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Multisymplectic analysis of the Short Pulse Equation and Numerical Applications

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Abstract

The multisymplectic analysis of the Short Pulse Equation known in nonlinear optics is used in order to construct a geometric multisymplectic integrator of it. A brief comparison of its effectiveness relative to the pseudo-spectral integration scheme is presented.

1 Introduction

The multisymplectic Hamiltonian formalism has emerged from geometric theories in the calculus of variations [1]. It has been a subject of numerous investigations recently [2–9] including possible applications to field quantization [10–13]. The multisymplectic approach to the construction of geometric numerical integrators of PDEs was proposed in [14]. The application of the closely related "multi-symplectic" structure in wave propagation has been pioneered by Bridges [15].

In this contribution we apply the multisymplectic formalism to the short pulse equation (SPE) known in nonlinear optics. The short pulse equation has appeared recently [16, 17] as a description of ultra-short pulses when the standard nonlinear Schrödinger equation cannot be applied because the slowly varying envelope approximation it is based on is not valid anymore. In [18, 19] the integrability of this equation has been proven, and in [20] an example of the exact solution has been constructed. In [21] three integrable two component generalizations of SPE have been found.

Here we apply the multisymplectic formalism in order to construct a multisymplectic geometric integrator for SPE. This work is a part of the investigation of the properties of ultra-short pulses in nonlinear optics with the help of SPE and its generalizations which requires a stable and robust numerical integration scheme for SPE.

The multisymplectic formulation of SPE is discussed in Sect. 2. In Sect. 3 we construct the simplest multisymplectic integrator and briefly compare its effectiveness with the well known pseudo-spectral numerical integration [22].

2 The multisymplectic formulation of SPE

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{1}$$

can be written in the form

$$\phi_{xt} - \phi - \frac{1}{6}(\phi_x^3)_x = 0 \tag{2}$$

if we introduce the potential ϕ

$$u := \phi_x. \tag{3}$$

This equation can be derived from the first order Lagrangian

$$L = \frac{1}{2}\phi_t\phi_x - \frac{1}{24}\phi_x^4 + \frac{1}{2}\phi^2. \tag{4}$$

Using the standard multisymplectic (De Donder-Weyl) Hamiltonian formalism, we introduce the polymomenta

$$\begin{aligned} p^t &:= \frac{\partial L}{\partial \phi_t} = \frac{1}{2}\phi_x, \\ p^x &:= \frac{\partial L}{\partial \phi_x} = \frac{1}{2}\phi_t - \frac{1}{6}\phi_x^3, \end{aligned} \tag{5}$$

and the (De Donder-Weyl) Hamiltonian

$$H_{DW} := p^t\phi_t + p^x\phi_x - L = 2p^xp^t + \frac{2}{3}(p^t)^4 - \frac{1}{2}\phi^2. \tag{6}$$

Then the multisymplectic (De Donder-Weyl) Hamiltonian equations take the form

$$\begin{aligned} \partial_x p^x + \partial_t p^t &= -\frac{\partial H}{\partial \phi} = \phi, \\ \partial_x \phi &= \frac{\partial H}{\partial p^x} = 2p^t, \\ \partial_t \phi &= \frac{\partial H}{\partial p^t} = 2p^x + \frac{8}{3}(p^t)^3. \end{aligned} \tag{7}$$

This set of first order equations is equivalent to SPE written in terms of the potential function $\phi(x,t)$, Eq. 2. It is well known that these equations can be obtained from the geometrical formulation of first order variational problems using the Poincare-Cartan form and its exterior derivative (the multisymplectic form) [1, 9].

$$\Omega = d\phi \wedge dp^x \wedge dt + d\phi \wedge dp^t \wedge dx - dH \wedge dx \wedge dt. \tag{8}$$

In order to establish a connection with the multi-symplectic formulation of Bridges [15] which has became more popular in discussions of geometric integrators of PDEs, let us introduce the set of variables $Z^v := (\phi, p^x, p^t)$. Then the DW Hamiltonian equations can be written in matrix form

$$\beta^t\partial_t Z + \beta^x\partial_x Z = \nabla_Z H, \tag{9}$$

where the β-matrices

$$\beta^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \beta^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{10}$$

can be identified with the so-called Duffin-Kemmer-Petiau matrices (in 2D) [23] which fulfill the DKP algebra relations $(a, b, c = (x, t))$.

$$\beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = -\beta^a \delta^{bc} - \beta^c \delta^{ab}. \tag{11}$$

This form of DW Hamiltonian equations generalizes the Hamiltonian equations in mechanics written in the form

$$\omega \partial_t Z = \partial_Z H,$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic matrix and $Z := (p, q)$.

Associated with the above two antisymmetric matrices β are two pre-symplectic forms

$$\begin{aligned} \kappa^x &= \frac{1}{2} dz \wedge \beta^x dz = -dp^x \wedge d\phi, \\ \kappa^t &= \frac{1}{2} dz \wedge \beta^t dz = dp^t \wedge d\phi. \end{aligned} \tag{12}$$

The structure given by two pre-symplectic forms κ^x and κ^t is called multi-symplectic by Bridges [15]. In the notations introduced by Bridges (1997) $\beta^x = K$ and $\beta^t = M$ and $H = -S$. These notations are now standard in the papers devoted to the geometric (multisymplectic) integrators of PDEs [24–28]. In this notation the fundamental *multisymplectic conservation law* is written in the form:

$$d/dt\kappa^t + d/dx\kappa^x = 0. \tag{13}$$

3 A multisymplectic integrator for SPE

The simplest realization of the multisymplectic integrator is constructed by the discretization of DW Hamiltonian equations using the midpoint method in both x and t directions:

$$\begin{aligned} &\phi_{i,j} \approx \phi(i\Delta x, j\Delta t), \\ &\phi_{i+1,j+1/2} := \frac{1}{2}(\phi_{i,j} + \phi_{i,j+1}), \\ &\phi_{i+1/2,j+1/2} := \frac{1}{4}(\phi_{i,j} + \phi_{i,j+1} + \phi_{i+1,j} + \phi_{i+1,j+1}). \end{aligned} \tag{14}$$

We obtain:

$$\frac{p^x_{i+1,j+\frac{1}{2}} - p^x_{i,j+\frac{1}{2}}}{\Delta x} + \frac{p^t_{i+\frac{1}{2},j+1} - p^t_{i+\frac{1}{2},j}}{\Delta t} = \phi_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{15a}$$

$$\frac{\phi_{i+1,j+\frac{1}{2}} - \phi_{i,j+\frac{1}{2}}}{\Delta x} = 2p^t_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{15b}$$

$$\frac{\phi_{i+\frac{1}{2},j+1} - \phi_{i+\frac{1}{2},j}}{\Delta t} = 2p^x_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{8}{3}(p^t_{i+\frac{1}{2},j+\frac{1}{2}})^3. \tag{15c}$$

As a consequence of the statement proven by Bridges and Reich [24] this integrator fulfills the discretized multisymplectic conservation law.

3.1 The numerical implementation

We solve the initial boundary value problem for Eq. 1 using the above multisymplectic integrator. We know the initial value $u(x, t = 0)$, its discretization $u_{i,j=0}, i = 1, ..., N$, and the vanishing values of the solutions on the right boundary (the wave propagates from the right to the left). Hence, we know $p^t_{i,j=0}$ and $\phi_{i,j=0}, i = 1, N$, and $p^t_{N,j} = \phi_{N,j} = p^x_{N,j} + p^x_{N,j+1} = 0, j = 0, ..., M$. From the known values at three mesh points $(i+1, j)$, $(i+1, j+1)$, and (i, j) (see Fig. 1) we calculate new values at the point $(i, j+1)$, i.e. given $p^t_{i+1,j}$, $p^t_{i+1,j+1}$, $p^t_{i,j}$, $\phi_{i+1,j}$, $\phi_{i+1,j+1}$, $\phi_{i,j}$, and $p^x_{i+1,j+1} + p^x_{i+1,j}$, we obtain $p^t_{i,j+1}$, $\phi_{i,j+1}$ and $p^x_{i,j+1} + p^t_{i,j}$. Then the integrator yields

$$\begin{aligned}
&(p^t_{i,j+1})^3 + 3(p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})(p^t_{i,j+1})^2 \\
&+3\left((p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})^2 + 4\left(\frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2}\right)\right) p^t_{i,j+1} \\
&-\frac{24}{\Delta t}(\phi_{i+1,j+1} - \phi_{i,j}) - 12\Delta x(\phi_{i+1,j} + \phi_{i+1,j+1}) \\
&+12\left(\frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2}\right) p^t_{i+1,j+1} + 6(\Delta x)^2(p^t_{i+1,j} + p^t_{i,j}) \\
&+24(p^x_{i+1,j} + p^x_{i+1,j+1}) + (p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1})^3 = 0,
\end{aligned} \tag{16a}$$

$$\begin{aligned}
\phi_{i,j+1} &= (\phi_{i+1,j} - \phi_{i,j} + \phi_{i+1,j+1}) \\
&-\Delta x\,(p^t_{i+1,j} + p^t_{i,j} + p^t_{i+1,j+1}) - \Delta x\, p^t_{i,j+1},
\end{aligned} \tag{16b}$$

$$\begin{aligned}
(p^x_{i,j+1} + p^x_{i,j}) &= (p^x_{i+1,j+1} + p^x_{i+1,j}) - \frac{\Delta x}{\Delta t}(p^t_{i,j} + p^t_{i+1,j} - p^t_{i+1,j+1}) \\
&+\frac{\Delta x}{\Delta t}p^t_{i,j+1} - \frac{\Delta x}{2}(\phi_{i+1,j} + \phi_{i,j} + \phi_{i+1,j+1} - \frac{\Delta x}{2}\phi_{i,j+1}.
\end{aligned} \tag{16c}$$

We first calculate $p^t_{i,j+1}$ from the cubic Eq. (16a) (the root which ensures the continuity of the solution is selected). Then Eq. (16b) yields $\phi_{i,j+1}$ and Eq. (16c) yields $p^x_{i,j+1} + p^x_{i,j}$ (see Fig. 1). Thus, we obtain $u_{i,j+1} = 2p^t_{i,j+1}$.

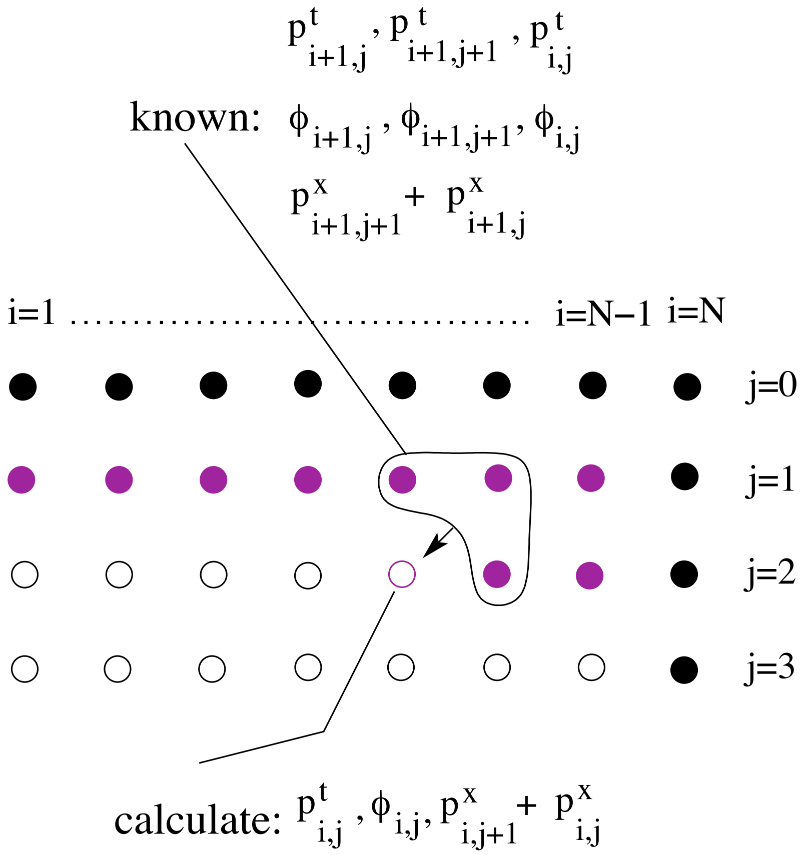

Figure 1: The discretization mesh.

In order to test the effectiveness of the method, we numerically propagate the known Sakovich' exact solution of SPE [20] to $t = 100$. The evolution of the Sakovich exact solution (with $m = 0.2$) is shown on Fig. 2 at $t = 0$ and $t = 100$.

The exact solution is compared with the numerical solutions obtained using the multisymplectic scheme and the pseudo-spectral scheme. We compare the error of the methods, and the CPU time required to reach $t = 100$ at different values of discretization steps Δt and Δx. The error of numerical integration is given by the standard deviation:

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1} N \left(u_{i,j} - \bar{u}_{i,j}\right)^2}, \tag{17}$$

where $u_{i,j}$ is the numerical solution and $\bar{u}_{i,j}$ is the exact Sakovich' solution at time $t = j\Delta t$.

The results of the multisymplectic integration for different values of Δt and $\Delta x = X_{max}/N$ ($X_{max} = 400$) are shown of Fig. 3. As expected, the error decreases with Δx and Δt decreasing. The multisymplectic method appears to be more effective than the pseudo-spectral method. For example, for $N = 2^{17}$ and $\Delta t = 0.0001$ the error of the multisymplectic scheme $\sigma \approx 6.5 \times 10^{-6}$, while for the pseudo-spectral

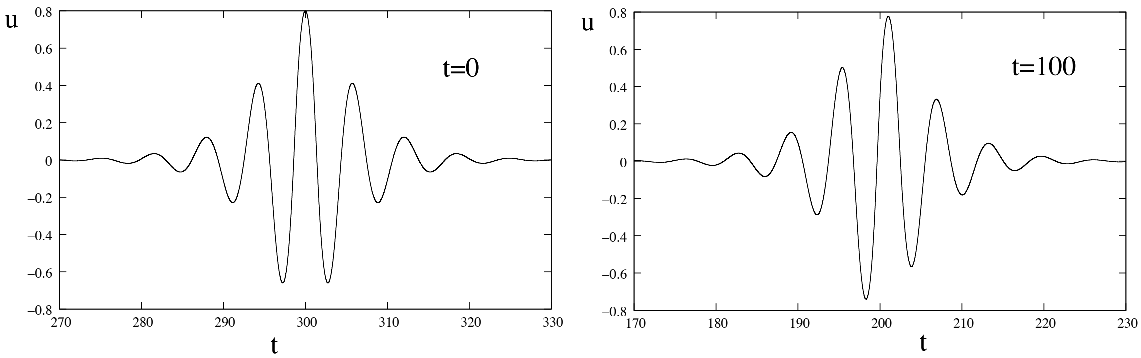

Figure 2: The evolution of the Sakovich' solution (with $m = 0.2$) for $t = 0$ and $t = 100$.

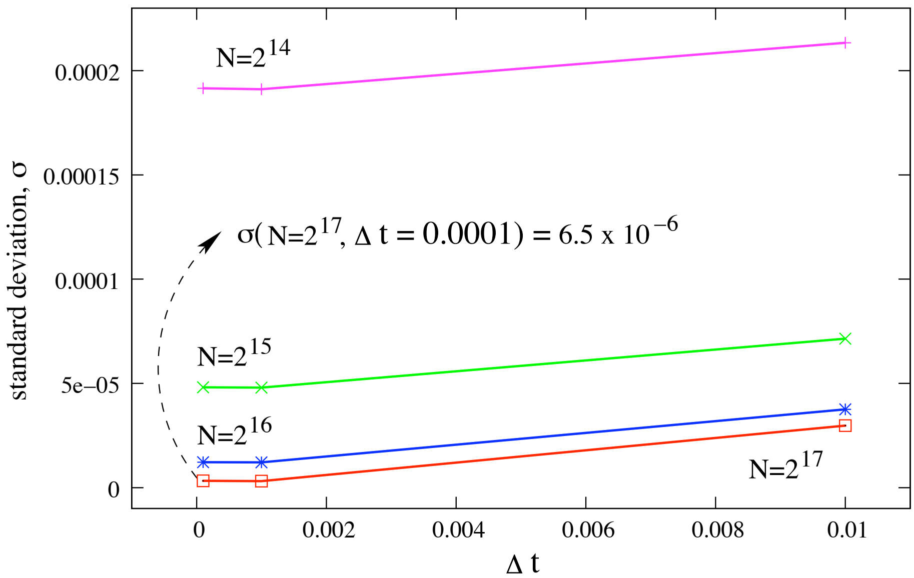

Figure 3: The dependence of the error of the multisymplectic integrator from Δt for different values of Δx.

method $\sigma \approx 7 \times 10^{-5}$. The CPU time required by the multisymplectic methods is 40000 sec, while the pseudo-spectral method requires $\approx$ 100000 sec (on 3GHz Pentium 4 PC).

In conclusion, we have used the multisymplectic formulation of SPE in order to construct the geometric multisymplectic integrator of SPE. We have compared the effectiveness of the corresponding numerical scheme with the pseudo-spectral method which uses the Runge-Kutta integration. The multisymplectic integration appears to be an order of magnitude more precise and approximately 2.5 times faster at long propagation times than the pseudo-spectral method. A comparison with the exact solution of SPE shows that the multisymplectic integration is stable and robust and preserves the energy functional.

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