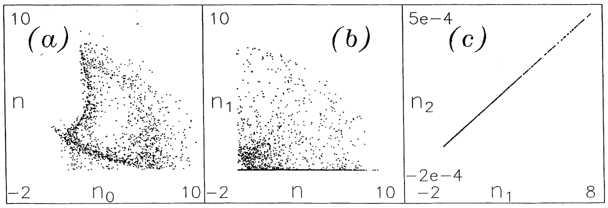

Fig. 7. MS in a mutually coupled system (9): (a) chaotic attractor $n = |a_{1,1}|$ vs. $n_0 = |a_0|$; (b) chaotic attractor $n_1 = |a_{2,1}|$ vs. n, condensation of points near $n_1 = 0$ indicates on–off intermittency in the time series of n_1; (c) MS between two spin-wave pairs from the same group, $n_2 = |a_{2,2}| = 6.31 \times 10^{-5} n_1$.

the $i = 2$ group was illustrated using experimentally observable quantities – absolute values of amplitudes of spin-wave pairs, but it occurs also for real and imaginary parts of $a_{2,j}$. This is an example of MS in a mutually coupled system.

We also investigated Eq. (9) without the nonlinear terms $V_{02}a_{2,j}^2$ in the first equation, as a unidirectionally coupled drive–response system. It turns out that such a system is very similar to the z-driven Lorenz system (6). For $\varepsilon > \varepsilon_c$, i.e. below the occurrence of blowout bifurcation, the solutions for $a_{2,j}$ converge to 0, and for $\varepsilon < \varepsilon_c$ – diverge to infinity. Hence GMS is again sensitive to small changes of the system parameters. In both cases there is always linear dependence between individual amplitudes of spin waves belonging to one group, i.e. even in the case of destruction of the response system attractor some features of GMS between the drive and response systems survive. Similarity between this system and Eq. (6) is caused by the fact that in Eq. (9) the equations for amplitudes of spin-wave pairs are again linear with respect to $a_{i,j}$ and invariant under the scaling transformation $a_{i,j} \to A a_{i,j}$. This result shows that the drive system dynamics is modified substantially by the influence of the response system for $\varepsilon < \varepsilon_c$ which can lead to the stabilization of GMS between the drive and response systems in the region of parameters above the blowout bifurcation.

4. Summary and conclusions

In this paper two generalizations of the concept of MS in chaotic systems were considered. First, two kinds of GMS were defined in unidirectionally coupled systems in which the drive and response systems were different chaotic systems or subsystems: the constant GMS and the sized GMS. In the case of constant GMS the function between the points on the drive and response system attractors was defined up to a constant shift which depended on the initial conditions, and in the case of sized GMS – up to a constant scaling factor. The auxiliary system approach applied to these cases results in MS between the response and auxiliary system if there is GMS between the drive and response system. However, in the case of a multi-attractor response system this method may be incapable of detecting GMS. Constant GMS can be robust against changes of the drive and response system parameters and noise, but we also analyzed an example from which it follows that, e.g. addition of small noise with non-zero mean or small shift in the mean values of the drive system variables can lead to the disappearance of GMS. This is due to the loss of stability of the response system. In this situation, certain properties of MS between the response and auxiliary system are preserved, but there is no GMS as there is no stable response system attractor. Sized GMS is, in general, sensitive to the changes of system parameters, and the response system attractor may be destroyed due to the blowout bifurcation. Again, certain properties of MS between the response and auxiliary systems retain though there is no GMS between the drive and response systems. The fact that the response and auxiliary systems can still exhibit certain indications of MS though there is no GMS with the drive system is caused by the invariance of the response system equations against shifts or scaling transformations. As pointed out in Ref. [29] these symmetries are more fundamental for the occurrence of MS than the existence of the zero largest CLE. Hence zero largest CLE between the response and auxiliary system is not a sufficient condition for the occurrence of GMS (see the case of Eqs. (4) and (5)), and also if this exponent is positive on average, some remnants of MS between the response and auxiliary system are still visible (see the case of Eqs. (6) and (7)).

Second, we defined and analyzed MS in mutually coupled systems, which were drive–response systems with the influence of the response system on the drive system added. In such systems, built in a systematic way, groups of response subsystems may be identified which show MS within groups. An example from solid-state physics of such MS was given. On its basis it was also shown that addition of nonlinear coupling from the response systems to the drive system stabilizes the response system attractor above the blowout for $\varepsilon < \varepsilon_c$. Instead of the divergence of the response system variables to infinity they show on–off intermittency, and, moreover, the property of MS between response systems is preserved. It turns out, then, that coupling the drive system to the response system can stabilize GMS between the drive and response systems (in the sense discussed in Section 2.2). An interesting point is that this does not require any modification of the response system. As pointed out at the end of Section 2.2, the occurrence of on–off intermittency above blowout bifurcation, and resulting stabilization of sized GMS, requires some nonlinearity [40]. However, if this nonlinearity were added to the response system equations, it could destroy GMS by, e.g. turning it into stronger GS, since this could violate the invariance of the response system equations under the scaling transformation. So it seems that in order to have sized GMS (but not GS) which is robust against changes of system parameters it is useful to couple the drive system to the response system via appropriately chosen nonlinearity. It can be seen that the problem of influence of nonlinearities on the stability of MS and GMS is quite interesting and deserves a separate study.

MS observed in Eq. (9) is an extension of the well-known results from the theory of stationary state in nonlinear ferromagnetic resonance above the Suhl instability threshold to the case of chaotic spin-wave dynamics. For example for the stationary state in parallel pumping it is known [51–53] that if many spin-wave pairs have the same critical rf field amplitude, all of them will be excited above the instability threshold. The sum of absolute values of amplitudes of spin-wave pairs is fixed, while, depending on the initial conditions, the individual amplitudes are different. In chaotic state similar results can be interpreted as an example of MS among spin-wave pairs. MS can lead to the decrease of the correlation dimension of attractors observed in chaotic spin-wave dynamics [54–56]. Results concerning MS in nonlinear ferromagnetic resonance will be published separately.

We think that the generalizations of MS introduced in the present paper may provide a starting point for future investigation of MS and GMS in various physical systems. In particular, it would be interesting to check if there is any possibility to distinguish between GS and GMS only on the basis of time series for the drive and response systems, without the auxiliary system [4]. Other problems which deserve further investigation are connected with the influence of asymmetric noise and various nonlinearities on the stability of MS and GMS. It would be also interesting to find examples of GMS in chaotic systems other than analyzed in this paper; as pointed out in Section 2.2, some form of MS can be present in systems consisting of many interacting modes.

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