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On–Off Intermittency in a High-dimensional Model of Randomly Driven Parallel Pumping†

A. KRAWIECKI and A. SUKIENNICKI‡

Institute of Physics, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland

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Abstract—A model of parallel pumping with many interacting parametric spin-wave pairs is investigated numerically. Two ways of obtaining on–off intermittency in the time series of absorption are discussed: random or chaotic modulation of the rf field amplitude or of the dc field, slow in comparison with the rf field frequency. If the possibility of thermal excitation of spin waves is neglected, only one spin-wave pair is excited above the intermittency threshold and exhibits bursts. In the opposite case, a phenomenon of widening of the phase space in the presence of thermal noise is observed: a packet of parametric spin waves is excited with frequencies close to half the pumping frequency. This modifies quantitatively, but not qualitatively the characteristics of on–off intermittency in the presence of thermal noise. © 1997 Elsevier Science Ltd

1. INTRODUCTION

Recently, there has been growing interest in the investigation of systems exhibiting on–off intermittency (OOI) [1–9]. This kind of intermittency is characterized by a sequence of (flat) laminar phases, during which the observed signal remains practically constant, and bursts, when the signal behaves chaotically [1]. This phenomenon occurs if, for certain values of the control parameter, there exists an invariant subspace with dimension lower than that of the phase space of a system, in which a global chaotic attractor is located. When the control parameter is increased above a certain critical value (the OOI threshold), this attractor loses stability. Then the phase trajectory is no longer attracted to the invariant subspace, but departs from it occasionally; the more frequently, the greater is the difference between the actual and critical value of the control parameter. If the distance from the invariant subspace is measured as an observable, this results in a sequence of flat phases and bursts. The loss of stability by the attractor is an example of the so-called blowout bifurcation [2].

In Ref. [1], a distinctive geometric mechanism was proposed which explains the occurrence of bursts. The influence of the chaotic motion within the invariant subspace on the observed dynamics in the directions perpendicular to the invariant subspace may be described by means of an effective time-dependent bifurcation parameter. This parameter is a function of system variables and behaves chaotically in time. If the amplitude of its chaotic oscillations is high enough, a bifurcation occurs in the subspace orthogonal to the invariant one from the stable fixed point (flat state) to chaotic bursts. A direct application of this idea is possible in systems in which, for a certain value of the control parameter, a stable fixed point loses stability and a new fixed point or a limit cycle appears. If the control parameter (normally constant in time) is replaced by a properly chosen chaotic time series, the fixed point is

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‡Also at Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02 668 Warsaw, Poland.

sometimes stable and sometimes not. Then, if the cases of the stability loss occur frequently enough, or, in other words, if the amplitude of chaotic oscillations of the control parameter is high enough, OOI appears. An example of such a system is the logistic map with random control parameter [3, 4]

$$y_{n+1} = \varepsilon x_{1,n} y_n(1 - y_n) + \xi x_{2,n}, \quad \varepsilon > 0. \tag{1}$$

Here, $x_{1,n}$,$x_{2,n}$ are uncorrelated random variables with uniform distribution in the interval [0,1] and the last term represents thermal noise with amplitude ξ. It should be noted that the map in equation (1) is two-dimensional, with two independent dynamical variables $x_{1,n}$,y_n. Let us consider the case with zero thermal noise: $\xi = 0$. Then $y_n = 0$ is a fixed point for the logistic map which is stable if $\varepsilon x_{1,n} < 1$ and unstable if $\varepsilon x_{1,n} > 1$. It turns out [3] that if $\varepsilon < e$ then y_n always approaches this fixed point, independently of the choice of the initial condition. If $\varepsilon > e$, then y_n may remain practically equal to zero for long stretches of time, but occasionally departs from the fixed point rather abruptly, producing a burst. Thus, for $\varepsilon < e$, there is a global one-dimensional stable attractor of the map (1) within the invariant subspace $y_n = 0$ which loses stability for $\varepsilon > e$. This attractor is the unit interval [0,1]. Within this attractor, the dynamics is described by $x_{1,n}$ which in this case is a random variable, but may be replaced with any chaotic variable. The time-dependent control parameter $\varepsilon x_{1,n}$ in this description becomes the time-dependent bifurcation parameter, and ε is the control parameter for the map (1) whose critical value for the occurrence of OOI (the OOI threshold) is $\varepsilon_c = e$.

The characteristics of OOI derived for the case of the map (1) are as follows [3]. If $\xi = 0$ then, for given ε, the probability that a laminar phase has length τ is $P(\tau) = A \exp(-B\tau|\varepsilon - \varepsilon_c|^2)\tau^{-3/2}$, or $P(\tau) \propto \tau^{-3/2}$ for sufficiently small $\tau,|\varepsilon - \varepsilon_c|$. The mean laminar phase length changes with the control parameter as $\langle\tau\rangle = \alpha + \beta|\varepsilon - \varepsilon_c|^{-1}$. Here A, B, α and β are constants. For any $\xi > 0$ [4], the OOI threshold decreases abruptly to $\varepsilon_c = 1$. In this case, the $\tau^{-3/2}$ law for the laminar phase lengths distribution is valid only for small τ and the exponential decay of $P(\tau)$ begins at $\tau \simeq \tau^*$ substantially smaller than that in the noise-free case. The characteristic time τ^* scales with the noise amplitude as $\tau^* \propto [\ln(y_t/\xi)]^2$, where y_t is the onset threshold for a burst, which may be chosen quite freely. There exists strong, mainly numerical evidence that, apart from some exceptional cases [5], the above characteristics remain valid for all systems with OOI, though sometimes they may be difficult to measure [6, 7].

In this paper, OOI is studied in a certain model of parallel pumping. Parallel pumping means ferromagnetic resonance with the magnetic rf field applied parallel to the dc field. In contrast with conventional experiments with parallel pumping, in the model considered here either the high-frequency rf field amplitude is modulated with some slowly varying in time random or chaotic function or the dc field has a slowly varying in time random or chaotic component. Nonlinear interactions of many pairs of spin waves (SW) excited by the rf field are taken into account. This model is an extension of the one previously studied, in which only one pair of SW was excited by the rf field [10]. Besides the characterization of OOI in a system not previously studied, a phenomenon of widening of the phase space of the system with OOI due to the presence of thermal noise is found. A good review of the theory of parallel pumping is [11]; derivation of the model equations for the SW amplitudes in Section 2 closely follows the one presented in that paper for the case of constant rf field amplitude.

2. THE MODEL

High-power ferromagnetic resonance has been for more than fifteen years a useful tool in the study of nonlinear phenomena, including multistability [12], auto-oscillations of the magnetization and chaos [12–18], hyperchaos [19], control of chaos [20, 21], various kinds of intermittency [22–24], and chaotic transients [25, 26]. OOI was also predicted theoretically [27] and observed experimentally [28] in this system. All the work mentioned above was performed with the rf field amplitude constant in time. In parallel pumping with a certain stochastic component of the rf field amplitude, only the stationary state was investigated [29]. Recently a case in which the rf field amplitude possessed a random or chaotic component varying in time slowly in comparison with SW relaxation time was analyzed. It was shown that, both in perpendicular resonance and parallel pumping, OOI should then be observed in the time series of absorption [10]. However, the model considered in the previous work seems to be oversimplified as it does not take into account nonlinear SW interactions. This drawback is removed in the model considered in the present paper.

In order to construct the model, let us assume first that the rf field amplitude h_z is constant in time. Both the dc field H_0 and rf field $h_z \cos \omega t$ (whose frequency ω is of the order of some GHz) are applied along the z-axis which is an easy axis of an anisotropic ferromagnet. As the rf field amplitude exceeds a certain threshold h_z^{thr}, the so-called parallel pumping instability occurs [30]: the rf field excites pairs of SW with half the pumping frequency and opposite wave vectors. These SW pairs interact with one another via four-mode interactions. The respective Hamiltonian $\mathcal{H}$ of the S-theory of interacting SW is then [11]

$$\mathcal{H} = h_z \cos \omega t \sum_{\mathbf{k}} (J_{\mathbf{k},-\mathbf{k}} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* + \text{c.c.}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}'} (T_{\mathbf{k},\mathbf{k}'} a_{\mathbf{k}}^* a_{\mathbf{k}'}^* a_{\mathbf{k}} a_{\mathbf{k}'} + S_{\mathbf{k},\mathbf{k}'} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* a_{\mathbf{k}'} a_{-\mathbf{k}'}). \tag{2}$$

Here, $\mathbf{k}$ is the wave vector of a SW with frequency $\omega_{\mathbf{k}}$, $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ are complex SW amplitudes, $J_{\mathbf{k},-\mathbf{k}}$ are coefficients of interactions between SW pairs and the rf field, $T_{\mathbf{k},\mathbf{k}'}$ and $S_{\mathbf{k},\mathbf{k}'}$ are coefficients of diagonal four-mode interactions, and the asterisk denotes the complex conjugate. The overall SW dispersion relation is

$$\omega_{\mathbf{k}}^2 = \gamma^2 (H_i + Dk^2)(H_i + Dk^2 + 4\pi M_0 \sin^2 \theta_{\mathbf{k}}), \tag{3}$$

where H_i is the resultant dc magnetic field consisting of the external, demagnetization and anisotropy fields, D is the exchange constant, M_0 is the saturation magnetization, γ is the gyromagnetic ratio and $\theta_{\mathbf{k}}$ is the angle between the dc field and $\mathbf{k}$. An observed quantity is usually absorption in the sample which is proportional to the sum of the absolute values of SW amplitudes. Equations of motion for the SW amplitudes are obtained as $\dot{a}_{\mathbf{k}} = -\eta_{\mathbf{k}} a_{\mathbf{k}} + \mathrm{i}\partial\mathcal{H}/\partial a_{\mathbf{k}}^*$ (and their complex conjugates), where the dot denotes the time derivative and $\eta_{\mathbf{k}}$ is phenomenological SW damping. Taking into account the fact that the amplitudes of the SW belonging to a pair $\mathbf{k}$, $-\mathbf{k}$ differ only by a constant phase factor $q_{\mathbf{k}}$, i.e. $a_{-\mathbf{k}} = a_{\mathbf{k}} \exp(\mathrm{i}q_{\mathbf{k}})$, these equations may be rewritten in new variables $b_{\mathbf{k}}(t) = a_{\mathbf{k}}(t)\exp(\mathrm{i}q_{\mathbf{k}}/2)\exp(\mathrm{i}\omega t/2)$, where $b_{\mathbf{k}}$ vary slowly in time in comparison with the rf field frequency [11]:

$$\dot{b}_{\mathbf{k}} = (-\eta_{\mathbf{k}} + \mathrm{i}\Delta\omega_{\mathbf{k}}) b_{\mathbf{k}} + \mathrm{i}\frac{h_z}{2} J_{\mathbf{k}} b_{\mathbf{k}}^* + \mathrm{i} \sum_{\mathbf{k}'} (2T_{\mathbf{k},\mathbf{k}'} |b_{\mathbf{k}'}|^2 b_{\mathbf{k}} + S_{\mathbf{k},\mathbf{k}'} b_{\mathbf{k}'}^2 b_{\mathbf{k}}^*). \tag{4}$$

Here, $J_{\mathbf{k}} = J_{\mathbf{k},-\mathbf{k}} + J_{-\mathbf{k},\mathbf{k}}$ and $\Delta\omega_{\mathbf{k}} = \omega_{\mathbf{k}} - \omega/2$. The parallel pumping instability threshold is $h_z^{\text{thr}} = \min 2\eta_{\mathbf{k}}/|J_{\mathbf{k}}|$, where the minimum is taken over all SW pairs with $\omega_{\mathbf{k}} = \omega/2$. The pair for which this expression is at a minimum is called a critical pair.

First it is necessary to reduce the dimensionality of the (infinite) system of equation (4). The SW amplitudes differ significantly from zero only in the neighbourhood of the resonance surface $\omega_{\mathbf{k}} = \omega/2$. After introducing new coordinates in $\mathbf{k}$-space, with θ and ϕ being the spherical coordinates at the resonance surface and κ the coordinate perpendicular to the resonance surface, it is possible to retain the dependence on κ only in $\Delta\omega_{\mathbf{k}}$, and replace other parameters with their values at the resonance surface. Because the system has a rotational symmetry with respect to the anisotropy axis, this yields $T_{\mathbf{k},\mathbf{k}'} = T(\theta,\theta',\phi'-\phi)$, $S_{\mathbf{k},\mathbf{k}'} = S(\theta,\theta',\phi'-\phi)$ and $J_k = J(\theta)\exp(2i\phi)$. The dependence on ϕ is removed from equation (4) by replacing the sum over ϕ' by an integral, introducing new variables $b_\kappa(\theta) = b_{\mathbf{k}}\exp(-i\phi)$ and rescaled coefficients of four-mode interactions

$$T(\theta,\theta') = \int_0^{2\pi} d(\phi'-\phi)T(\theta,\theta',\phi'-\phi),$$
$$S(\theta,\theta') = \int_0^{2\pi} d(\phi'-\phi)S(\theta,\theta',\phi'-\phi)e^{2i(\phi'-\phi)}. \tag{5}$$

In order to remove the dependence on θ, a more radical assumption is necessary. It was shown experimentally [31] and explained theoretically [11] that, above the instability threshold in the stationary state, only the critical pairs of SW are excited (normally, the ones with $\theta_{\mathbf{k}} = \theta_{\text{crit}} = 90°$); excitation of other SW is possible only if the rf field amplitude exceeds significantly the instability threshold. Amplitudes of all other SW are then equal to zero and equation (4) assumes the form [11]

$$\dot{b}_\kappa = (-\eta + i\Delta\omega_\kappa)b_\kappa + i\frac{h_z}{2}Jb_\kappa^* + i\sum_{\kappa'}(2T|b_{\kappa'}|^2 b_\kappa + Sb_{\kappa'}^2 b_\kappa^*). \tag{6}$$

In this equation $b_\kappa = b_\kappa(\theta_{\text{crit}})$, etc. If the state of the system is not stationary (i.e. absorption in the sample is time-dependent), there is no theoretical background for the above simplification and the only reason for its application is the possibility of performing numerical calculations using equation (6).

It should be noted that in equation (6), $\Delta\omega_\kappa$ may differ from zero; thus the excitation of SW pairs with non-zero detuning from half the pumping frequency is allowed. In the following, SW will be labelled by κ_j, $j = -N,-N+1,\dots,N$, where $2N+1$ is the dimension of the system (6) (or the number of SW pairs) and their detunings will be placed symmetrically around zero: $\Delta\omega_{\kappa_j} = j\Delta\omega$. Here, $\Delta\omega$ is a parameter connected with a characteristic dimension of the sample d (e.g. the diameter of a spherical sample). In finite samples, SW do not form a continuous spectrum (3) because the wave vector lengths assume values from a discrete set $k \simeq n\pi/d$. Thus the frequency distance between neighbouring modes is $\Delta\omega \simeq (\partial\omega_{\mathbf{k}}/\partial k)\pi/d$, where the derivative is taken at $\omega_{\mathbf{k}} = \omega/2$ and $\theta = \theta_{\text{crit}}$.

Equations (6) have a stationary solution $b_\kappa = 0$ (zero absorption) which is stable for $h_z < h_z^{\text{thr}}$ and unstable otherwise. There are two independent parameters which may be varied externally to obtain OOI in this system, in the same way as in the map (1): the rf field amplitude and the dc field. In the former case, the rf field time dependence may be assumed to be $h_z(t)\cos\omega t$, where $h_z(t)$ is the time-dependent rf field amplitude which varies in time randomly or chaotically, slowly in comparison with $\cos\omega t$. In the latter case, the dc field which has a certain random or chaotic time-dependent component $h_0(t)$ may be written as $H_0 + h_0(t)$, where H_0 is the constant part and $h_0(t) \ll H_0$. Though both the dc and rf fields are applied along the anisotropy axis, these two cases must be distinguished.

In the first case, the amplitude of the high-frequency oscillations is modulated, but the SW spectrum (3) remains unchanged. Because this modulation is slow, it is still possible to exclude the fast time dependence from the SW amplitudes by changing the variables from $a_{\mathbf{k}}$ to $b_{\mathbf{k}}$, as in the case of constant rf field amplitude. Other assumptions made during the derivation of equation (6) are still valid; thus this equation may be simply rewritten with the replacement $h_z \to h_z(t)$. Introducing new variables $u_{\kappa_i} = T^{1/2} b_{\kappa_i}$ and a dimensionless rf field amplitude $\varepsilon(t) = h_z(t)/h_z^{\text{thr}}$, one gets

$$\dot{u}_{\kappa_i} = (-\eta + \mathrm{i}\Delta\omega_{\kappa_i})u_{\kappa_i} + \mathrm{i}\varepsilon(t)\eta u^*_{\kappa_i} + \mathrm{i}\sum_i \left(2u_{\kappa_i}|u_{\kappa_i}|^2 + \frac{S}{T}u^*_{\kappa_i}u^2_{\kappa_i}\right) + \xi_{\kappa_i,\text{therm}}. \tag{7}$$

In this equation $\xi_{\kappa_i,\text{therm}}$ represents phenomenological thermal noise. By including this term, one accounts for the fact that, even with the rf field turned off, SW are thermally excited to some equilibrium level $u_{\kappa_i,\text{therm}}$. Assuming that $\xi_{\kappa_i,\text{therm}} = \xi[1 + (\Delta\omega_{\kappa_i}/\eta)^2]^{1/2}$ and neglecting all nonlinear terms in equation (7), one gets $|u_{\kappa_i,\text{therm}}| = u_{\text{therm}} = \xi/\eta$. This expression is constant in time because processes of thermal excitation are much faster than the time dependence of b_{κ_i}.

If the dc field has some time-dependent component, the SW frequencies become time-dependent too (through H_i in equation (3)), so in this case the quantities $\Delta\omega_{\kappa_i}$ in equation (6) become randomly or chaotically time-dependent. They may be written in the form $\Delta\omega_{\kappa_i} + \delta\omega(t)$, where $\delta\omega(t) \simeq \gamma h_0(t)$. Thus equation (6) may be rewritten as

$$\dot{u}_{\kappa_i} = [-\eta + \mathrm{i}(\Delta\omega_{\kappa_i} + \delta\omega(t))]u_{\kappa_i} + \mathrm{i}\varepsilon\eta u^*_{\kappa_i} + \mathrm{i}\sum_i \left(2u_{\kappa_i}|u_{\kappa_i}|^2 + \frac{S}{T}u^*_{\kappa_i}u^2_{\kappa_i}\right) + \xi_{\kappa_i,\text{therm}}, \tag{8}$$

where $\varepsilon = h_z/h_z^{\text{thr}}$ is a constant in this case.

Equations (7) and (8) are the system of equations in which OOI is studied in Section 3. For the purpose of numerical calculations in Section 3, it is convenient to divide both sides of these equations by η and work with renormalized time $t' = \eta t$. The parameter values may be estimated as follows. The coefficients S and T given in equation (5) may assume both positive and negative values and depend strongly on the wave vector length of the excited SW pairs. All numerical calculations in Section 3 were performed with $S/T = 1$, but the overall behaviour of the solutions turned out to be insensitive to the choice of this parameter. The SW damping η is of the order of $10^6\,\text{s}^{-1}$ in yttrium iron garnet (YIG). Then, in the room temperature, $u_{\text{therm}} = \xi/\eta \simeq 10^{-8}\,\text{s}^{-1/2}$ [10], but higher (even unphysical) thermal excitation levels were also considered because then the intermittency characteristics differ sharply from the ones without thermal noise. The parameter $\Delta\omega$ ranged from $10^{-3}\eta$ to 1.0η; typically results are presented for 0.025η (SW densely distributed in ω-space or big samples) and 1.0η (discrete SW spectrum or small samples).

The $\varepsilon(t)$ and $\delta\omega(t)$ functions were assumed as follows. Let $\chi(t_1,t_2)$ be the characteristic function of the time interval (t_1,t_2), i.e. $\chi = 1$ for $t_1 \le t < t_2$ and $\chi = 0$ otherwise. The time-dependent bifurcation parameters in equations (7) and (8) were assumed as random square waves, i.e. functions changing their values every $\Delta t = \Delta t'/\eta$ time units, which may be written as

$$\begin{Bmatrix} \varepsilon(t) \\ \delta\omega(t) \end{Bmatrix} = \begin{Bmatrix} \varepsilon \\ \delta\omega \end{Bmatrix} \sum_{n=0}^{\infty} \begin{Bmatrix} x_n \\ x_n - 0.5 \end{Bmatrix} \chi(t_n, t_n + \Delta t), \quad t_{n+1} = t_n + \Delta t. \tag{9}$$

In equation (9), x_n is a random variable with uniform distribution in the interval [0,1], as in equation (1), and ε, $\delta\omega$ are control parameters for equations (7) and (8), respectively. The function $\varepsilon(t)$ was chosen to be always positive, with mean $\varepsilon/2$, while $\delta\omega(t)$ assumes both positive and negative values and has zero mean. With such a choice of $\varepsilon(t)$ and $\delta\omega(t)$, equations (7) and (8) are in fact turned into a discrete-time map (as in the experiment [9]), but such a random square wave may serve as an approximation for any stochastic or chaotic signal with a finite amplitude of oscillations and varying in time with a characteristic time scale Δt (see Section 4 for further discussion). Between the time steps the system (7,8) was integrated numerically using a fourth and fifth order Runge–Kutta method with permanent step size and error control. It turns out that, in order to observe OOI in sufficiently wide intervals of the control parameters ε and $\delta\omega$, the time step Δt must be comparable or greater than the SW relaxation time $1/\eta$ [10]. Thus the results of Section 2 were obtained for $\Delta t' = 2.0$ unless pointed out otherwise. The number of interacting SW in the system (7,8) ranged from $2N+1=25$ up to 121; in particular, more SW were usually taken into consideration when the solutions with small frequency distance between neighbouring modes $\Delta\omega/\eta \ll 1$ and non-zero thermal noise $u_{\text{therm}} \neq 0$ were investigated. The numerical results presented in Section 3 are checked to be independent of the number of equations, of course within some reasonable borders. For the purpose of characterization of OOI, absorption in the sample was defined simply as a sum of absolute values of SW amplitudes, i.e. the proportionality factor was assumed to be unity.

3. RESULTS

3.1. *On-off intermittency thresholds*

Let us consider equations (7) and (8) with $u_{\text{therm}} = 0$. For the time dependence of the bifurcation parameters as in equation (9), the critical values of the control parameters for the occurrence of OOI, ε_c and $\delta\omega_c$ may be estimated after linearizing these equations around the fixed point $u_{\kappa_j} = 0$ [3]. The eigenvalues of the stability matrix for $t_n \leq t < t_{n+1}$ are, in the case of equation (7),

$$\lambda_{\pm}^{j,n} = -\eta \pm \sqrt{\varepsilon^2\eta^2x_n^2 - \Delta\omega_{\kappa_j}^2} \tag{10}$$

and, in the case of equation (8),

$$\lambda_{\pm}^{j,n} = -\eta \pm \sqrt{\varepsilon^2\eta^2 - [\Delta\omega_{\kappa_j} + \delta\omega(x_n - 1/2)]^2}, \tag{11}$$

where j is the SW index. As $\text{Re}\,\lambda_{-}^{j,n} < 0$ then the most rapidly growing (or most slowly decreasing) part in the solution for u_{κ_j} after n_0 time steps Δt in the linear approximation will be proportional to $\exp(\sum_n^{n_0} \text{Re}\,\lambda_+^{j,n}\Delta t) \rightarrow \exp(n_0\Delta t\langle \text{Re}\,\lambda_+^{j,n}\rangle)$, as $n_0 \rightarrow \infty$, where the $\langle ... \rangle$ denotes the average over n. Thus u_{κ_j} will decrease to zero if $\langle \text{Re}\,\lambda_+^{j,n}\rangle < 0$ and increase if $\langle \text{Re}\,\lambda_+^{j,n}\rangle > 0$. The OOI threshold for the jth SW pair is then defined by the condition $\langle \text{Re}\,\lambda_+^{j,n}\rangle = 0$. Moreover, it can be easily noted that $\langle \text{Re}\,\lambda_+^{j,n}\rangle$ is at a maximum for $j=0$ in both cases of equations (7) and (8), so only the SW pair with $\Delta\omega_{\kappa_j} = 0$ will exhibit OOI just above the intermittency threshold and ε_c and $\delta\omega_c$ calculated for this SW pair will be the critical values of the control parameters for the whole system (7,8). This conclusion is confirmed by numerical calculations (see Section 3.2).

For the case of random rf field amplitude, equation (7), and $j=0$, $\langle \text{Re}\,\lambda_+^{j,n}\rangle = -\eta + \varepsilon\eta\langle x_n\rangle = -\eta + \varepsilon\eta/2$ (see equation (9)) and $\varepsilon_c = 2$ or $h_{z,c} = 2h_z^{\text{thr}}$. So OOI appears when the mean value of the rf field amplitude exceeds the parallel pumping instability threshold. This result was obtained earlier in Ref. [10] and confirmed by numerical calculations with $\Delta t'$ ranging from 0.5 up to 20.0.

Let us now turn to the case of equation (8). In order to simplify the notation, let us introduce a new random variable $z_n = \delta\omega(x_n - 0.5)$ with zero mean and uniform distribution $\rho(z) = 1/\delta\omega$ in the interval $[-\delta\omega/2, \delta\omega/2]$. Then $\langle \text{Re}\,\lambda_+^{j,n}\rangle = \langle -\eta + \text{Re}\sqrt{\varepsilon^2\eta^2 - z_n^2}\rangle$, and two separate cases must be considered. If $\delta\omega/2 \leq \varepsilon\eta$, the expression under the square root is always positive and

$$\begin{aligned}\langle\sqrt{\varepsilon^2\eta^2 - z_n^2}\rangle &= \int_{-\delta\omega/2}^{\delta\omega/2} \sqrt{\varepsilon^2\eta^2 - z^2}\,\frac{1}{\delta\omega}\,dz = \eta F_1(\varepsilon, \delta\omega/\eta) \\ &= \eta\left[\frac{1}{2}\sqrt{\varepsilon^2 - \frac{1}{4}\left(\frac{\delta\omega}{\eta}\right)^2} + \frac{\varepsilon^2\eta}{\delta\omega}\arcsin\left(\frac{\delta\omega}{2\varepsilon\eta}\right)\right].\end{aligned} \tag{12}$$

If $\delta\omega/2 > \varepsilon\eta$, the expression under the square root is positive only if $-\varepsilon\eta \le z_n \le \varepsilon\eta$ and otherwise the square root is purely imaginary, thus

$$\langle \mathrm{Re}\sqrt{\varepsilon^2\eta^2 - z_n^2}\rangle = \int_{-\varepsilon\eta}^{\varepsilon\eta} \sqrt{\varepsilon^2\eta^2 - z^2}\frac{1}{\delta\omega}\mathrm{d}z = \eta F_2(\varepsilon,\delta\omega/\eta) = \eta\frac{\varepsilon^2\pi^2\eta}{2\delta\omega}. \tag{13}$$

For a given ε, the critical value of the control parameter $\delta\omega_c$ may then be evaluated from the condition

$$F(\varepsilon,\delta\omega_c/\eta) = 1, \qquad F(\varepsilon,\delta\omega/\eta) = \begin{cases} F_1(\varepsilon,\delta\omega/\eta), & \delta\omega/\eta < 2\varepsilon, \\ F_2(\varepsilon,\delta\omega/\eta), & \delta\omega/\eta > 2\varepsilon. \end{cases} \tag{14}$$

A numerical solution of the above equation is depicted in Fig. 1. For $\varepsilon < 1$, there is no OOI at all; $\delta\omega_c$ increases monotonically with ε and depends linearly on the SW damping η. Contrary to the previous case, OOI is obtained for $\delta\omega < \delta\omega_c$, i.e. for a decreasing control parameter. The OOI threshold was also calculated numerically from equation (8) with $N = 0$ (single SW pair) by observing the long-time behaviour of the solution (Fig. 1). It turns out that, for moderate $\Delta t'$, the actual values of $\delta\omega_c$ are greater than the ones predicted theoretically but, for increasing $\Delta t'$, both numerical and theoretical values coincide. This is so probably because, for $\Delta t' \sim 1.0$, the influence of the eigenvalues $\lambda_-^{j,n}$ on the solution for u_{κ_j} cannot be completely neglected. The increase of the number of SW in equation (8) does not influence the OOI threshold. A more general conclusion which may be drawn from this result is that in time-continuous systems the critical value of the control parameter for the occurrence of OOI may depend on the characteristic time of chaotic oscillations of the time-dependent bifurcation parameter.

3.2. *Results of numerical modelling*

In this section, the results of numerical modelling of OOI with the use of equation (7) are presented. Solutions in the case of the dc field with random time-dependent component have properties identical to these in the case of randomly modulated rf field amplitude, so only this latter case is discussed in detail.

If equation (7) is solved without thermal noise ($u_{\mathrm{therm}} = 0$) then, for $\varepsilon > 2.0$, OOI in the time series of absorption appears. Laminar phases, during which all SW amplitudes are practically equal to zero and thus absorption is negligible, can be easily distinguished from

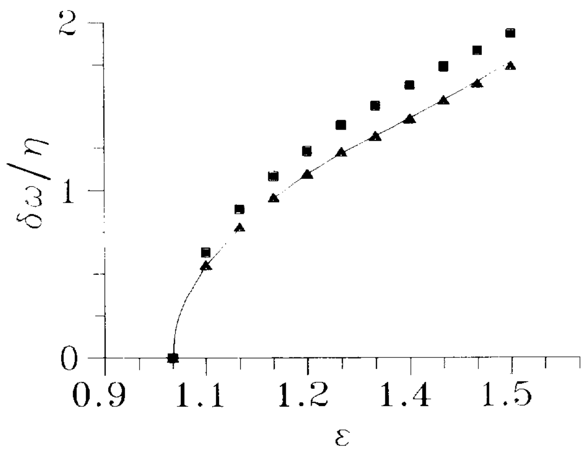

Fig. 1. The OOI threshold for eqn (8): squares, $\Delta t' = 2.0$; triangles, $\Delta t' = 20.0$; solid line, theoretical results of eqn (14).

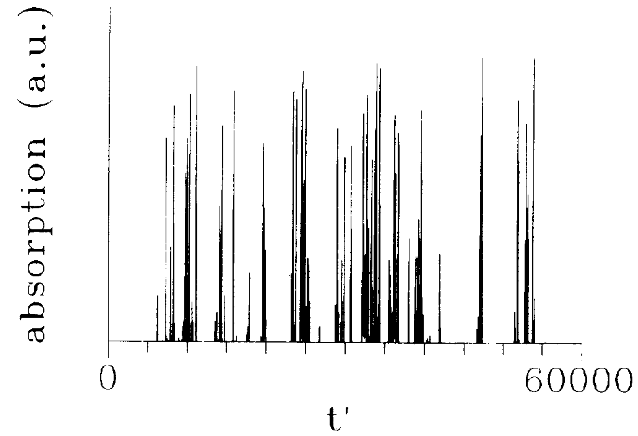

Fig. 2. Time series of absorption for eqn (7), with $\varepsilon = 2.033$, $u_{therm} = 0$.

the bursts, during which absorption behaves chaotically (see Fig. 2). Only one pair of SW is excited during the bursts, independently on $\Delta\omega$, at least within the borders stated in Section 2. Just above the OOI threshold, this is the pair with zero detuning from half the pumping frequency, $j = 0$. However, the index j of the excited pair decreases from 0 to $-N$ with the rise of ε and hence, for greater values of the control parameter, amplitudes of SW pairs with greater detuning exhibit bursts. Amplitudes of all other SW decay to zero. In fact, in order to avoid the convergence of the solution to zero during long laminar phases $u_{therm} = 10^{-300}$ was assumed. This can be more easily observed if $\Delta\omega \ll 1$, because, for $\Delta\omega = 1.0$ and $2.0 < \varepsilon < 2.2$, for example, always the mode with $j = 0$ is excited and only for greater values of the control parameter SW with $j = -1, -2, \ldots$ exhibit bursts. For a given ε, the mean length of laminar phases is independent of $\Delta\omega$ and decreases with the rise of the rf field amplitude proportionally to $|\varepsilon - \varepsilon_c|^{-1}$ (see Fig. 3). Similarly, the probability distribution of the laminar phase lengths for a given ε obeys identical scaling law $P(\tau) \propto \tau^{-3/2}$, for all $\Delta\omega$. Moreover, these scaling laws are quantitatively identical to the ones obtained in Ref. [10] within a

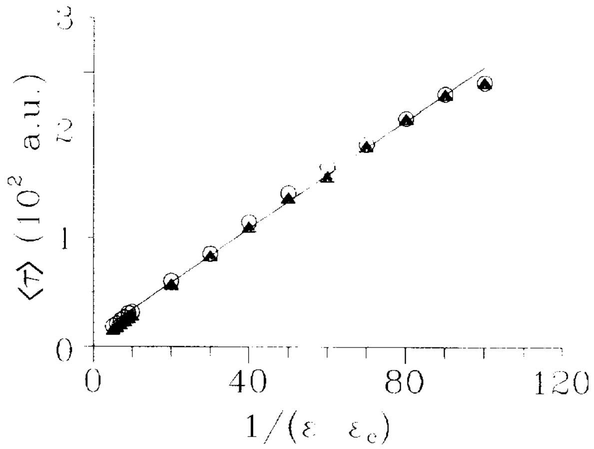

Fig. 3. Scaling of the mean laminar phase length $\langle\tau\rangle$, with ε (in the time series of absorption): circles, $\Delta\omega = 1.0$; triangles, $\Delta\omega = 0.025$; solid line, the least squares fit for $\Delta\omega = 1.0$. Here and in all other figures, the onset threshold for a burst was 0.39; the maximum values of absorption were of the order of 1.0×10^3.

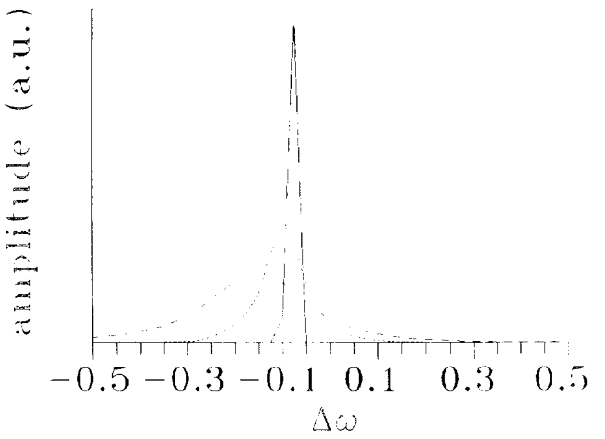

Fig. 4. Mean values of the SW amplitudes over a long time interval as a function of their detuning from half the pumping frequency $\Delta\omega$: solid line, $u_{\mathrm{therm}} = 0$; dotted line, $u_{\mathrm{therm}} = 1.0 \times 10^{-7}$; dashed line, $u_{\mathrm{therm}} = 1.0 \times 10^{-3}$.

simplified model with only one SW pair for the case of zero thermal noise. This is so because the distribution and mean length of laminar phases depend on the equations of motion linearized around the invariant manifold $u_{\kappa_i} = 0$ [3]. If only one SW pair is excited the linearization of the full system (7) and of the model with one SW pair yield identical equations. Hence, the description of OOI in randomly driven parallel pumping in the previous work [10] yields results quantitatively correct in the case of zero thermal noise.

Let us now consider the case $u_{\mathrm{therm}} \neq 0$. The OOI threshold is decreased to $\varepsilon = 1.0$ and the system behaviour becomes strongly dependent on the parameter $\Delta\omega$. For instance if $\Delta\omega = 1.0$ still only one SW pair (with $j = 0$) is excited during the bursts, and other SW remain at the thermal level u_{therm}. But if $\Delta\omega \ll 1.0$ a whole packet of SW with frequencies close to $\omega/2$ is excited (see Fig. 4). A maximum of the SW packet in the ω-space is shifted towards greater detunings from $\omega/2$ in comparison with the case without thermal noise, and the packet as a whole widens with the rise of u_{therm}, i.e. more and more modes exhibit significant bursts with the rise of thermal noise. One can say that, in both cases of small and big $\Delta\omega$, the packet of excited SW should have a finite width, but if SW form a discrete spectrum (e.g. in small samples) the frequency difference between neighbouring modes may exceed this width and it is too big for more than one mode to be excited. Thus it turns out that addition of thermal noise results not only in the decrease of the OOI threshold, but also in the widening of the phase space part visited by the system trajectory during the time evolution.

For both $\Delta\omega \sim 1$ and $\Delta\omega \ll 1$, the scaling law $\langle\tau\rangle \propto |\varepsilon - \varepsilon_c|^{-1}$ is invalid, but (keeping in both cases the same onset threshold for a burst and the same number $2N + 1$ of SW) the mean laminar phase length in the time series of absorption, e.g. for $\Delta\omega = 0.025$, is substantially shorter than for $\Delta\omega = 1.0$ (see Fig. 5) and the relative difference increases with increasing thermal noise. The power law $P(\tau) \propto \tau^{-3/2}$ for the probability distribution of the laminar phase lengths is replaced by an exponential fall for large τ (see Fig. 6). The characteristic duration of the laminar phase τ^* which separates the power and exponential scaling areas (signed by an arrow in Fig. 6) is again shorter in the system with $\Delta\omega = 0.025$ than with $\Delta\omega = 1.0$. In both cases, τ^* scales with the square of the logarithm of u_{therm} which is a little generalized form of the scaling behaviour characteristic of OOI: $\tau^* = A \ln^2 u_{\mathrm{therm}} + B \ln u_{\mathrm{therm}} + C$, where A, B, and C are fitting parameters, different for various $\Delta\omega$ (see Fig. 7). On the basis of these results, it may be concluded that systems which have identical

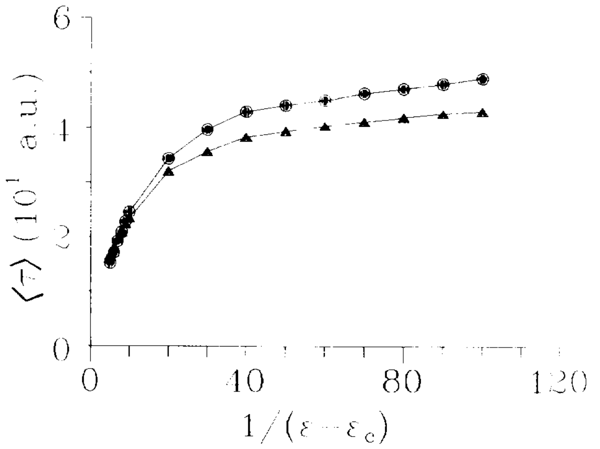

Fig. 5. As in Fig. 3, but for $u_{therm} = 1.0 \times 10^{-7}$.

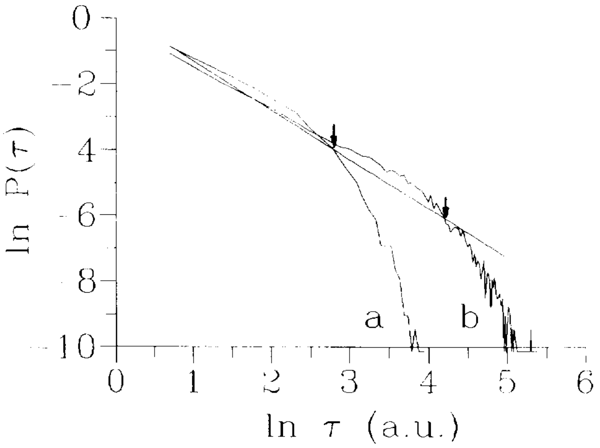

Fig. 6. Probability distribution of the laminar phase lengths (in the time series of absorption) for $u_{therm} = 1.0 \times 10^{-3}$ and $\varepsilon = 2.033$: (a) $\Delta\omega = 0.025$; (b) $\Delta\omega = 1$; straight line, a perfect $\tau^{-3/2}$ scaling.

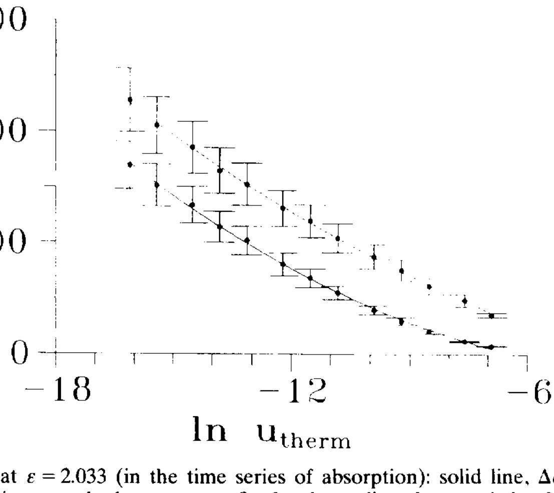

Fig. 7. Scaling of τ^* vs u_{therm} at $\varepsilon = 2.033$ (in the time series of absorption): solid line, $\Delta\omega = 0.025$; dotted line, $\Delta\omega = 1.0$; the lines are the least squares fits for the scaling characteristic of OOI.

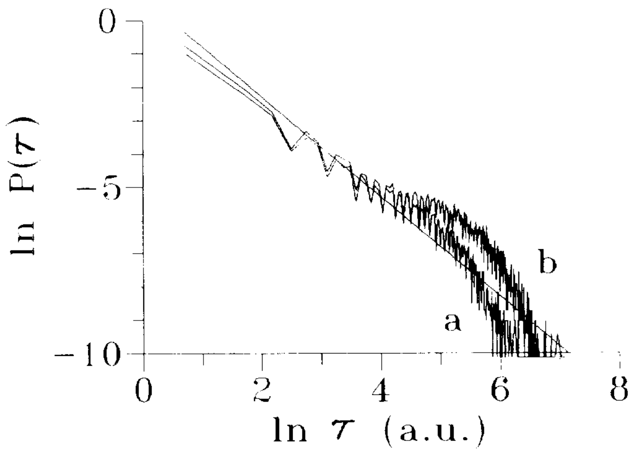

Fig. 8. As in Fig. 6, but for the time series of individual SW: (a) $j = -2$, (b) $j = 6$; $u_{therm} = 1.0 \times 10^{-5}$, $\Delta\omega = 0.025$.

characteristics of OOI without thermal noise may behave in a different manner when thermal noise is added. Thus the description of OOI in the previous paper [10] in parallel pumping with thermal noise taken into account yields results which are only qualitatively correct.

It is also important to note that the laminar phase lengths in the time series of amplitudes of individual SW also obey the distributions characteristic of OOI. For example, the distribution of laminar phase lengths for two different SW with various detunings $\Delta\omega_{\kappa_i}$ is shown in Fig. 8. For these SW the onset thresholds for a burst were assumed equal. It can be seen that the laminar phases in the time series of amplitudes of SW far from the maximum of the SW packet (Fig. 4) are longer on average. This is connected with the fact that such SW are more weakly excited in comparison with the ones close to the maximum of the packet. The characteristic duration of the laminar phase τ^* for individual SW also obeys the scaling law for OOI quite well (see Fig. 9). It turns out that the SW do not burst independently: during the burst all SW become excited, though the width of the SW packet varies in time. The shape of the SW packet envelope during the burst is very similar to the one depicted previously in Fig. 4.

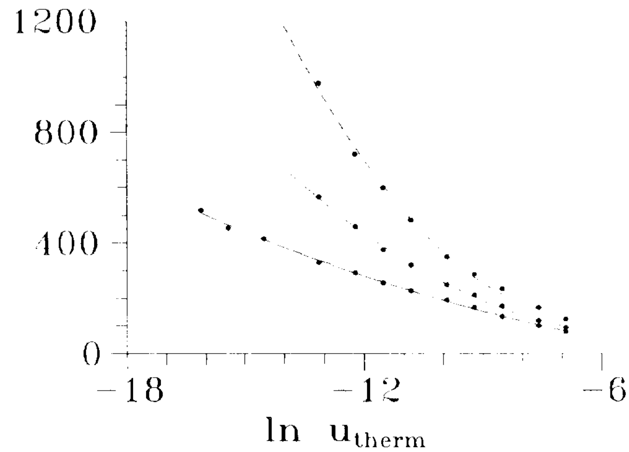

Fig. 9. As in Fig. 7, but for the time series of individual SW: solid line, $j = -2$; dotted line, $j = -8$; dashed line, $j = 6$; $\Delta\omega = 0.025$. The error estimates were omitted.

4. DISCUSSION AND CONCLUSIONS

In this paper, the results of our previous work on OOI in nonlinear ferromagnetic resonance with randomly or chaotically varying in time rf field amplitude [10] have been generalized by including nonlinear interactions among parametrically excited SW pairs. Another method of obtaining OOI in such a system, namely adding a random or chaotic component to the magnetic dc field, was also discussed. It was shown that OOI can appear if the characteristic period of the random or chaotic oscillations of the respective bifurcation parameter is comparable to or greater than the SW relaxation time. The mechanism for the occurrence of bursts is analogous to the case of the logistic map with random control parameter (1). The qualitative predictions of the simplified model with one SW pair were confirmed, but quantitative differences were found, in particular in the case when the possibility of thermal excitation of SW which form a quasi-continuous spectrum is taken into account. Scaling laws of OOI in the system studied are of the same kind as for a simple one-dimensional map (1).

Some remarks concerning the numerical results of Section 3.2 are necessary. The behaviour of the system in equation (7) in the case $u_{\rm therm} = 0$ is analogous to the one found when the stationary state of equation (4) is investigated [11]: just above the parallel pumping instability threshold the SW pair with $\Delta\omega_\kappa = 0$ is excited and with the rise of the rf field amplitude SW with non-zero detuning become excited. This is a result of the four-mode interactions among SW which shift the effective detuning of the modes, $\Delta\omega_{\kappa_i} \to \Delta\tilde{\omega}_{\kappa_i} = \Delta\omega_{\kappa_i} + 2T\sum_j |u_{\kappa_j}|^2$ (see equation (4)) so that SW with $\Delta\tilde{\omega}_{\kappa_i} = 0$ are always excited. The difference with the case analysed in Ref. [29] should also be noted. In that paper, the rf field amplitude varied randomly and fast in comparison with the SW relaxation time, and hence the stationary state appeared above the parallel pumping instability threshold and a whole packet of SW with frequencies from a certain interval of non-zero width was excited. Here, for the rf field amplitude varying slowly in comparison with $1/\eta$, the excited SW have rather a singular distribution in the ω-space.

The excitation of a whole SW packet for small $\Delta\omega$ in the presence of thermal noise may have been expected on the basis of the investigation of the transient behaviour of the SW amplitudes after switching on the rf field [32, 33]. It is known that in this case, before the stationary state is established with only one SW pair excited above the thermal level, a transient is observed. First, a packet of SW is excited with the envelope shape very similar to the one in Fig. 4. In the course of time, this packet becomes narrower and at least the distribution of SW amplitudes becomes singular. OOI in the model discussed in the present paper may be observed simply if the period Δt of the random oscillations of, for example, the rf field amplitude is long enough to ensure the possibility of exciting or decaying of the SW packets, but short enough that the stationary state could not be achieved between t_n and $t_n + \Delta t$.

From the mathematical point of view, equations (7) and (8) are systems of non-equivalent oscillators coupled via nonlinear interactions. Without thermal noise, the OOI threshold is different for every oscillator (SW pair) with different $\Delta\omega_{\kappa_i}$ (see Section 3.1). One can argue that the excitation of the whole SW packet in the case $u_{\rm therm} \neq 0$ is another example of the decrease of the OOI threshold, this time for every individual SW pair, due to the inclusion of thermal noise. This conjecture is supported by the observation that the intermittency characteristics of every individual SW pair amplitude obey the scaling relations typical of OOI (see Section 3.2). Thus it seems that the observed phenomenon of widening of the phase space of the system with OOI in the presence of thermal noise is not a particular feature of the system considered, but should appear in various large ensembles of interacting, non-equivalent subsystems. In such systems, addition of thermal noise can lead to two qualitative changes in the OOI picture: the decrease of the overall OOI threshold and the widening of the part of the phase space visited by the system trajectory. So far, OOI in high-dimensional systems was investigated only in coupled map lattices [34–38] which are ensembles of identical subsystems (maps), so any similar phenomenon cannot be observed in them. The results of the present paper suggest that new phenomena connected with OOI may be also found, for example, in the ensembles of interacting non-equivalent oscillators.

From the observer's point of view, the excitation of a whole SW packet manifests itself as if greater thermal noise was present in the system (for a given onset threshold for a burst, the mean laminar phase length is shorter in equation (7) with $\Delta\omega \ll 1$ than with $\Delta\omega \sim 1$, etc.). This may cause difficulties in measuring the OOI characteristics in real high-dimensional systems, where thermal noise is always present.

It would be interesting to check if the behaviour of the system (7) and (8) depends on the particular form of the time dependence of $\varepsilon(t)$ and $\delta\omega(t)$. In Ref. [10], it was shown that, if the rf field amplitude is modulated by one of the variables of the Lorenz system, the OOI characteristics change only quantitatively, not qualitatively. This is in agreement with the conjecture that these characteristics are insensitive to the particular form of the time dependence of the bifurcation parameter [3]. Preliminary numerical calculations in which the random square wave (9) was replaced by one of the variables of the Lorenz system indicate that this is true also for equations (7) and (8). In particular, the phenomenon of widening of the phase space in the presence of thermal noise was also observed in this case.

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