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Ferromagnetic phase transition in Barabási–Albert networks

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Abstract

Ising spins put onto a Barabási–Albert scale-free network show an effective phase transition from ferromagnetism to paramagnetism upon heating, with an effective critical temperature increasing as the logarithm of the system size. Starting with all spins up and upon equilibration pinning the few most-connected spins down nucleates the phase with most of the spins down.

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Networks with more complicated connectivities than periodic lattices have been investigated in detail recently. For example, the Barabási–Albert network [1] is grown such that the probability of a new site to be connected to one of the already existing sites is proportional to the number of the previous connections to this already existing site: the rich get richer. Similar networks have been investigated [2] to check if destruction of a few computers will split the percolating cluster of the internet (i.e., the set of all computers in the world connected directly or indirectly with each other). Networks exist also in social models where vertices are individuals or organizations, and links correspond to social relationships between them [3]. However, as far as we know, all studies of the scale-free Barabási–Albert networks considered only the *topology* and no *interactions* between linked vertices.

Here we investigate the ordering phenomenon in this Barabási–Albert network, if Ising spins are put onto the sites (vertices) of the network. We assume ferromagnetic

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coupling between linked spins and positive temperature T of the system. Such a magnet would show paramagnetism if the whole network is still connected but only weakly such that thermal fluctuations destroy the spontaneous magnetization. (Bose–Einstein condensation on similar networks was already studied before [4], as was the Ising model on small-world networks [5], and other Ising models are in preparation [6].) In the social example one could identify $\exp(-\text{const}/T)$ with the probability that members of the same social group are not convinced to share the same opinion. (Spin variables $S_i = \pm 1$ then correspond to two possible opinions of the group members on the same subject.)

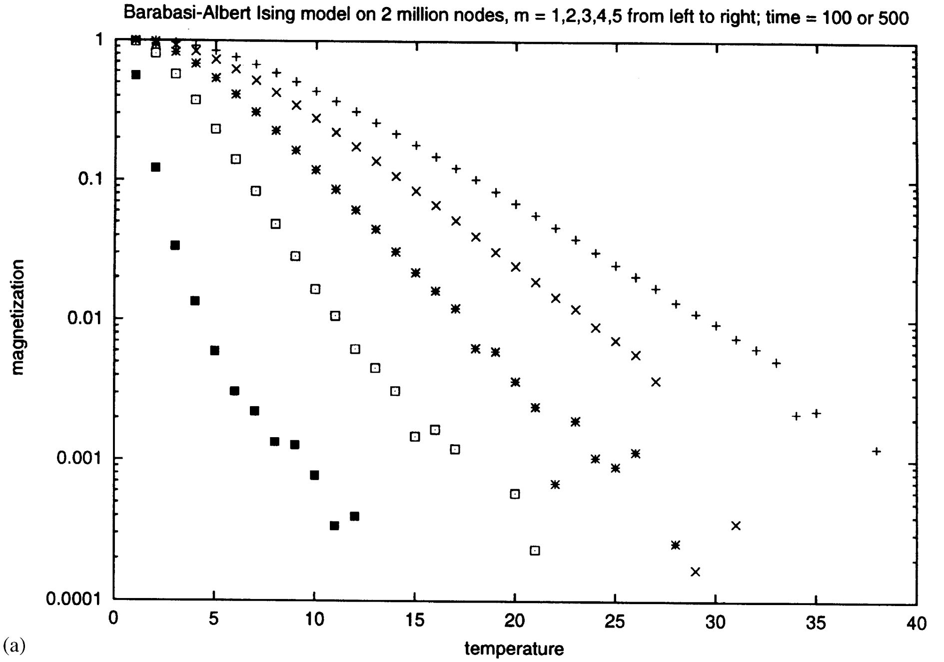
Thus we create a Barabási–Albert network of N sites added to an initial core of m fully connected sites. Each of the N new nodes is connected to m randomly selected previous nodes. (We allow more than one connection to the same site for the later added nodes; mostly we take $m = 5$.) Then we freeze this network structure, put an Ising spin $S_i = \pm 1$ onto every site, with all spins up initially. Then with the standard heat bath Monte Carlo algorithm spins search for thermal equilibrium at temperature T (all temperatures are given in units of coupling constant over Boltzmann constant). Fig. 1a shows the resulting magnetization (averaged over the last half of 500 sweeps through the network) versus temperature; it seems to decrease exponentially with increasing temperature, until due to finite-size effects it oscillates about zero. This effective Curie temperature $T_C(N)$ is fitted in Fig. 1b onto $2.6\ln(N) - 3$ for $5 \leqslant N \leqslant 5{,}000{,}000$. (A percolation threshold vanishing as $1/\ln(N)$ was given in Ref. [7, Eq. 144].)

Fig. 2 shows that in the ferromagnetic region the spins with few neighbours flip up and down while those with many neighbours point up most of the time.

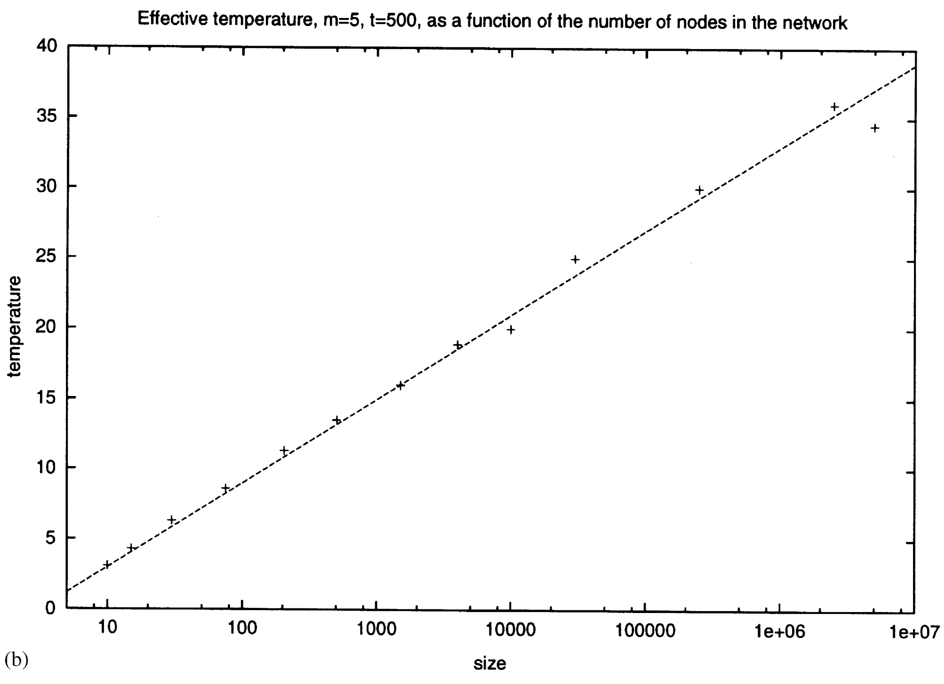
Analogous to the appearance and spreading of new opinions in society [8], we now try to flip the spontaneous magnetization for $T < T_C$ by forcing the most-connected spin permanently down; then we pin in the same way the second-most-connected spin, and so on, all in time intervals of 50 iterations. We see that, in general, removal of the few leading spins having hundreds of neighbours is sufficient to flip the magnetization. (Since we do not apply any magnetic field and have no fixed boundary conditions, we expect the flipping of the magnetization to be a nucleation event, which would happen even if we flip only one randomly selected spin provided we wait long enough.) Fig. 3 shows the magnetization versus time averaged over 100 samples. Higher temperatures require fewer leading spins to be pinned.

Now we present a simple mean field theory for some of these effects. Let us consider the BA network with the characteristic constant m and the corresponding Ising model with the ferromagnetic coupling constant $J = 1$. The probability that a node has a degree k is given by $P(k) \simeq Ak^{-3}$ when $k \geqslant m$ and $P(k) = 0$ when $k < m$. For large networks ($N \to \infty$) the normalization constant equals $A \simeq 2m^2$. In the mean field approximation (MFA) we can simplify interactions among each group of spins with a fixed value of k by the effective field kM, where M is the mean magnetization (per one spin). It follows that

$$M = \int P(k)\tanh(\beta Mk)\,\mathrm{d}k$$



(a)



(b)

Fig. 1. (a) M versus temperature for 2 million nodes and various m, for $250 \leqslant t \leqslant 500$ (shorter times t were used far below T_C). (For $m=1$ even 60 million nodes were simulated.) (b) Effective T_C versus $m+N$ for $m=5$ and various N, averaged over up to 1000 samples.

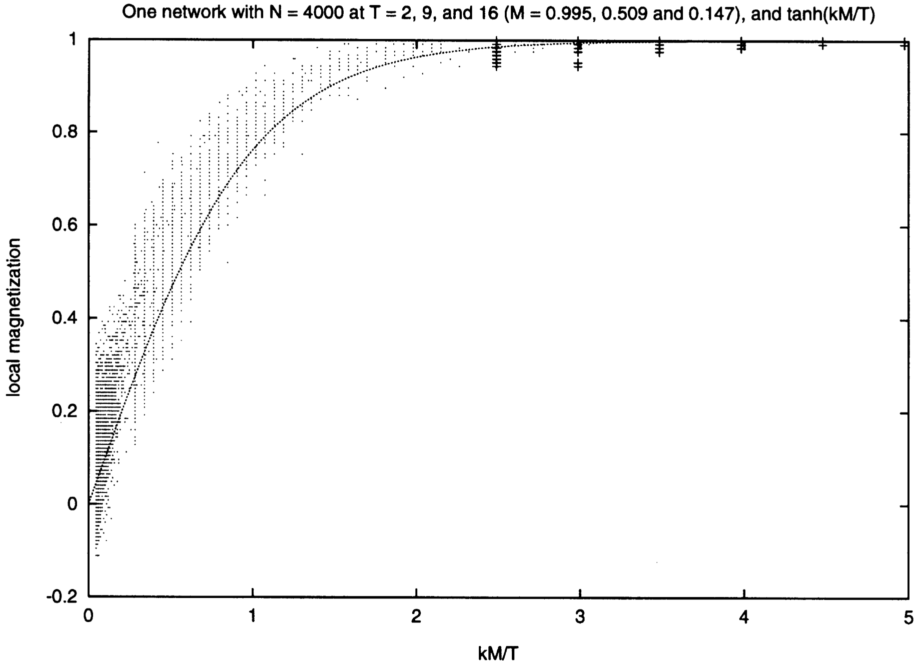


Fig. 2. Correlation between the number of neighbours and the local magnetization $\langle S_i \rangle$ for one network of $N = 4000$ at $T = 2$, 9 and 16. Average over $250 < t \leqslant 500$ iterations. The curve is the mean field prediction $\tanh(\beta k M)$. $N = 2,000,000$ gives similar effects.

($\beta = 1/T$). This is a transcendental algebraic equation for $M(T)$. It is easy to find analytically the critical temperature T_C that corresponds to the case when left-hand side (lhs) and right-hand side (rhs) are tangent at $M = 0$. Differentiating both sides over M and putting $M = 0$ we get $1 = A \int_m^\infty \beta k^{-2}\, \mathrm{d}k$; thus $T_C = 2m$. The result can be written as $T_C = \langle k \rangle \simeq 2m =$ the mean degree of the node. This is the typical MFA result. Fig. 1 shows it to be correct in order of magnitude and in its increase with increasing m. But the MFA does not describe the logarithmic size effect on $T_C(N)$, due perhaps to exponentially rare and small regions of densely connected spins [9]. A different mean field theory by G. Bianconi (private communication), and S.N. Dorogovtsev et al. [10], gave the $\log(N)$ dependence of Fig. 1b.

Knowing the mean magnetization $M(T)$ one can calculate the correlation $\mu(k)$ of local magnetization to the spin degree k. In fact, in MFA we can write $\mu(k) = \tanh[\beta k M(\beta)]$. It follows that for all temperatures $T < T_C$ this dependence is a universal function of $u = \beta k M(\beta)$. This prediction is consistent with Fig. 2.

The effect of pinning of the most important spins (Fig. 3) can be also described analytically and it follows that there occurs a *discontinuous* phase transition from the "spin up" phase to the "spin down" phase by a well-defined critical number of pinned spins.

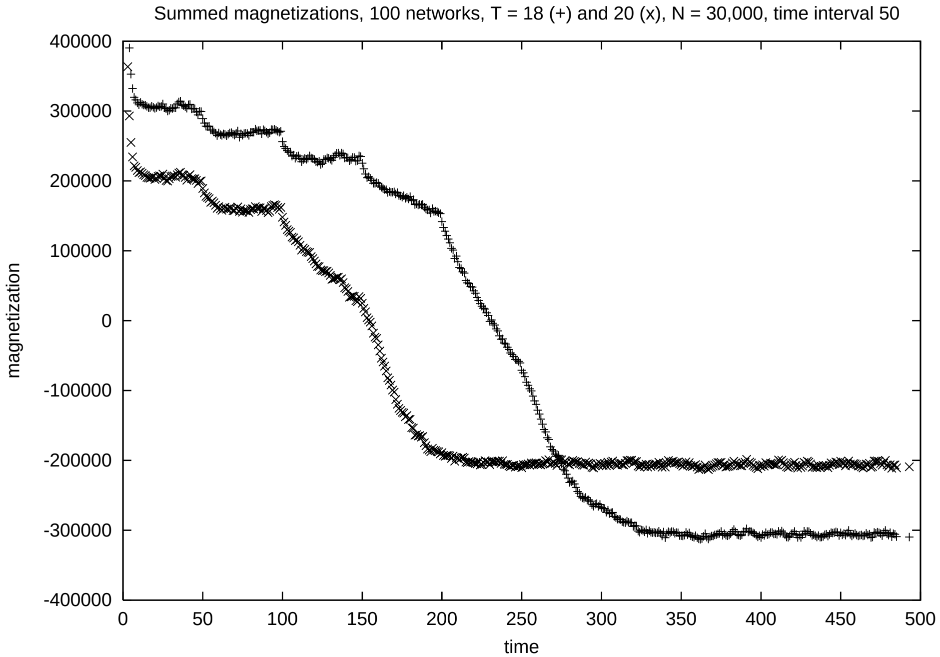


Fig. 3. Total magnetization versus time, summed over 100 networks of $N = 30,000$ when after every 50 iterations the most-connected free spin is forced down permanently. For lower temperatures, the sign change of the magnetization happens later. $N = 2,000,000$ gives similar effects.

Pinning one spin of degree k to the state $S = -1$ is equivalent in MFA to lowering the mean magnetization by $\mu(k)/N$ and introducing to the system the *external* magnetic field of the magnitude $J = 1$ that is oriented "down". This field is felt *only* by k other spins thus its mean value for the whole system equals $b(k) = -k/N$. If we pin j of the most connected spins, then it means that we pin all spins of the degree $k > \kappa$ where

$$j = NA \int_{\kappa}^{\infty} k^{-3}\,\mathrm{d}k \ .$$

It follows that we decrease the mean magnetization (per one spin) by

$$\delta M(j) = A \int_{\kappa}^{\infty} (\mu(k) + 1)k^{-3}\,\mathrm{d}k \simeq \frac{2j}{N} \ ,$$

where we have assumed that all pinned spins were completely ordered before pinning, $\mu(k) = 1$. The effective *internal* field $B(j)$ acting in the system from the pinned spins can be calculated as

$$B(j) = -A \int_{\kappa}^{\infty} \mu(k)k^{-2}\,\mathrm{d}k \simeq -2m\sqrt{j/N} \ .$$

It is important to stress that $\delta M \propto 1/N$ but $B(j) \propto \sqrt{1/N}$ and this internal field causes much larger decrease of magnetization ΔM than the direct effect of pinning. For small j we have $\Delta M(j) = \chi B(j)$, where χ is the system initial susceptibility. In

MFA, it can be calculated as the mean value of the derivative of local magnetization $\mu(k)$ over the field $B(j)$ and it equals to

$$\chi = \frac{2m^2}{T}\int_m^\infty \frac{\mathrm{d}k}{k^3\cosh^2(\beta Mk)} .$$

The last integral cannot be calculated analytically. Pinning several spins means influencing the system by a large internal field $B(j)$ that can even cause a flip of the total magnetization. The value for the minimal flipping field B^* can be obtained from the equation

$$M^* = A\int_m^\infty k^{-3}\tanh[\beta(M^*k + B^*)]\,\mathrm{d}k$$

and the corresponding tangency relation at the critical point M^*:

$$1 = \beta A\int_m^\infty k^{-2}\cosh^{-2}[\beta(M^*k + B^*)]\,\mathrm{d}k .$$

The above equations imply conditions for B^* and M^*. After some algebra one can find the following relation between B^* and M^*:

$$B^* = -mM^* + T\,\mathrm{arctanh}(M^*/2) .$$

Combining the last result with the relation for $B(j)$, we get the following equation for the minimal number j^* of pinned spins needed to invert the system magnetization:

$$\sqrt{\frac{j^*}{N}} = \frac{M^*}{2} - \frac{T}{2m}\,\mathrm{arctanh}\,\frac{M^*}{2} \simeq (M^*/2)(1 - T/T_c) ,$$

where the last approximation is valid only for the small $M^* = 0.03\ldots0.1$ of Fig. 3 and we used the above $T_C = 2m$. This prediction $j^*/N \sim 10^{-4}\ldots10^{-3}$ agrees reasonably with our simulations.

In summary, we combined the Ising model with the Barabási–Albert network and found that depending on the convincing power one either has a majority opinion or two equally widespread opposing opinions, i.e., a Curie point.

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