Bridging local self-similarity and global scale-invariance in fractal complex networks

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We show that fractality in complex networks arises from the geometric self-similarity of their built-in hierarchical community-like structure, which is mathematically described by the scale-invariant equation for the masses of the boxes with which we cover the network when determining its box dimension. This approach - grounded in both scaling theory of phase transitions and renormalization group theory - leads to the consistent scaling theory of fractal complex networks, which reveals a collection of scaling exponents and different relationships between them. The exponents can be divided into two groups: microscopic (hitherto unknown) and macroscopic, characterizing respectively the local structure of fractal complex networks and their global properties. Interestingly, exponents from both groups are related to each other and only a few of them (three out of seven) are independent, thus bridging the gap between local self-similarity and global scale-invariance of fractal networks. We successfully verify our findings in real networks situated in various fields (information – the World Wide Web, biological – the human brain, and social – scientific collaboration networks) and in several fractal network models.

I. REVISING PARADIGMS OF FRACTAL COMPLEX NETWORKS

It will soon be two decades since it was first shown that some real networks (such as the World Wide Web [WWW] and different biological networks) have fractal properties [1, 2]. This means that, when covered with non-overlapping boxes, with the maximum distance between any two nodes in each box less than l_B, they exhibit power-law scaling [1–5]:

$$N_B(l_B)/N \simeq l_B^{-d_B}, \tag{1}$$

where $N_B(l_B)$ is the number of boxes of a given diameter, and d_B is the fractal (or box) dimension of the network of size N. Such fractal networks are also said to be *self-similar*, because their power-law degree distributions,

$$P(k) \sim k^{-\gamma}, \tag{2}$$

remain invariant under a renormalization scheme [6, 7], according to which a new network emerges from the original one when nodes belonging to the same box in the original network are replaced by one supernode in the renormalized network. In this case, the supernode is connected to another supernode if in the original network there is at least one link between the nodes of the corresponding boxes.

Here, at least two critical remarks can be made. The first remark is that an analogous invariance of the degree distribution with respect to the box-covering renormalization scheme is also observed in networks that do not satisfy Eq. (1) (in this respect, well-known examples are the internet and Barabási-Albert (BA) networks [2, 8, 9]). The second remark is that it is not entirely clear, what structural characteristics of fractal networks exhibits geometric self-similarity and remain invariant [10] under the described renormalization. Clearly, the power-law node degree distribution cannot be considered such a characteristic because it is intrinsically invariant under the rescaling of the degree [11]. Its invariance under box-covering renormalization may only suggest the existence of some (presumably) degree-dependent network measure, whose self-similarity under the renormalization procedure could result in the observed invariance of the degree distribution. One argument supporting this statement is that random networks, where the degree distribution is not a power law, can also exhibit fractal properties (in this regard, the best example is the giant component of classical random graphs near the percolation transition).

If the above remarks, indicating an incomplete understanding of fractality in complex networks, are reasonable, pertinent questions would be: What are the real origins and potential consequences of fractality in complex networks? What determines networks' fractal dimension? Indeed, several studies have been published throughout the years that focus on the exploration of the origins of fractality [12–18]. However, these efforts did not lead to consensus. Thus, there is a lack of realistic (and not just deterministic [19, 20], or reflecting the renormalization procedure [2, 21, 22]) fractal network models that would allow testing the role of fractality in the context of geometry-involving issues [23], such as navigability, localization of information sources, prediction of hidden network connections, etc. These are of particular importance when faced with the confirmed fractal properties of different information, biological, and even social networks (see e.g.[24, 25]). The goal of this article is to initiate far-reaching changes in this state of affairs.

In what follows, we will first argue that the correct scale-dependent network measure, which is self-similar (i.e. geometrically invariant) under the box-covering renormalization procedure, is the mass of the box: $m(L, k)$, which depends on the diameter L of the box and (what is of particular importance in complex networks) on the degree k of the best connected node (hub) inside this box. We will show that one of the consequences of this result is the previously discovered scaling relation between the degree k' of the supernode in the renormalized network and the degree k of the hub of the corresponding box in the network before renormalization with l_B-boxes: $k' = l_B^{-d_k} k$, where d_k is only one of four scaling exponents that characterize microscopic structure of the fractal complex network and determine its box dimension. We will also show that if the fractal complex network has a power-law node degree distribution (which is traditionally referred to as the *scale-free property*), then the mass box distribution also follows the power-law, and it is invariant under the box renormalization procedure. Furthermore, the characteristic exponents of both distributions are related to the microscopic scaling exponents describing the masses of the boxes, thus bridging the gap between local self-similarity and global scale invariance in fractal complex networks. Lastly, we successfully verify our findings in real networks situated in various fields (information – the World Wide Web, biological – the human brain, and social – scientific collaboration networks) and in several fractal network models.

II. LOCAL SELF-SIMILARITY AND GLOBAL SCALE-INVARIANCE IN FRACTAL NETWORKS

A. Geometric self-similarity

In classical fractals [26], which reproduce themselves at different space scales, self-similarity manifests itself in the scale-invariant equation [10], which describes how the mass $m(L)$ of the system changes with its linear size L:

$$m(bL) = \mu(b)\, m(L), \tag{3}$$

where $b > 0$. In theoretical physics, this type of equation is, for example, encountered in the theory of critical phenomena [11, 27]. Mathematically, this equation defines a homogeneous function. Its solution is simply a power law:

$$m(L) = AL^{d_f}, \tag{4}$$

which, in the case of fractals, determines their fractal dimension, $d_f = \ln\mu/\ln b$, and leads to the well-known scaling relation [28]:

$$m(bL) = b^{d_f} m(L), \tag{5}$$

which is simply the general form of Eq. (1), with $d_f = d_B$, $b = l_B^{-1}$, $m(L) = N$, and $m(bL) = N_B(l_B)$. In what follows, building on this observation, we will assume that Eq. (5) is a special case of a more general equation in which the masses of the system and its parts, which are further identified with the network size and the masses of different L-boxes extracted from this network, do not only depend on the diameter of the examined set of nodes but also on the degree of the best-connected node in this set.

Before proceeding, it is useful to recall the meaning of Eq. (5). In fact, this equation can be interpreted in two ways. More directly, it states that if one considers *a smaller part of the system*, let's say of size bL (with $b < 1$), then $m(bL)$, as compared to $m(L)$, is decreased by a factor $\mu(b) = b^{d_f}$. However, this equation seems to characterize the masses of *the system on two different scales* that is, at two stages of some self-similar renormalization procedure applied to that system. Accordingly, taking into account these two stages, the ratio $m(bL)/m(L)$ does not depend on L but only on the ratio b of the two scales. At this point, to make the equation more operationalizable, $m(bL)$ could be replaced by $m'(L')$, thus indicating that we are dealing with the system after renormalization. Thus, in what follows, when generalising the concept of geometric scale-invariance to the case of fractal networks, we will rather use the notation with the apostrophe.

To begin with, we rewrite Eq. (5) in the form:

$$m'(L', k') = l_B^{-d_B} m(L, k), \tag{6}$$

where d_B is the box dimension of fractal networks, and $m'(L', k')$ stands for the mass of the super-box that is, the box of diameter L' and the super-hub of degree k' that results from l_B-renormalization of the box of mass $m(L, k)$. Eq. (6) therefore poses:

$$L' = L/l_B, \tag{7}$$

which is a direct consequence of the assumed renormalization procedure, and

$$k' = k/l_B^{d_k}, \tag{8}$$

which is the long-confirmed scaling relation observed in fractal complex networks, but with respect to Eq. (6), it becomes an inevitable result of dimensional analysis. Since Eq. (6) defines a generalized homogeneous function [27] of the form:

$$m(L, k) = BL^{\alpha} k^{\beta}, \tag{9}$$

after its substitution to (6), we obtain the following scaling relation:

$$d_B = \alpha + \beta d_k. \tag{10}$$

The obtained relation, Eq. (10), is one of the most important results of this article. According to this relation, the box dimension of fractal networks is only determined by the scaling exponents characterizing the microscopic structure of the network, which, however (as with classical fractals), reproduces itself at the meso- and macroscales (see Fig. 1). In particular, the exponent α describes how the mass of the box, Eq. (9), varies with its diameter, cf. Eq. (5):

$$m(bL, k) = b^{\alpha} m(L, k), \tag{11}$$

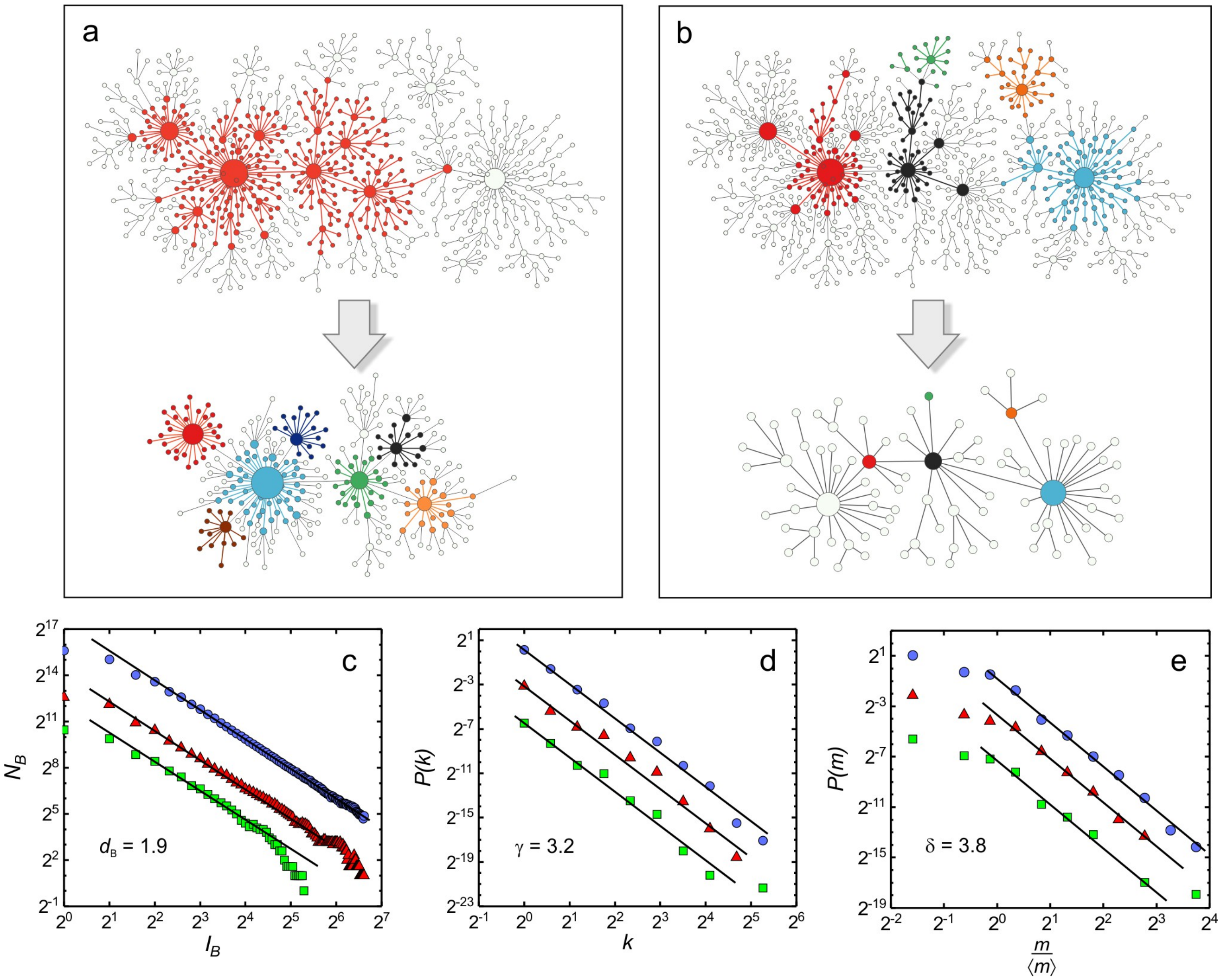

FIG. 1. **Schematic illustration of the idea of geometric self-similarity in complex networks on the example of the fractal model of nested BA networks** (for the definition of the model, see Supplementary Material). **a**: **Fractal complex network subdivided into parts—boxes of a given diameter—each of which is (at least approximately) a reduced-size copy of a whole network**. In the top picture shown, the largest box (the red one) extracted from the original network is treated as a new network (shown below) and divided into new smaller boxes (some of them are marked with different colours). The macroscopic characteristics of this new network (represented by green squares in the accompanying graphs) are similar to those of the original network (indicated by navy circles in these graphs). **b**: **The renormalization procedure applied to the same network as in part a.** The top original network is divided into boxes of a fixed diameter (some of them are marked with different colours). In the new network after renormalization (shown below), these boxes are replaced by nodes. Again, the macroscopic characteristics of this new network (represented by red triangles in the accompanying graphs) are similar to those of the original network. **c**, **d**, **e**: **Macroscopic characteristics of the studied networks** corresponding to the number of boxes $N_B(l_B)$ needed to cover the network as a function of the maximum diameter l_B in the box, the node degree distribution—$P(k)$, and the mass box distribution—$P(m)$, respectively. To construct these graphs, a nested BA network of size $N \simeq 5 \cdot 10^4$ was created (this data are marked with navy circles). To analyze the self-similarity of the network parts, the original network was covered with boxes of diameter $l_B = 40$, and the largest box of size $M \simeq 1.4 \cdot 10^3$ was extracted as a new network (this data are marked with green squares). To create a renormalized network of size $N' \simeq 6.1 \cdot 10^3$, the original one was covered with boxes of size $l_B = 6$, i.e. $N' = N_B(6)$ (these data are marked with red triangles).

where $b > 0$. In a similar vein, the second addend in (10), which is further called the mass exponent (in analogy to the degree exponent, d_k),

$$d_m = \beta d_k, \tag{12}$$

characterizes, how the local network density changes as a result of renormalization:

$$m'(L, k') = l_B^{-d_m} m(L, k). \tag{13}$$

B. Scale-free property

At this point, we would like to emphasize the lack, in our considerations so far, of scale-free node degree distributions, whose invariance due to the renormalization procedure is considered an attribute of fractal networks [1–3]. Interestingly, this lack clearly shows the otherwise obvious fact that fractal networks may not have the scale-free property. Nevertheless, when they reveal the property, then both the node degree distribution, $P(k) \sim k^{-\gamma}$, and the box mass distribution, $P(m) \sim m^{-\delta}$, are invariant under the box-covering renormalization procedure, with the invariance of $P(k)$ being rather a consequence of the previously discussed self-similarity of boxes (6)-(10) and scale-free property of $P(m)$, and not the other way around.

In particular, it can be shown that the well-known scaling relation between the three indices [1],

$$\gamma = 1 + \frac{d_B}{d_k}, \tag{14}$$

results from the previously unknown relation for the characteristic exponent of the box mass distribution,

$$P(m) \sim m^{-\delta}, \tag{15}$$

namely:

$$\delta = 1 + \frac{d_B}{d_m}, \tag{16}$$

where d_m is the mass exponent given by Eq. (12). To prove this, below we present the reasoning leading to Eq. (14), pointing out its inaccuracies and correcting them accordingly.

Thus, to obtain Eq. (14), one starts with the following equation [1]:

$$n(k)dk = n'(k')dk', \tag{17}$$

where $n(k)$ (respectively, $n'(k')$) is the number of nodes with k (respectively, k') links in the network before (after) renormalization. Then, substitutions are made in this equation: $n(k) = NP(k)$ and $n'(k') = N'P'(k')$, where N and $N' = N_B(l_B) = Nl_B^{-d_B}$ (1) stand for the number of nodes in the network before and after renormalization, respectively. These substitutions lead to the following density balance equation:

$$P(k)dk = l_B^{-d_B} P'(k')dk', \tag{18}$$

from which Eq. (14) is derived under the assumptions that $P(k) \sim k^{-\gamma}$ and $P'(k') \sim k'^{-\gamma}$, and that Eq. (8) is met between k and k'. Here, the question arises as to whether Eq. (18) applies to all nodes in the studied networks, as described by the node degree distributions $P(k)$ and $P'(k')$, or perhaps only to hubs in the boxes, as suggested by Eq. (8). However, if these equations were applicable only to hubs in the boxes, then Eq. (17), which is the starting point of the presented reasoning, should rather be written as:

$$n(m)dm = n'(m')dm', \tag{19}$$

where $n(m)$ and $n'(m')$ represent the number of $L-$boxes of mass m and m', respectively, in the original network and in the network after l_B-renormalization, with m and m' depending on each other according to Eq. (13). Interestingly, making the appropriate substitutions in this equation (i.e. $n(m) = N_B(L)P(m)$ and $n'(m') = N'_B(L)P'(m')$, where $N_B(L) = NL^{-d_B}$ and $N'_B(L) = N_B(l_B)L^{-d_B}$), one obtains the density balance equation analogous to Eq. (18):

$$P(m)dm = l_B^{-d_B} P'(m')dm'. \tag{20}$$

Lastly, assuming the invariant character of the box mass distribution that is, $P(m) \sim m^{-\delta}$ and $P'(m') \sim m'^{-\delta}$ one obtains the scaling relation (16), from which Eq. (14) naturally follows, when $\beta = (\gamma-1)/(\delta-1)$ (22), which is one more scaling relation characterizing the structure of fractal complex networks.

III. FROM MICROSCOPIC TO MACROSCOPIC SCALING EXPONENTS IN REAL AND MODEL-BASED FRACTAL NETWORKS

All scaling exponents discussed in this article, which describe fractal complex networks, can be divided into two groups. The first group refers to the macroscopic characteristics of the network (d_B, γ, and δ), and the second group includes the exponents that characterize the network structure at the microscopic level (d_k, d_m, α and β). Interestingly, exponents from both groups are related to each other and, as in the scaling theory of critical phenomena, only a few of them, three to be exact, are independent. The choice of the three fundamental exponents depends on the focus of the study. Here, to validate our results in real and model-based fractal networks, we take the easier to measure macroscopic exponents as independent. This choice results in the following set of test relations, cf. Eqs.(14) and (16):

$$d_k = \frac{d_B}{\gamma-1}, \qquad d_m = \frac{d_B}{\delta-1}, \tag{21}$$

and, cf. Eqs. (10) and (12):

$$\alpha = \frac{\delta-2}{\delta-1}\, d_B, \qquad \beta = \frac{\gamma-1}{\delta-1}, \tag{22}$$

of which only the relation for d_k (21) has been verified in real [1] and model [2] networks, and the results of the validation of relations (22) are summarized below.

The real networks analyzed in this paper come from various fields and represent information, social, and biological networks. We analyzed: 1) a sample of the WWW with nodes corresponding to web pages and links

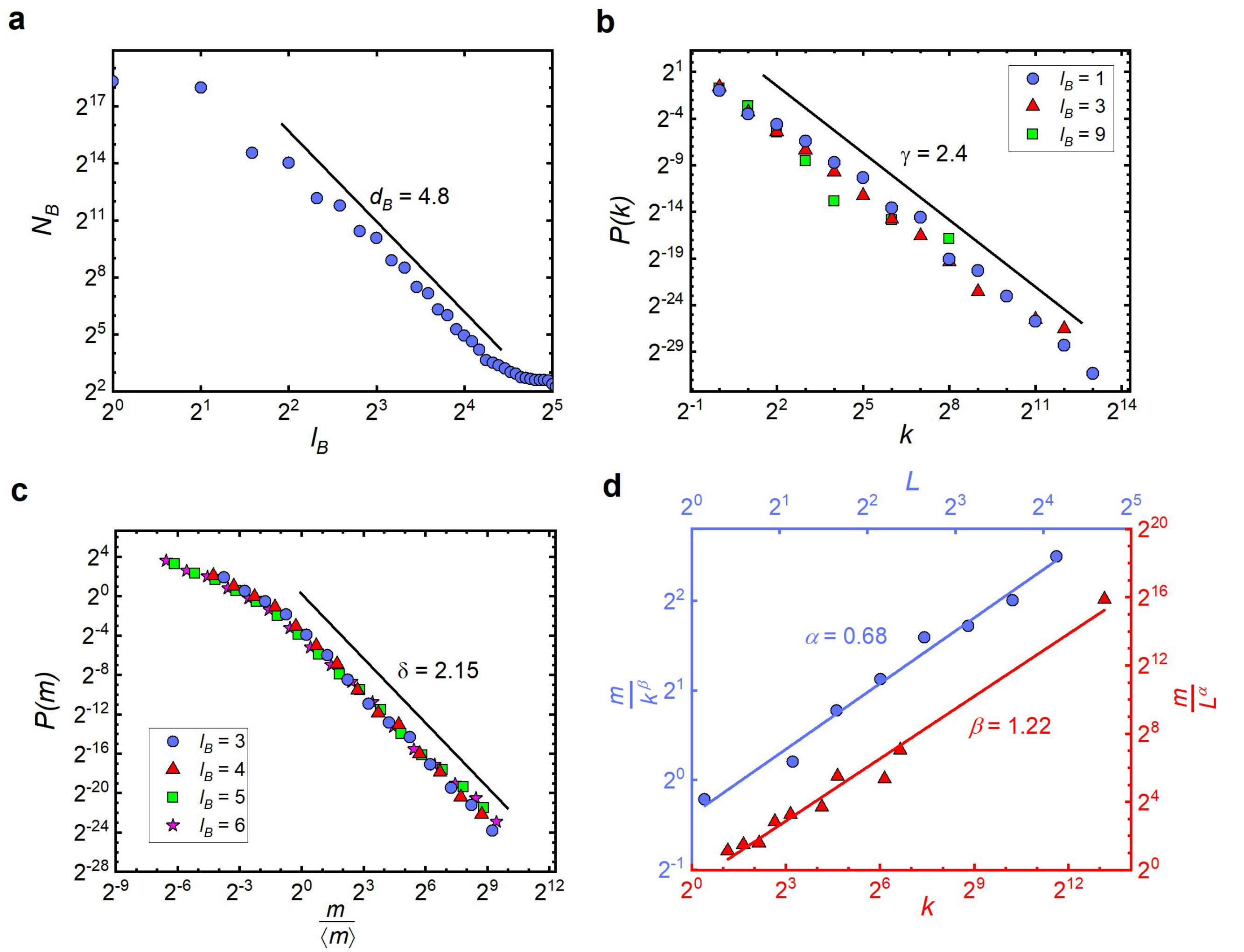

FIG. 2. **Scale-invariant and self-similar scaling in the WWW**. **a**: A log-log plot of N_B versus l_B revealing the fractal nature of the studied network according to Eq. (1). **b**: Invariance of the node degree distribution $P(k)$ under the renormalization for different box sizes l_B. **c**: Invariance of the mass box distribution $P(m)$. **d**: Scaling of the masses of boxes according to Eq. (9). (See the description given in the main text.)

standing for hyperlinks [29]; 2) a coauthorship network (DBLP), where nodes are scientists and edges are placed between two scientists if they have co-authored a paper [30, 31]; 3) a functional brain network (BRAIN), which reflects the correlation between the activity of different areas in the human brain [32, 33]. In addition to real networks, we have also analyzed several fractal network models, including our own network model, which is based on nested BA networks [34], the Song-Havlin-Makse (SHM) model [2] and (u, v)-flowers [19]. Detailed information on all these networks (real and synthetic) can be found in Supplementary Material.

Table I presents the theoretical and empirical values of the scaling exponents of all analyzed networks. The theoretical values, which are given in brackets, are of two types. For the deterministic model-based networks—the SHM model and (u, v)-flowers—their values can be calculated using the appropriate formulas, the details of which are provided in Supplementary Material. For real networks and for the numerical model of nested BA, the theoretical values of α and β were calculated from Eqs. (22) using the empirical values of the macroscopic exponents.

Correspondingly, the empirical values of the scaling exponents were calculated from Fig. 2 and Figs. S2-S6 in Supplementary Material according to the following protocol (the same for each network): First, we determined the box dimension d_B of these networks resulting from tiling the network with boxes of different sizes l_B. To this end, we used the algorithm developed by Song et al. [35]. We showed that the value of d_B after renormalization (even multiple times) remains the same as before renormalization (see, e.g. Fig. 1 and 2(a)). We then examined the invariance of distributions $P(k)$ and $P(m)$ under the networks' renormalization procedure with boxes of different sizes. It should be noted that in all networks we studied, both distributions are scale-invariant, with well-defined

TABLE I. **Empirical and theoretical values of the scaling exponents for various fractal networks**. In the table, N is the number of nodes in the analyzed network, $\langle k \rangle$ is the average node degree, and d corresponds to the diameter of the network.

network	N	$\langle k \rangle$	d	d_B	γ	δ	α	β
WWW	325728	4.6	46	4.8	2.4	2.2	0.68 (0.63)	1.22 (1.22)
DBLP	2523	2.5	62	2.0	3.2	3.4	1.23 (1.17)	0.86 (0.92)
Brain	2920	4.7	77	2.2	2.8	2.3	0.57 (0.51)	1.39 (1.38)
Nested BA	50000	2	475	1.92	3.2	3.8	1.24 (1.23)	0.84 (0.79)
SHM model	78126	2	4373	1.46	3.32	3.32	0.82	0.96
$s=2$, $a=3$, $n=2s+1=5$		(tree)		$\left(\frac{\ln n}{\ln a} \simeq 1.46\right)$	$\left(1+\frac{lnn}{\ln s} \simeq 3.32\right)$	$\left(1+\frac{\ln n}{\ln s} \simeq 3.32\right)$	$\left(\frac{\ln(n/s)}{\ln s} \simeq 0.83\right)$	(1)
(u,v)-**flowers**	43692	3	416	2.0	3.0	3.0	0.99	0.98
$u=2$, $v=2$, $w=u+v=4$				$\left(\frac{\ln w}{\ln u} = 2\right)$	$\left(1+\frac{lnw}{\ln 2} = 3\right)$	$\left(1+\frac{lnw}{\ln 2} = 3\right)$	$\left(\frac{\ln(w/2)}{\ln u} = 1\right)$	(1)

characteristic exponents γ and δ (see e.g. Fig. 2b-c). Lastly, having determined the macroscopic scaling exponents: d_B, γ, and δ, we were able to calculate the theoretical values of the local exponents—α and β, Eqs. (22)—which we used to obtain the adequately rescaled masses of boxes to determine their empirical values (see e.g. Fig. 2d). In particular, to obtain the empirical value of α, the masses of all the internally connected boxes, obtained during tiling the network with different l_B-boxes, were divided by the hub's degree raised to the power of the theoretically obtained β. Such rescaled masses m/k^{β} were then plotted against the actual diameters of the boxes, $L < l_B$, which had been specified individually for each box. A similar procedure was applied to determine the empirical value of β.

IV. PERSPECTIVES

The origins and consequences of fractality are one of the three main research directions in the geometry of complex networks [23], next to the hyperbolic geometry of hidden network spaces [36, 37] and the geometry induced by dynamic processes in networks [38–40]. Although these three geometries, due to the various definitions of distance in each of them, are defined differently, there is no doubt that they must be closely related to each other. While these relationships have yet to be explored, evidence of their existence can be found in our results.

For example, when examining deterministic models of fractal networks (SHM model and (u, v)-flowers, see Supplementary Material), we noticed that while macroscopic scaling exponents are very stable in the sense that they do not depend on the box-covering method [35, 41], this may not be the case for microscopic exponents. In particular, in the mentioned models, gathering nodes according to their kinship—which is the most optimal, because it corresponds to the smallest number of boxes—gives the values of microscopic exponents closest to their theoretical predictions. Since the degree of kinship can be thought of as a distance in some metric space—the space of kinship—this observation is important. In fact, the fractality of these models may be considered a feature they inherit from their kinship spaces. Here, natural questions arise, such as whether the fractality of real complex networks may result from the properties of hidden metric spaces. Similar studies on community structure confirm the existence of such a relationship [42–44]. The mention of the community structure is not entirely accidental here, because, as the example of the DBLP network shows—in which the removal of weak ties reveals its fractal properties (see also [25, 31])—the fat-tailed community size distribution [45, 46] may result from the scale-invariant distributions of box masses observed in (not necessarily tree-like) fractal skeletons [13, 14] of these networks.

The second thread that we would like to emphasize concerns the geometry induced by diffusion-like dynamic processes in networks [38–40]. In classical fractals, this kind of geometry is closely related to the cluster-growing method of calculating their fractal dimensions, which is actually a way of measuring the distance [26]. In complex networks, due to the misunderstanding of the idea of geometric self-similarity, establishing an analogous relationship has not yet been possible [1]. It seems that the scaling theory of fractal complex networks presented in this paper has the potential to break this impasse. This is even more likely since in its general findings, with box masses depending not only on the diameter of the boxes but also on the degree of the best-connected node inside the box, the theory refers to the well-established heterogeneous (degree-based) mean-field theory commonly used to study dynamical processes on complex networks [47].

V. METHODS

The datasets used and the complete Python code for all calculations can be obtained from [48].

VI. ACKNOWLEDGEMENTS

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