# Self-organized criticality and coevolution of network structure and dynamics

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We investigate, by numerical simulations, how the avalanche dynamics of the Bak-Tang-Wiesenfeld sandpile model can induce emergence of scale-free networks and how this emerging structure affects dynamics of the system.

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Since Bak, Tang, and Wisenfeld's (BTW) discovery of self-organized criticality (SOC) [1], the phenomenon has received enormous attention from researchers. During these almost 20 years dozens of original sandpile model's variants [1,2] have been studied [3–6] and a number of SOC examples in the real world have been discovered. One of the most remarkable features that characterizes self-organized criticality is the events' power law distribution. This feature, combined with the abundance of real-world networks with scale-free (SF) degree distribution [7–10], may give rise to the suspicion that a relationship exists between the two issues. Although a few papers have mentioned this idea already (see, for example, Ref. [11]), Hughes *et al.* [12,13] have established a real link between the two phenomena for the first time in a rather complex model for cascades of magnetic field lines' reconnection in the solar atmosphere. Here we would like to present a simple mechanism of such a phenomenon.

At the beginning let us recap the rules of the sandpile model, which is a simple, intuitive example of self-organized criticality. It is a cellular automaton whose configuration is determined by the integer variable $c_i$ (the "sand column's" height) at every node $i$ in the network. Depending on network's structure, minor differences in definition can occur. Here we follow the BTW model's definition for random networks with a given degree distribution $p(k)$ [15]. The dynamics are defined by the following simple rules: A grain of sand is added at a randomly selected node $i$: $c_i \to c_i+1$. A sand column with a height $c_i \geqslant k_i$, where $k_i$ is equal to degree of node $i$, becomes unstable and collapses by distributing one grain of sand to each of its $k_i$ neighbors. This may cause some of them to become unstable and to collapse in the next time step. This in turn can lead to an avalanche of subsequent instabilities. During the evolution a small fraction $f$ of grains is lost, which prevents the system from becoming overloaded. When the avalanche dies another grain of sand is added.

In the sandpile model the avalanches' sizes' distributions (measured as the total number of topplings in the avalanche), the avalanches' areas (the number of distinct nodes participating in a given avalanche) and the avalanches' durations, as well as many other statistics, follow power law distributions [1,2].

Studies of sandpile dynamics carried out to date show that the measured distributions' characteristic exponents depend on the network's dimension and topology.

The one-dimensional (1D) BTW model can be exactly solved and yields an avalanche area distribution $P_a(a,L) = a^{-1}F_a(a/L)$ where $a$ is an avalanche's area (total number of lattice sites toppling at least once during the avalanche), $L$ is the system's size and $F_a(a/L)$ is a finite size scaling function [18]. Despite plenty of numerical studies of the two-dimensional (2D) BTW model and its analytical tractability [19], the scaling behavior in this case is probably not yet completely understood [20–24] partly due to the fact that avalanches involve a large fraction of multiple topplings. It follows that an avalanche's area $a$ differs from an avalanche's size $s$. A result is a finite size scaling for $P_a(a,L)$ with the characteristic exponent $\tau_a=6/5$, while the distribution $P_s(s,L)$ exhibits multifractal scaling behavior [22]. The three-dimensional (3D) cubic BTW model does not suffer from the problems caused by multiple topplings and is one of the simplest undirected models of self-organized criticality with nontrivial (non-mean-field) critical behavior. In this case $\tau=4/3$ [1]. For $D \geqslant 4$, a mean-field description of avalanche propagation is adequate, and the corresponding exponents are the same as those of the clusters' sizes in critical percolation theory [25]. The exponent $\tau$'s mean-field value obtained from exact solutions on the Bethe lattice [26] and on the full graph [27] is $3/2$. Similarly to Erdős-Rényi (ER) random networks, $\tau=1.5$ [14].

Recently, Goh *et al.* have studied sandpile dynamics on scale free networks $p(k) \sim k^{-\gamma}$ [15,16] and they have shown that the avalanche's area exponent $\tau$ is independent of the average network connectivity $\langle k \rangle$ and changes with the degree distribution's exponent $\gamma$. They have obtained

$$\tau = \frac{\gamma}{\gamma-1} \tag{1}$$

in the range $2<\gamma<3$ and $\tau=1.5$ for $\gamma>3$. The question we ask in the paper is the following: how does the avalanche's area exponent behave when the network's topology depends on sandpile dynamics; i.e., when mutual interactions occur between the network's structure and network's dynamics?

In Ref. [11], Bianconi and Marsili proposed a simple model in which the network reorganizes its structure as a consequence of avalanches of rewiring processes. The only parameter of the model that influences the rewiring, and consequently, the network's structure, is a type of probability in which a chosen node becomes unstable and has to be re-

 

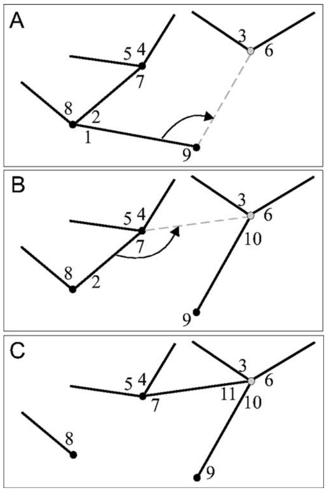


FIG. 1. Example of rewiring process. A gray colored node started an avalanche of size 2. The number close to each end of the link describes a moment of its last rewiring.

wired. Choosing this probability as a power law sets the system in a critical state and forces the network to take a power law degree distribution.

In the present paper, instead of forcing the network to stay in a critical state, we allow the system to naturally evolve toward the critical region. In our model:

(1) the considered networks' degree distribution changes due to the sandpile's avalanches' distribution on this network and

(2) the avalanches' size distribution changes because the network structure evolves.

These two mechanisms influence each other and lead to the equilibrium point at which the avalanches' distribution's and the degree distribution's shapes become very similar.

In order to complete our model's rules, apart from the sandpile model's rules recapitulated above, we define the rewiring process in the following way: each end of a link has been assigned a value specifying the last time when it was rewired. After an avalanche of area $A$, the number of $A$ "oldest" ends of links (from the whole network) are rewired to the node which triggered the avalanche. An example of a rewiring process is shown in Fig. 1.

A node can become unstable by losing links, i.e., $c_i \geqslant k_i$ because $k_i$ has been decreased. Such a node will participate in the next avalanche caused by the addition of a new grain. If a node loses all its links it has no possibility of creating an avalanche or of connecting to other nodes. Taking the above rules literally would mean to let this node redistribute 0 grains indefinitely. To avoid this problem we assume $c_i=1$ for this node's critical height. This means that if a grain is added to such a node, it will generate an avalanche of size 1, which will lead to $k_i \to k_i+1$.

In our studies all networks have: $\langle k \rangle=4$ and $f=10^{-4}$. We tested different sizes of networks and we checked that the number of nodes $N=10^5$ is sufficiently large to be sure that the exponents are not affected by the scaling function's presence. It is important to stress that because this model is more complex than the majority of standard SOC models it is difficult to determine an explicit form of the finite sized scaling function. Instead we compared the exponents $\tau$ of different sized networks.

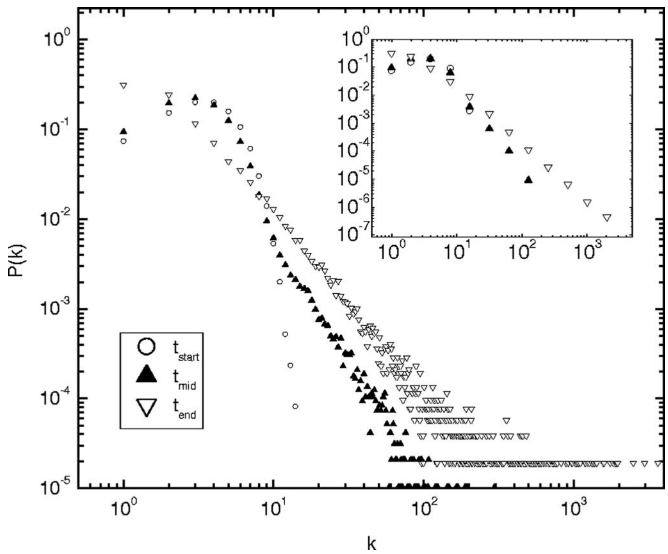


FIG. 2. Node degree distributions of rewired network in three time moments: $t_{start}=0$, $t_{mid}=2500$, and $t_{end}=10\,000$. Inset: the same data binned logarithmically.

We start our simulation with an Erdős-Rényi random network (which corresponds to $\gamma=\infty$ in static SF networks). The time unit used was simply one avalanche and we have carried out our simulations for $t=10\,000$ steps. Figure 2 presents three snapshots of the node degree distribution in three different moments: $t_{start}=0$, $t_{mid}=2500$, and $t_{end}=10000$. Distributions at time $t$ were calculated from the time period from $t$ to $t+500$. Comparing the snapshots shows that the network reorganizes itself from Poissonian, through a mixture of Poissonian and scale free and finally settles on a pure scale-free degree distribution with the well established characteristic exponent $\gamma \cong 2.1$. At the same time the avalanches' area distribution's characteristic exponent increases from $\tau=1.5$, which is the known result for ER random networks [14], to $\tau \cong 2.1$. Figure 3 shows that in the course of simulation both exponents converge to approximately the same equilibrium value.

Figure 4 presents the convergence process in a more detailed way. We define there a new parameter $\tilde{\gamma}(t)$ that in some sense may be understood as the characteristic exponent of fat-tailed degree distributions and may be compared to $\gamma$. $\tilde{\gamma}(t)$ is simply obtained from the second moment of the degree distribution, which is known from simulations. Given $\langle k^2 \rangle$, we numerically solve the equation below for $\tilde{\gamma}(t)$

$$\langle k^2 \rangle = \sum_{k=1}^{N} k^2 p_{an}(k), \tag{2}$$

where

$$p_{an}(k) = \langle k \rangle \frac{(\tilde{\gamma}-2)^{\tilde{\gamma}-1}}{(\tilde{\gamma}-1)^{\tilde{\gamma}-2}} \frac{\Gamma\left(k-\tilde{\gamma}+1, \langle k \rangle \frac{\tilde{\gamma}-2}{\tilde{\gamma}-1}\right)}{\Gamma(k+1)} \tag{3}$$

is the static model's known analytic solution [17]. The new parameter has been introduced because in the simulation's

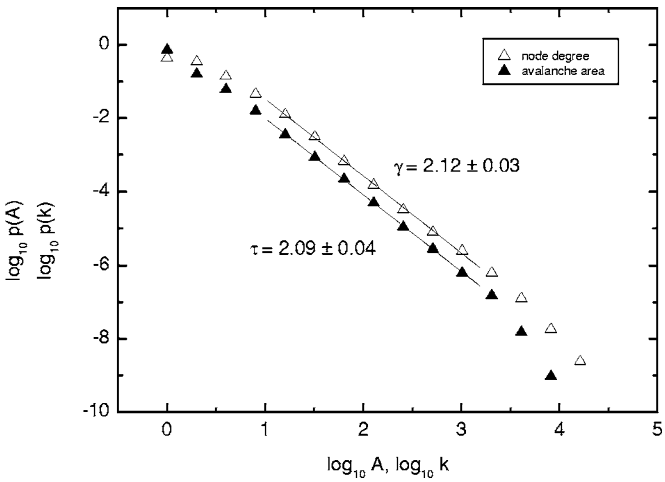


FIG. 3. Distributions of avalanche area and node degree in time $t_{end}$. Data are logarithmically binned. Lines are linearly fitted with the values indicated at the figure. A *K-S* test gives a $p$ value 0.997 at the level of significance $\alpha=0.05$. In other words, the difference between the samples is not significant.

intermediate times the degree distribution does not follow a pure power law. In the mentioned figure it is apparent that the value of the exponent $\tilde{\gamma}$ (open triangles) decreases from $\infty$ to 2.1. Simultaneously, the parameter $\tau$ characterizing the sandpile's dynamics (solid squares) increases from 1.5 and finally settles at the equilibrium value of 2.1. The values of $\tau$ agree fairly well with the values of $\tau_{theor}$ (solid line) calculated from relation (1) derived for static SF networks by Goh *et al.* [15]. The last observation suggests that during simulation the system moves close to the trajectory given by the formula (see Fig. 5)

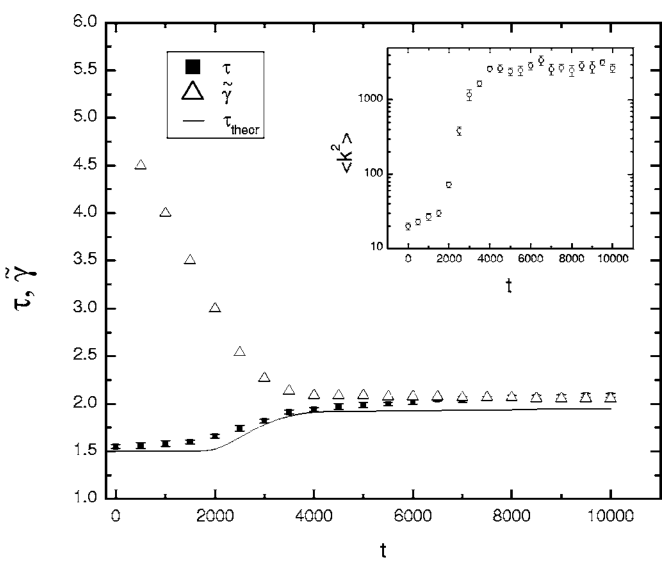


FIG. 4. Process of equilibration of exponents $\tilde{\gamma}$ and $\tau$. Solid line presents theoretical $\tau_{theor}$ obtained from $\tilde{\gamma}$ and Eq. (1). Inset: second moment of the degree distribution in time.

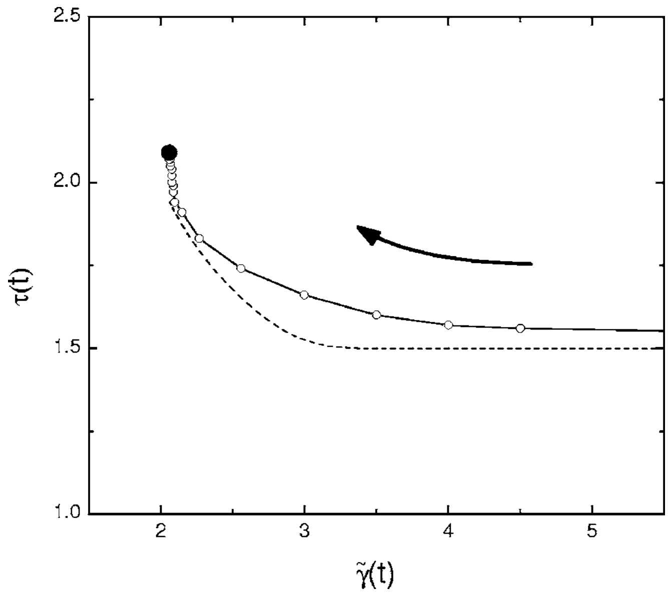


FIG. 5. Dependence of avalanche area exponent $\tau$ on parameter $\tilde{\gamma}$ of generated scale-free network. The arrow shows direction of a movement in space of parameters during the process of equilibration. Measurements are done in equal time steps $\Delta t=500$ and marked as open circles. The black dot depicts the fixed point of the process. Dashed line presents theoretical $\tau_{theor}$ obtained from $\tilde{\gamma}$ and Eq. (1).

$$\tau(t) = \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t) - 1}, \tag{4}$$

and an approximate equilibrium point may be calculated from the above equation when taking into account the observed numerical relation $\tilde{\gamma} \cong \tau$.

According to a simple theory of complex networks' coevolution, such systems' final critical state should be characterized by equation $\tilde{\gamma}=\tau=2$ and the last quality can be robust against different threshold assignment strategies in sandpile dynamics. In order to support the last statement let us mention two papers [12,16] in which we have found probable symptoms of such a universality. Paper [16] studies a class of sandpile models. In this class a node's threshold height is set as $k^{1-\eta}$, where $0 \leq \eta < 1$ is a parameter of the class. The avalanches' size exponent is received as $\tau=(\gamma-2\eta)/(\gamma-1-\eta)$ for $2 \leq \gamma < 3$. Assuming that our model's self-organization and rewiring process make $\tau=\gamma$, then $\tau=\gamma=2$ is independently obtained on $\eta$. The second example is a model of rapid rearrangements in the Sun's corona's magnetic field flows' network [12]. The authors show that the considered network's link reconnections' avalanches and scale free structure co-generate each other. They also show that for the equilibrium, the degree distribution exponent is $\gamma=2$. Unfortunately, they do not present a reconnection distribution exponent which corresponds to $\tau$.

In fact, the precise value of the fixed point observed in our simulations is a bit larger than 2, about 2.1. We have no unique explanation for this discrepancy. It can result from finite size effects; however, we have verified that for the studied networks of order $N=10^5$ the exponents did not change with the system size $N$. The difference may also originate from structural degree-degree correlations that occur in the neighborhood of equilibrium. It is well know [8,10] that in the vicinity of $\gamma=2$ scale-free networks must be highly disassortatively correlated but the way we perform rewiring includes a small random contribution. To ensure such correlations perhaps requires rewiring the least disassortative link instead of the oldest one which was, however, much more difficult to implement in a fast computing code. The discrepancy's other possible explanation may arise from the fact that the theoretical formula (4) has been obtained [15] using a theory of multiplicative branching processes that, similarly to our simulations, does not take the mentioned degree-degree correlations into account.

To conclude, in this paper we have presented by numerical simulations how Bak-Tang-Wiesenfeld's sandpile model's avalanche dynamics and the network's structure may influence each other. Such an interplay between dynamics and structure leads to self-organization in which the avalanches' distribution's and degree distribution's shapes become similar. We suspect that the value of both exponents $\gamma=\tau\cong 2$ may be universal for a large class of SOC phenomena in which the critical behavior occurs not "on" the network's structure but "in" the structure.

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