Wykład II

- Funkcja gamma Eulera,
- Funkcja beta Eulera,
- Wzór Stirlinga.

Funkcja gamma Eulera

$$\boxed{\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0}$$

- Uogólnienie silni

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x} dt = -e^{-t} t^{x}\Big|_0^\infty + x\int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x)$$

$$\Rightarrow \Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)\ldots 2\cdot 1\cdot \Gamma(1)$$

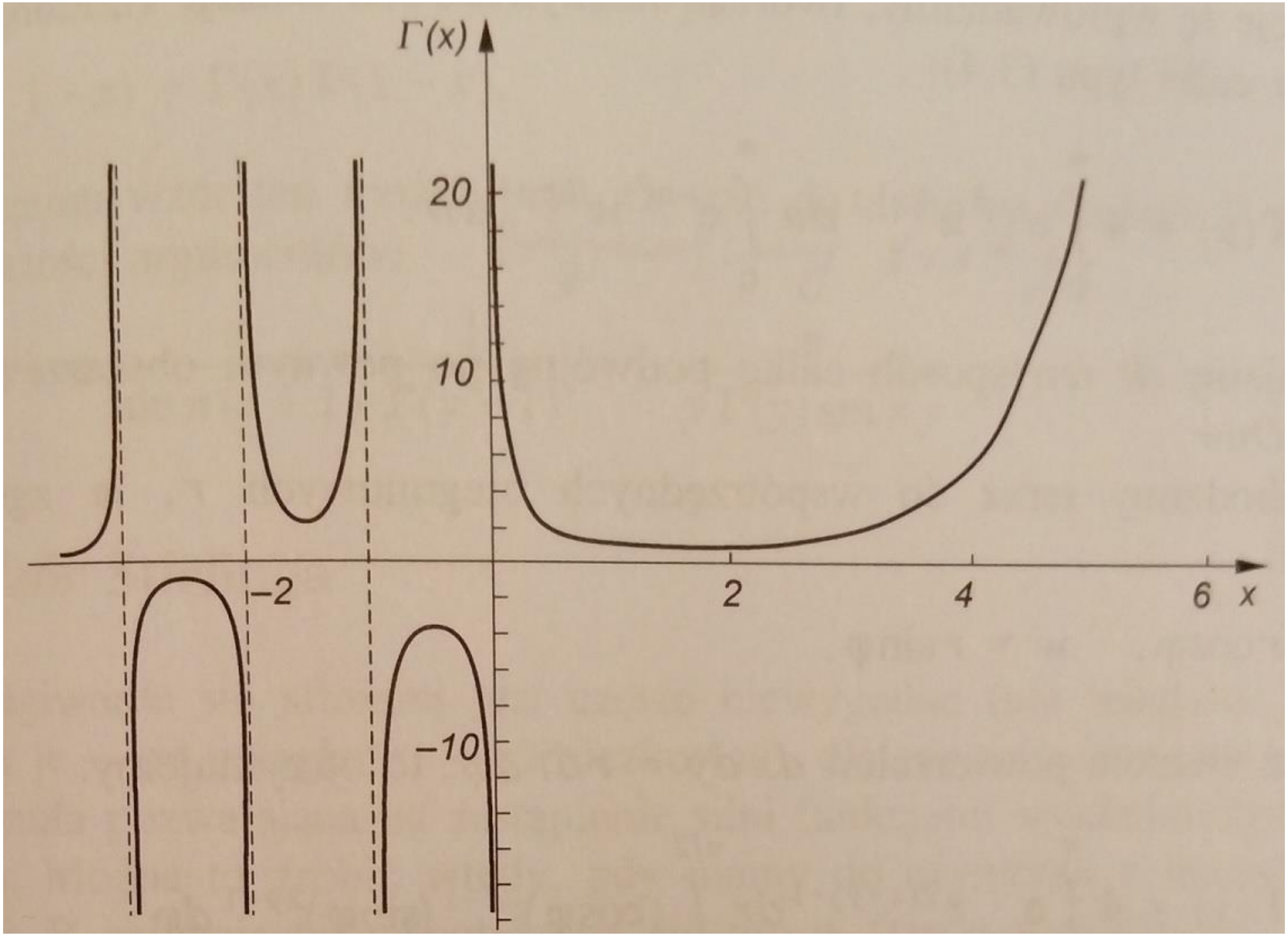
$$\Gamma(1) = 1 \Rightarrow \Gamma(n+1) = n!$$

- Rozszerzenie na obszar $x < 0$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad -1 < x < 0$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \Rightarrow \Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad -2 < x < -1$$

$$\vdots$$



Funkcja gamma Eulera

- Wartości funkcji gamma dla argumentów połówkowych

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \left| t = u^2 \right| = 2\int_0^\infty e^{-u^2} u^{2x-1} du$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du = \sqrt{\pi}$$

$$\Rightarrow \Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\dots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}$$

Funkcja beta Eulera

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty e^{-u^2} u^{2x-1} du \int_0^\infty e^{-w^2} w^{2y-1} dw$$

$$u = r\cos\varphi, \quad w = r\sin\varphi, \quad dudw \to rdrd\varphi$$

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty dr e^{-r^2} r^{2(x+y)-1} \int_0^{\pi/2} d\varphi (\cos\varphi)^{2x-1} (\sin\varphi)^{2y-1} =$$

$$\Gamma(x+y)\cdot 2\int_0^{\pi/2} d\varphi (\cos\varphi)^{2x-1} (\sin\varphi)^{2y-1}$$

$$\boxed{\beta(x,y) = 2\int_0^{\pi/2} d\varphi (\cos\varphi)^{2x-1} (\sin\varphi)^{2y-1} = \left| s = \cos^2\varphi \right| = \int_0^1 s^{x-1} (1-s)^{y-1} ds}$$

$$\boxed{\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

Można wykazać, że

$$\beta(x,1-x)=\frac{\pi}{\sin(\pi x)}$$

$$\left.\begin{array}{c}\beta(x,1-x)=\Gamma(x)\Gamma(1-x)\\ x=1+y,\quad y>0\end{array}\right\}\Rightarrow\Gamma(-y)=\frac{\pi}{\sin[\pi(y+1)]}\frac{1}{\Gamma(y+1)}=-\frac{\pi}{y\Gamma(y)\sin(\pi y)}$$

Wzór Stirlinga

Przybliżenie silni dla dużych argumentów

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \Rightarrow \Gamma(x+1) = \int_0^\infty e^{-t} t^x dt.$$

Dokonujemy podstawienia $t = x + u\sqrt{x} \Rightarrow dt = \sqrt{x}du$

$$\begin{aligned}
\Gamma(x+1) &= \int_{-\sqrt{x}}^\infty e^{-(x+u\sqrt{x})}(x+u\sqrt{x})^x \sqrt{x}du = \int_{-\sqrt{x}}^\infty e^{-(x+u\sqrt{x})} e^{x\ln(x+u\sqrt{x})}\sqrt{x}du = \sqrt{x}e^{-x}\int_{-\sqrt{x}}^\infty e^{-u\sqrt{x}} e^{x\ln\left[x\left(1+u/\sqrt{x}\right)\right]}du \\
&= \sqrt{x}e^{-x}\int_{-\sqrt{x}}^\infty e^{-u\sqrt{x}} e^{\ln x^x} e^{x\ln\left(1+u/\sqrt{x}\right)}du = x^x\sqrt{x}e^{-x}\int_{-\sqrt{x}}^\infty e^{-u\sqrt{x}} e^{x\ln\left(1+u/\sqrt{x}\right)}du.
\end{aligned}$$

Korzystamy z rozwinięcia na szereg Taylora

$$\ln(1+y) \overset{y\to 0}{\approx} y - \frac{y^2}{2} + \ldots \Rightarrow \ln\left(1+\frac{u}{\sqrt{x}}\right) \overset{x\to\infty}{\approx} \frac{u}{\sqrt{x}} - \frac{1}{2}\frac{u^2}{x} + \ldots$$

$$\Gamma(x+1) = x^x\sqrt{x}e^{-x}\int_{-\sqrt{x}}^\infty e^{-u\sqrt{x}}e^{u\sqrt{x}}e^{-u^2/2}du x^x\sqrt{x}e^{-x}\int_{-\sqrt{x}}^\infty e^{-u^2/2}du \overset{x\to\infty}{\approx} x^x\sqrt{x}e^{-x}\int_{-\infty}^\infty e^{-u^2/2}du = \sqrt{2\pi}x^{x+1/2}e^{-x}$$

$$x = n \Rightarrow \Gamma(n+1) \approx n! \approx \sqrt{2\pi}n^{n+1/2}e^{-n}$$

$$\ln n! \approx \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln n - n \overset{n\gg 1}{\approx} n\ln n - n.$$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