Control of chaotic solitons by a time-delayed feedback mechanism

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We investigate a control of chaotic solitons using a modified time-delayed autosynchronization method. Numerical results for the maximal Lyapunov exponent are in very good agreement with analytical theory developed originally for low-dimensional systems. The control is most efficient when the spatial distribution of the control force is proportional to the kink translational mode. Observations of the motion of the kink center and the total power transmitted to the system lead to the same values of the maximal Lyapunov exponent.

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Recently, the control of spatiotemporal chaos has attracted more and more interest [1–21]. Examples of possible applications include such diverse cases as stabilization of periodic patterns in optical turbulence [1–4], selection of spatiotemporal current densities in semiconductors [5], tracking of the no-motion state in the Rayleigh-Bernard experiment [6], as well as the control of the El Nino model [8]. A common feature of all of these systems is their complex spatiotemporal behavior, which cannot be captured by low-dimensional dynamics. Existing control methods mirror in part approaches developed for low-dimensional systems and can be divided into *nonfeedback* and *feedback* classes. Stabilization of the desired spatiotemporal state by the former methods is achieved due to a weak driving signal that mimics the target pattern. This driving signal usually does not disappear after the controlled system reaches the final state. Feedback methods include *pinning* schemes that make use of appropriate perturbations of system variables [8,12–14] or system parameters [15,16] that are proportional to temporal differences between the actual and the desired state. The perturbations are applied at a proper number of *controlling points* (for discrete space models) [9] or *sensors* (for continuous space models) [14]. The main disadvantage of such an approach is the necessary identification of the desired spatiotemporal state, which, in practice, is only possible for simple stationary states with high-spatial symmetry. This weakness does not appear for methods based on *time-delayed feedback* (TDF) [17–19], which make use of the autosynchronization effect, which can appear when the control force is proportional to the difference between the present and a past state of the system. The value of the time delay between both of these states should be equal to the period (or to its multiple) of the stabilized periodic orbit [20]. An additional space-dependent part of the feedback force allows us to stabilize time *and* space periodic patterns [1,21] or even experimentally observed traveling waves [22]. A similar effect results as a combination of the TDF method with Fourier filters [2,4].

At present, three main problems that are to be solved for control of chaos in spatially extended systems are: How to collect a minimal amount of information needed to perform control procedure? Can one control the whole system by influencing only a part of its degrees of freedom, e.g., pinning a few sites of a coupled map lattice? Are there any optimal amplitudes and spatial distributions of control forces? Previous studies in this field have been based mainly on numerical simulations and up to now there are no theoretical investigations (beside cases of stationary states [23]) of these issues.

In this paper, we will apply the TDF for the control of a spatially extended *soliton bearing* φ^4 model in the presence of a phase boundary and a periodic pumping force. In fact, we make use of an important feature of soliton systems, which is the separability of their solutions into distinct classes [24,25]. This attribute is characteristic for a class of partial differential equations integrable by the inverse scattering transform [24], e.g., sine-Gordon or nonlinear Schrödinger equation and it is only approximately valid for the considered model. We will show that for the control of chaotic soliton motion, it is enough to observe the center of the kink soliton and we will find a characteristic dependence of the control efficiency on the shape and the amplitude of the control force. Our numerical results are in a very good agreement with analytic calculations that are based on TDF theory developed recently for low-dimensional systems [26,27] using the Floquet theory. We stress here that we could use this theory due to specific character of soliton bearing systems where most of the degrees of freedom are nonactive and it is enough to control the soliton coordinates.

Let us consider a model of a classical scalar field $\varphi(x,t)$ governed by the equation of motion

$$\varphi_{xx}-\varphi_{tt}-\gamma\varphi_t+\frac{1}{2}\varphi-\frac{1}{2}\varphi^3=-F(x)-G(x,t), \qquad (1)$$

where $\gamma>0$ is a damping constant, the static force $F(x)=B(4B^2-1)\tanh(Bx)$ represents a typical phase boundary centered at the point $x=0$, and $G(x,t)$ is a pumping force. For $G(x,t)=0$, there is a static soliton solution of Eq. (1) in the form of a kink

$$\varphi_k(x,t)=2B\tanh(Bx) \qquad (2)$$

that is pinned by the force $F(x)$ to the site $x=0$. The linear stability of this solution was analyzed in [28] and it was proved that in the limit of zero damping, the kink is stable provided that $B>1/2$. We choose the space-time-dependent pumping force $G(x,t)$ in such a way that its spatial part corresponds exactly to the ground-state function of the operator describing small oscillations (linear phonons) around the kink [28–30], i.e., $G(x,t)=P\cos(\omega t)\cosh^{-2}(Bx)$. Since the force $G(x,t)$ is proportional to the first derivative of the kink shape, thus in the limit of small amplitudes $P\ll 1$ it pumps energy mostly into the kink translational mode and it shifts the position of the kink center without large disturbances of the kink shape. This effect is however limited only to the case when the kink is close to the site $x=0$, because for larger distances, the kink translational mode changes significantly and nonlinear effects appear. Chaos in the model (1) was predicted in [28] and observed in [31,32,34] where several transitions between periodic, quasiperiodic, and chaotic kink motion were found. In [32], it was shown that the chaos can be suppressed using the concept of geometrical resonance [33].

It is well known [29] that dynamics of a soliton can be described by means of modes: translational, breathing, and radiating. For our paper, the translational mode has the main influence on soliton behavior [31], which can be characterized by the time dependence of the center of the soliton mass $x_c(t)$. Therefore, the dynamics of the whole phase space of our spatially extended system will be truncated to only one scalar quantity $x_c(t)$.

In order to stabilize the chaotic soliton we add to the right-hand side of Eq. (1) a force in the form

$$f_c(x,t)=\frac{K[x_c(t)-x_c(t-T)]}{\cosh^{\alpha}[B(x-x_c(t)-\Delta)]}. \qquad (3)$$

Here, $T=2\pi/\omega$ is the period of the periodic pumping force $G(x,t)$, while K, α, and Δ are constants. The nominator of $f_c(x,t)$ can be considered as the standard control term used in the time-delayed feedback for a low-dimensional dynamical system with the position of the soliton center $x_c(t)$ as the control variable and the parameter K as the control amplitude. The role of the denominator is to spread out the control force along the whole soliton. The shape of this spreading is connected with the form of the kink translational mode, the width of the control force is governed by the parameter α and the parameter Δ describes the spatial shift between the control force and the soliton. One should take note that the center of the control force (3) is *moving* with the soliton [a static case will be considered in Eq. (6)]. In numerical simulations, we always use initial conditions in the form (2) with boundary conditions $\partial\varphi/\partial x=0$ for $x=\pm L$ where $2L$ was the total length of our system. An example of the successful control is presented in Fig. 1.

Let us start our investigations with the case $\Delta=0$. Figure 2(a) shows the dependence of the maximal Lyapunov exponent Λ of the controlled system on the rescaled control amplitude $\tilde{K}=KS(\alpha)$ for different values of the width parameter α. The Lyapunov exponent Λ has been calculated from observations of the motion of the kink center $x_c(t)$. The control amplitude K has been rescaled by the factor $S(\alpha)=\int_{-L}^{L}\cosh^{-\alpha}(x)dx$ proportional to the total area of the spatial part of $f_c(x,t)$. Obviously, the chaotic kink motion is suppressed when $\Lambda<0$, which is fulfilled only for some ranges of the parameter $\tilde{K}$. Values of the edges $\tilde{K}_{min}, \tilde{K}_{max}$ of the control regions depend strongly on the parameter α and the smallest value of $\tilde{K}_{min}$ occurs for $\alpha=2$. It means that the optimal shape of control force is the translational soliton mode. To describe analytically the numerical results, we apply the theory of TDF control developed in [26] for low-dimensional systems. In the linear approximation, the system's Lyapunov exponent should depend on the control amplitude $\tilde{K}$ as [26]

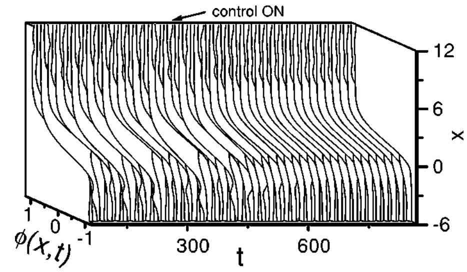

FIG. 1. Example of the control of chaotic soliton with parameters: $B=0.53$, $\gamma=0.15$, $\omega=0.36$, $P=0.36$, $K=0.17$, $\alpha=2$. The same parameters are used for the next figures. The field $\varphi(x,t)$ is depicted only for $t=nT/2$.

$$\Lambda(\tilde{K})=\lambda+\chi'\tilde{K}[1+\exp(-\Lambda T)\cos(T\Delta\Omega)], \qquad (4)$$

$$\Delta\Omega(\tilde{K})=-\chi'\tilde{K}\exp(-\Lambda T)\sin(T\Delta\Omega). \qquad (5)$$

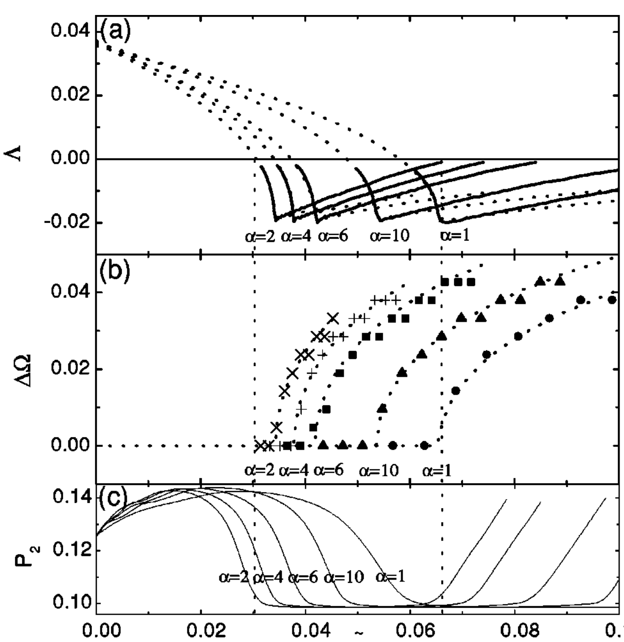

FIG. 2. Dependence of the maximal Lyapunov exponent (a), frequency deviation (b), and the power of nontranslational modes (c) on the normalized amplitude of the control force for different parameters, solid lines and points—numerical computations, dotted lines—theory. Dotted vertical lines show the region of successful control for $\alpha=2$.

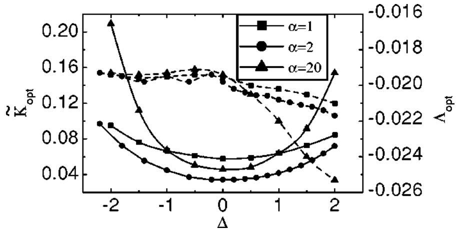

FIG. 3. Dependence of Λ_{opt} (dashed lines) and corresponding $\tilde{K}_{opt}$ (solid lines) on the shift parameter Δ for three different values the width parameter α.

Here, λ is the Lyapunov exponent of the uncontrolled system, χ' is the initial susceptibility of the Lyapunov exponent to the amplitude of the control force, and $\Delta\Omega$ is the deviation of revolution frequency of the perturbation around the unstable periodic orbit from the revolution frequency of the uncontrolled system [26]. The equation is valid provided that the corresponding Floquet multiplier of the uncontrolled system is negative, i.e., when the linear perturbation performs a flip around the orbit. Fitting for each plot $\Lambda(\tilde{K})$ (with values of α fixed) two unknown constants λ and χ', we obtained results that are depicted in Fig. 2(a). One can easily see that all curves merge within a very close neighborhood of the point $\Lambda(\tilde{K}=0)\approx 0.037$ corresponding to the Lyapunov exponent of the uncontrolled system. The quantitative agreement between theoretical and numerical results is within a few percent. Some discrepancy can be observed in the right part of each curve, especially for higher values of the parameter $\tilde{K}$, however, one should keep in mind that the theoretical predictions are just the first-order computation in the control amplitude and that the whole theory has been developed for low-dimensional dynamical systems. In Fig. 2(b), we show frequency deviations $\Delta\Omega$. Similarly as for the plots $\Lambda(\tilde{K})$, the theoretical curves obtained with the help of Eq. (5) fit very well to numerical calculations. The characteristic cusps appearing in Fig. 2(a) correspond exactly to these values of the control parameter $\tilde{K}$ at which the frequency deviations $\Delta\Omega$ start to differ from zero.

Treating the cusps of the plots in Fig. 2(a) as points of the optimal control, we find that although the optimal values of *amplitudes* $\tilde{K}_{opt}$ depend on the width parameter α, the values of the optimal *exponents* Λ_{opt} are nearly independent of this parameter. The situation changes, however, under the influence of the shift parameter Δ. Figure 3 shows that the control is more effective (Λ_{opt} is smaller) for positive values of Δ and large values of α. Since the investigated chaotic trajectory $x_c(t)$ (see Fig. 1) is shifted to positive x values, the control is more effective when the center of the control force is at the opposite side of the soliton than the phase boundary and pumping center.

As the next case, let us consider the control of the soliton using the force that is *pinned* to one fixed point $x=0$ in the space

$$f_c(x,t)=K[x_c(t)-x_c(t-T)]\cosh^{-\alpha}[Bx]. \qquad (6)$$

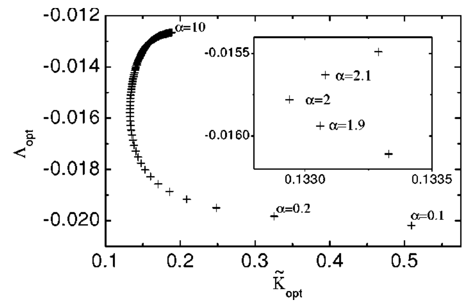

FIG. 4. Dependence of the optimal Lyapunov exponent Λ_{opt} on the rescaled amplitude $\tilde{K}$ of the control force for different values of the parameter α.

Now, the optimal Lyapunov exponent increases with the parameter α (Fig. 4) but similarly, as for the case of the moving control force (3), the lowest value of the $\tilde{K}_{opt}$ corresponds to the case $\alpha=2$. It means that depending on our aims, we can choose between better stability of the controlled system (then we use $\alpha\rightarrow 0$) or the smallest amplitude of the control force (then we use $\alpha=2$). Comparing Figs. 3 and 4, one observes that the moving control force (3) with non-negative values of the shift parameter Δ leads to lower values of Λ_{opt} and to higher values of $\tilde{K}_{opt}$ as compared to the pinned control force (6), except for the case $\alpha\rightarrow 0$, when both approaches give the same value of Λ_{opt}. In this sense, the moving control force (3) is superior over the pinned control force (6).

The amount of energy transferred to the system during the control can be considered as a measure of control quality. In our paper, the corresponding power $P_{in}(t)$ can be calculated as the space integral of the local power density $p_{in}(x,t) = f_c(x,t)\dot{\varphi}(x,t)$. Figure 5 shows the power $P_{in}(t)$ for different parameters $\tilde{K}$ together with corresponding Lyapunov exponents. We observed three phases during the control procedure. After a short initial phase when the soliton moves far from unstable periodic orbit, the phase of exponential decay of the power follows (the linear range in Fig. 5). During this phase, the soliton is directed by the action of the control force towards the periodic trajectory and this movements should be well described by the maximal Lyapunov exponent of the spatially extended system. In fact, we found that the values of slop coefficients corresponding to the second

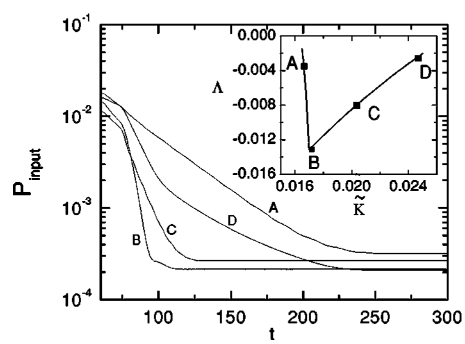

FIG. 5. Power transferred to the system as function of time for different parameters $\tilde{K}$. Inset shows corresponding values of Lyapunov exponents.

phases of the plots $P_{in}(t)$ are equal (within numerical accuracy) to the maximal Lyapunov exponents Λ estimated from the motion of the soliton center $x_c(t)$. We stress here that the power P_{in} defines the energy transfer to *all* degrees of freedom of our system. The fact that the decay of this power is well described by the exponent Λ corresponding to the translational soliton motion, means that other degrees of freedom can be discarded during the second phase of the control force action. In the final control phase, the power P_{in} no longer decreases, which corresponds to a balance between the stabilization action of the control force and destabilizing effects of created modes. Since we truncated the dynamics of the spatially extended system to the translational kink motion, the creation of spatial additional modes can be considered as a kind of noise acting on the moving soliton. In Fig. 2(c), one can see that this noise level is much lower when the chaotic motion of soliton is controlled. The existence of this third phase seems to be a characteristic feature of chaos control in spatially extended systems since control forces in TDF methods tend to zero in low-dimensional models in the absence of noise. In spatially extended systems, this limit can be reached only if all unwanted collective degrees of freedom are suppressed by the action of the control force, which is impossible in practice when the control force creates modes.

In conclusion, we have observed that it is possible to control chaos in a spatially extended soliton bearing model observing the position of the soliton center and applying the control force as the time-delayed feedback with a collective degree of freedom being the center of the soliton mass. The resulting values of the system maximal Lyapunov exponent are in very good agreement with the control theory developed recently for low-dimensional models. This exponent can be calculated either from observations of the kink trajectory or from the total power transfer. The optimal spatial distribution of the control force corresponds to the kink translational additional mode. We stress that the behavior of the whole spatially extended system has been approximated by the *effective* dynamics of a single pointlike particle. It is a peculiar feature of soliton bearing systems; they can be considered as a kind of bridge between low-dimensional and spatially extended models.

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[1] J. A. Sepulchre and A. Babloyantz, Phys. Rev. E **48**, 945 (1993).
[2] W. P. Lu, D. J. Yu, and R. G. Harrison, Phys. Rev. Lett. **76**, 3316 (1996).
[3] D. Hochheiser, J. V. Moloney, and J. Lega, Phys. Rev. A **55**, R4011 (1997).
[4] Peng-Ye Wang, Ping Xie, Jian-Hua Dai, and Hong-Jun Zhang, Phys. Rev. Lett. **80**, 4669 (1998).
[5] A. V. Mamaev and M. Saffman, Phys. Rev. Lett. **80**, 3499 (1998).
[6] G. Franceschini, S. Bose, and E. Scholl, Phys. Rev. E **60**, 5426 (1999).
[7] J. Tang and H. H. Bau, Phys. Rev. Lett. **70**, 1795 (1993).
[8] E. Tziperman, H. Scher, S. E. Zebiak, and M. A. Cane, Phys. Rev. Lett. **79**, 1034 (1997).
[9] Peng-Ye Wang and Ping Xie, Phys. Rev. E **61**, 5120 (2000).
[10] P. Parmananda, M. Hildebrand, and M. Elswirth, Phys. Rev. E **56**, 239 (1997).
[11] P. M. Gade, Phys. Rev. E **57**, 7309 (1998).
[12] T. Hikihara and Y. Ueda, Chaos **9**, 887 (1999).
[13] Hu Gang and Qu Zhilin, Phys. Rev. Lett. **72**, 68 (1994).
[14] L. Junge, U. Parlitz, and Z. Tasev, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 2265 (1999).
[15] L. Junge and U. Parlitz, Phys. Rev. E **61**, 3736 (2000).
[16] D. Auerbach, Phys. Rev. Lett. **72**, 1184 (1994).
[17] S. Sinha and N. Gupte, Phys. Rev. E **58**, R5221 (1998).
[18] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[19] K. Konishi and H. Kokame, Phys. Lett. A **248**, 359 (1998).
[20] K. Konishi, H. Kokame, and K. Hirata, Phys. Rev. E **62**, 1 (2000).
[21] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).
[22] W. Lu, D. Yu, and R. G. Harrison, Phys. Rev. Lett. **78**, 4375 (1997).
[23] P. Kolodner and G. Flatgen, Phys. Rev. E **61**, 2519 (2000).
[24] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[25] G. Eilenberger, *Solitons* (Springer-Verlag, Berlin, 1981).
[26] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).
[27] W. Just, E. Reibold, H. Benner, K. Kacperski, P. Fronczak, and J. Hołyst, Phys. Lett. A **254**, 158 (1999); W. Just, E. Reibold, K. Kacperski, P. Fronczak, J. A. Hołyst, and H. Benner, Phys. Rev. E **61**, 5045 (2000).
[28] J. A. Gonzalez and J. A. Hołyst, Phys. Rev. B **45**, 10 338 (1992).
[29] A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica **1**, 1 (1980).
[30] J. A. Hołyst and H. Benner, Phys. Rev. E **52**, 4583 (1995).
[31] J. A. Gonzalez, L. E. Guerrero, and A. Bellorin, Phys. Rev. E **54**, 1265 (1996); J. A. Gonzalez and B. A. Mello, Phys. Lett. A **219**, 226 (1996).
[32] J. A. Gonzalez, B. A. Mello, L. E. Guerrero, and L. I. Reyes, Phys. Rev. Lett. **80**, 1361 (1998).
[33] R. Chacon, Phys. Rev. Lett. **77**, 482 (1996).
[34] N. R. Quintero, A. Sanchez, and F. G. Mertens, Phys. Rev. Lett. **84**, 871 (2000).