

## ARE THERE SOLITON-LIKE SOLUTIONS OF COUPLED (1+1)- AND (2+1)-DIM NSE?

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We consider two coupled nonlinear Schrödinger equations, the (1+1) and the (2+1)-dimensional and concentrate basically on the question as to whether there exists a stable, self-trapped solution. The positive answer is obtained within the variational and the numerical method. Namely, it is observed that neither spreading nor catastrophic self-focusing can develop and an oscillating, self-trapped solution arises. Numerical results show, in contradiction to the variational ones, that amplitudes of those oscillations decrease with propagation distance and for sufficiently large distances they vanish to zero.

In this paper we consider two coupled nonlinear Schrödinger equations (NSE), the (2+1)- and the (1+1)-dimensional,

$$i \frac{\partial}{\partial \zeta} \Psi_1 + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} \Psi_1 + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \Psi_1 + (|\Psi_1|^2 + 2|\Psi_2|^2) \Psi_1 = 0, \quad (1a)$$

$$i \frac{\partial}{\partial \zeta} \Psi_2 + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \Psi_2 + (|\Psi_2|^2 + 2|\Psi_1|^2) \Psi_2 = 0, \quad (1b)$$

where  $\zeta$  is the longitudinal coordinate,  $\tau$  and  $\xi$  are two transverse coordinates, the temporal and the spatial. Eqs. (1a,b) can model, e.g. simultaneous propagation of two optical pulses in a Kerr-type planar waveguide with the assumption that the temporal duration of the first pulse, whose evolution is described by (1a), is in the picosecond range and its dispersion is anomalous, while the duration of the second pulse is large, thus its dispersion is small and has been neglected in (1b) [1]. The interaction between pulses has been assumed to be limited to cross-phase modulation, a nonlinear effect through which the phase of one pulse is affected by another pulse and which can cause a redistribution of energy within both pulses. Another effect, four-wave mixing, has been neglected, so that no energy transfer between the pulses is taken into consideration. Throughout the

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paper the pulse whose duration is assumed to be short (long) is referred to as the *short* (*long*) pulse.

In this paper we study solutions of Eqs. (1a,b) with the aid of the variational method and numerical simulations (using the Split-Step spectral method) and compare both results. As the ansatz in the variational method we take the Gaussian function:

$$\Psi_j(\zeta, \tau, \xi) = \sqrt{\kappa_j(\zeta)} \exp\left[-\frac{\tau^2(1 + iC_{\tau j}(\zeta))}{2w_{\tau j}(\zeta)}\right] \exp\left[-\frac{\xi^2(1 + iC_{\xi j}(\zeta))}{2w_{\xi j}(\zeta)}\right] e^{i\phi_j}, \quad (2)$$

which depends on 12 parameters, namely the temporal (spatial) width,  $w_{\tau j}$  ( $w_{\xi j}$ ), the temporal (spatial) chirp,  $C_{\tau j}$  ( $C_{\xi j}$ ), the amplitude,  $\kappa_j$  and the phase,  $\phi_j$ , of the  $j$ -th pulse,  $j = 1, 2$ . As the initial condition we take the Gaussian function given by (2) with the following parameters:  $w_{\tau j}(0) = w_{\xi j}(0) = 1$ ,  $C_{\tau j}(0) = C_{\xi j}(0) = 0$ ,  $\phi_j(0) = 0$ , where  $j = 1, 2$ . The amplitudes  $\kappa_j(0)$ ,  $j = 1, 2$  are varied in the analysis.

The variational ordinary differential equations (ODE) for the 12 parameters of the ansatz (2) have been derived already in [1, 2]. Here we rewrite only those ODE's which describe evolution of the temporal and spatial widths of the pulses:

$$\frac{d^2 w_{\tau 1}}{d\zeta^2} = \frac{1}{w_{\tau 1}^3} - \frac{1}{2} \frac{I_1}{w_{\tau 1}^2 w_{\xi 1}} - \frac{4I_2 w_{\tau 1}}{(w_{\tau 1}^2 + w_{\tau 2}^2)^{\frac{3}{2}} (w_{\xi 1}^2 + w_{\xi 2}^2)^{\frac{1}{2}}}, \quad (3a)$$

$$\frac{d^2 w_{\xi 1}}{d\zeta^2} = \frac{1}{w_{\xi 1}^3} - \frac{1}{2} \frac{I_1}{w_{\tau 1} w_{\xi 1}^2} - \frac{4I_2 w_{\xi 1}}{(w_{\tau 1}^2 + w_{\tau 2}^2)^{\frac{1}{2}} (w_{\xi 1}^2 + w_{\xi 2}^2)^{\frac{3}{2}}}, \quad (3b)$$

$$\frac{d^2 w_{\tau 2}}{d\zeta^2} = 0, \quad (3c)$$

$$\frac{d^2 w_{\xi 2}}{d\zeta^2} = \frac{1}{w_{\xi 2}^3} - \frac{1}{2} \frac{I_2 r}{w_{\tau 2} w_{\xi 2}^2} - \frac{4I_1 w_{\xi 2}}{(w_{\tau 1}^2 + w_{\tau 2}^2)^{\frac{1}{2}} (w_{\xi 1}^2 + w_{\xi 2}^2)^{\frac{3}{2}}}, \quad (3d)$$

where  $I_j := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_j(\zeta, \tau, \xi) d\tau d\xi = \kappa_j(0)$  is a constant of motion and represents the energy of the  $j$ -th pulse.

Let us consider first solutions of the (2+1)- and the (1+1)-dimensional NSE in the absence of coupling between them. Concerning the (2+1)-dimensional NSE (Eq. (1a) without the last term) it is known that stable, soliton-like solutions do not exist. Thus, depending on parameters of the system (in our case energy of the short pulse,  $\kappa_1$ ), either spatio-temporal spreading (for  $\kappa_1 < \kappa_{\text{th}}$ ) or catastrophic self-focusing occurring simultaneously in space and time (for  $\kappa_1 > \kappa_{\text{th}}$ ) develops;  $\kappa_{\text{th}}$  is the threshold of catastrophic self-focusing, which is usually computed with the aid of the method of moments [4, 5], the variational method [6], and also numerical simulations [7]. From the point of view of analytical estimations catastrophic self-focusing is identified with development of a singularity in the solution at a finite distance of propagation, (in the variational method it is equivalent to vanishing of both widths of a pulse). In numerical simulations catastrophic self-focusing can be identified with a discontinuity of the phase  $\phi(\tau, \xi, \zeta)$  of the amplitude  $\Psi := |\Psi|e^{i\phi}$  in the central point of the transverse plane,  $\tau = 0$ ,  $\xi = 0$ , and with non-monotonic behaviour of the intensity  $|\Psi|^2$  in the central point after catastrophic

self-focusing has been reached [8]. The threshold of catastrophic self-focusing given by the numerical analysis,  $\kappa_{\text{thN}} \approx 1.885$ , [2] is lower than the one given by the analytical estimations,  $\kappa_{\text{thV}} = 2$  [6].

The (1+1)-dim NSE (Eq. (1b) without the last term) depends only on one transverse variable  $\xi$ , and being an integrable system possesses a family of sech-shaped soliton solution [9]. From the variational method with the ansatz and the initial condition given by Gaussian function, which depends on two transverse variables,  $\tau$  and  $\xi$ , it follows that the temporal width of the pulse is constant while the spatial one oscillates. These oscillations are due to the fact that the shape of the Gaussian trial function differs from the exact soliton solution [10]. However, numerical simulations lead to a different behaviour; namely, the temporal width of the pulse appears to oscillate synchronically with the spatial width. Amplitudes of both oscillations decrease with the longitudinal variable  $\zeta$  and vanish at finite  $\zeta$  when the spatial soliton is formed [11].

Now, let us take into account the nonlinear coupling between Eqs. (1a) and (1b), i.e., the (2+1)- and the (1+1)-dimensional NSE and concentrate basically on the question as to whether there exists a stable, self-trapped solution. An analysis of simultaneous propagation of two optical pulses in a Kerr-type planar waveguide modelled by equations similar to (1a,b) has been already presented in [1, 2, 3], where it was assumed that dispersion of the long pulse is, in general, not negligible and belongs to the normal regime. In [3] the problem of spatio-temporal compression of the short pulse was investigated, while in [2] the aspect of catastrophic self-focusing, spatio-temporal splitting, and in the limiting case of vanishing dispersion of the long pulse, the possibility of formation of a self-trapped solution were studied. In the last paper it was shown, using the variational method, that the presence of coupling between the pulses can cause, among others, (i) an arresting of catastrophic self-focusing of the short pulse, or (ii) catastrophic self-focusing of the long pulse, which occurs when its spatial width vanishes to zero while its temporal width remains larger than zero on the whole propagation distance,  $\zeta$ . Regarding the case of vanishing dispersion of the long pulse, it was demonstrated, based on the variational results, that (iii) catastrophic self-focusing of both pulses does not develop until the parameters of the system are such that catastrophic self-focusing of the short pulse does not occur when it propagates as a single pulse, i.e., the condition  $\kappa_1 < 2.0$  is satisfied. This statement concerns also the cases when the energy of the long pulse is very large, e.g.  $\kappa_2 = 20$ , and when energy of the short pulse is just below the threshold of catastrophic self-focusing, e.g.  $\kappa_1 = 1.999$ . From the variational method it follows also that the evolution of the long pulse coupled to the short one is essentially similar to that of the single long pulse; namely, the temporal width does not depend on the longitudinal variable  $\zeta$ , as it is seen from Eq. (3c) while the spatial width undergoes periodic oscillations (see Fig. 5 b in [2] and also Figs. 1 b,d in this paper). The propagation of the short pulse coupled to the long one is, however, qualitatively different as compared to the behaviour of a single short pulse, namely, both temporal and spatial widths of the pulse undergo periodic oscillations (see Fig. 5a in [2] and Figs. 1 a,c in this paper). It was then concluded that (iv) neither spatio-temporal spreading nor catastrophic self-focusing of the short pulse can develop and a self-trapped solution arises.

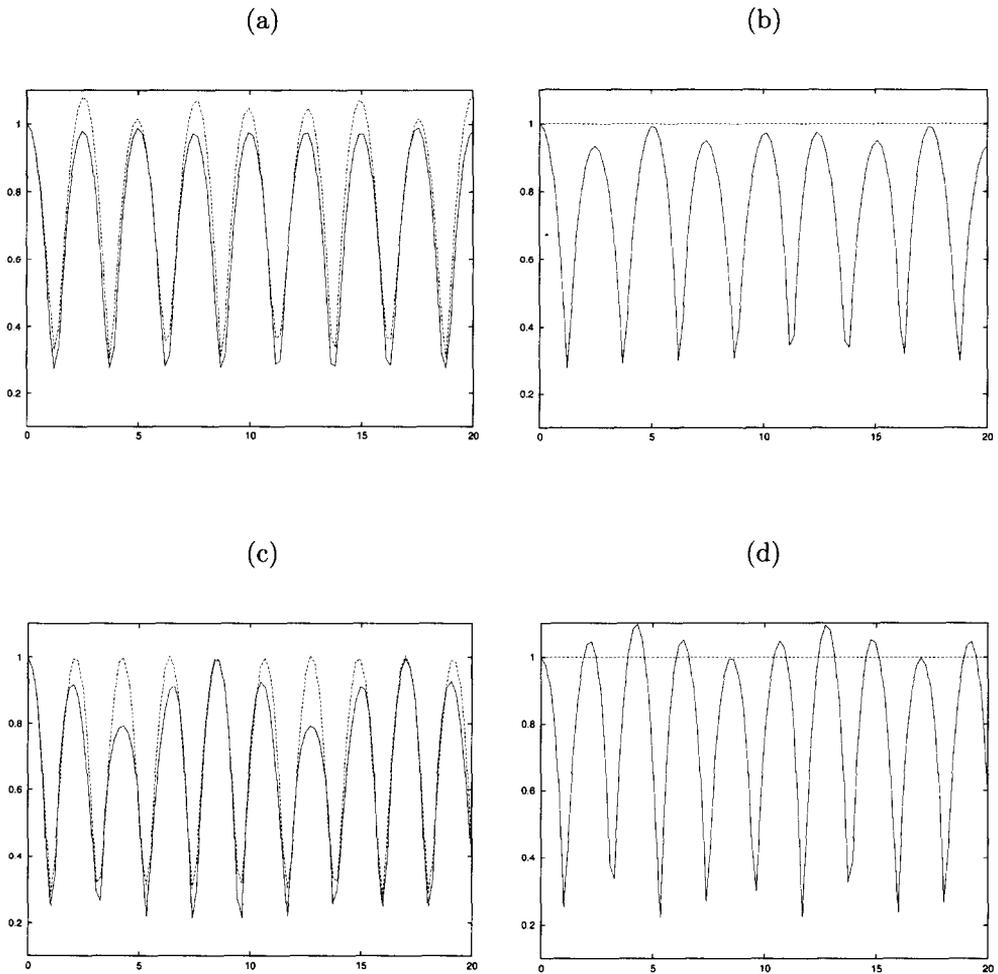


Fig. 1: The results obtained using the variational method displaying the dependence of the temporal,  $w_\tau$ , (the dotted lines) and spatial,  $w_\xi$ , (the continuous lines) widths of the short pulse, (a) or (c), co-propagating with the long pulse, (b) or (d). For (a) and (b)  $\kappa_1 = 1.0$ ,  $\kappa_2 = 1.5$ , for (c) and (d)  $\kappa_1 = 1.0$ ,  $\kappa_2 = 2.0$ .

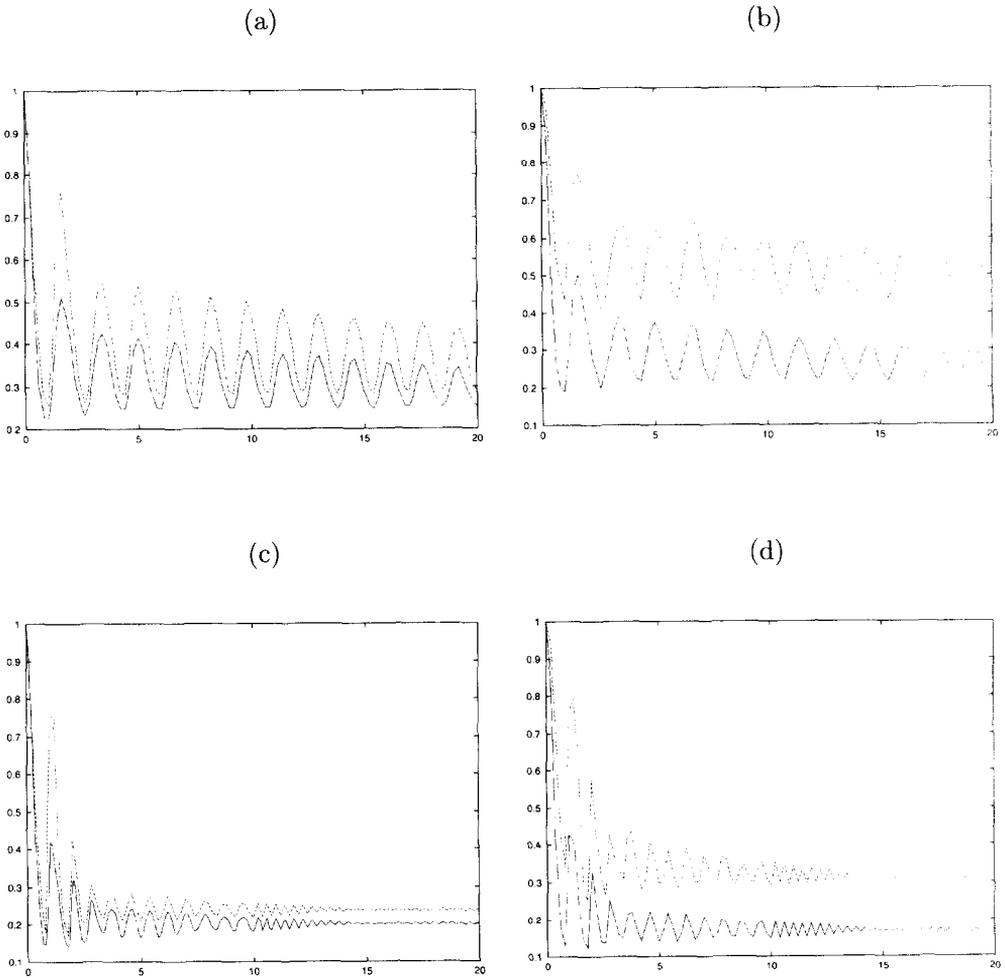


Fig. 2: The results obtained using the numerical simulations displaying the dependence of the temporal  $w_\tau$  (the dotted lines) and spatial  $w_\xi$  (the continuous lines) widths of the short pulse, (a) or (c), co-propagating with the long pulse, (b) or (d). For (a) and (b)  $\kappa_1 = 1.0$ ,  $\kappa_2 = 1.5$ , for (c) and (d)  $\kappa_1 = 1.0$ ,  $\kappa_2 = 2.0$ .

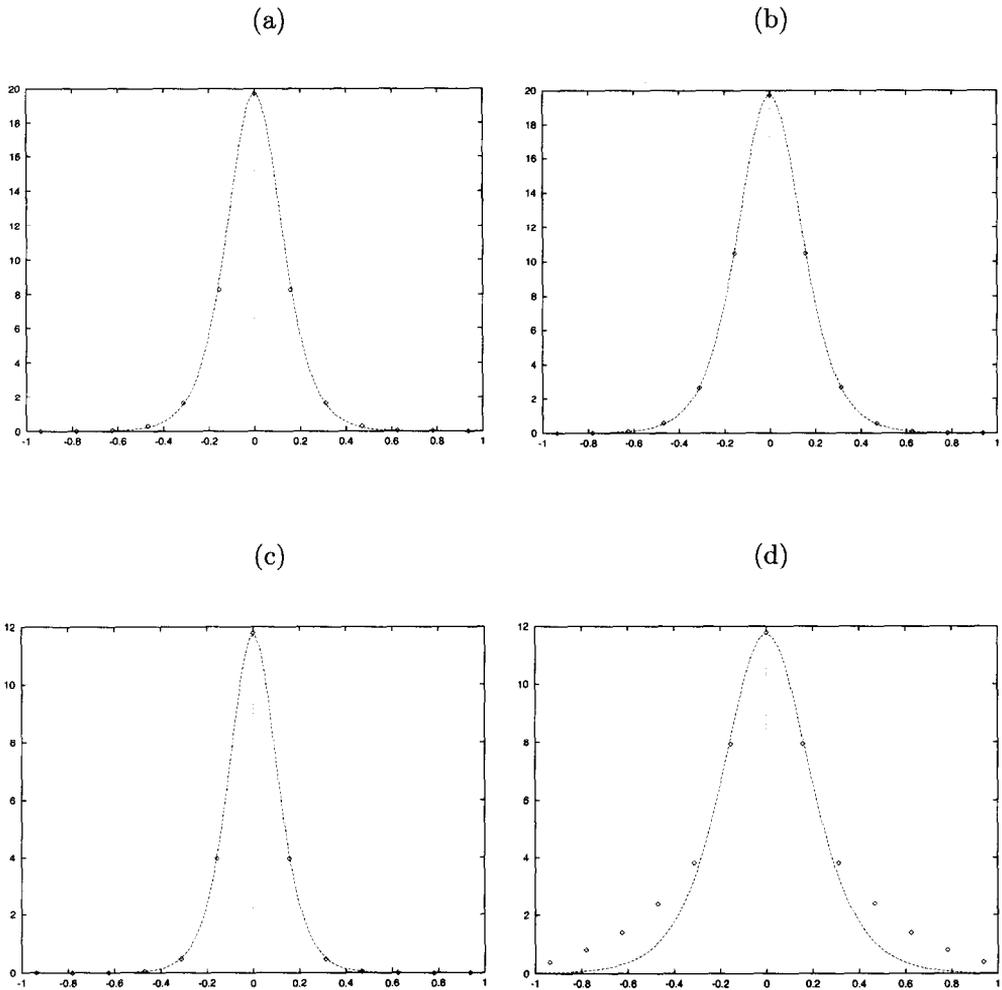


Fig. 3: The transverse spatial (a) and the temporal (b) cross-section of the short pulse propagating simultaneously with the long pulse for the propagation distance  $\zeta = 10$ . The transverse spatial and the temporal cross-section of the long pulse are shown, respectively, in Figs. (c) and (d). The parameters of the system correspond to Figs. 2 c, d, i.e.  $\kappa_1 = 1$  and  $\kappa_2 = 2.0$ . Square points on figures denote results of numerical simulations while curves represent the given  $\text{asech}(\beta x)$  function with the following parameters:  $\alpha = 19.75$ ,  $\beta = 6.2$  (a),  $\alpha = 19.75$ ,  $\beta = 5.3$  (b),  $\alpha = 11.75$ ,  $\beta = 7.2$  (c), and finally  $\alpha = 11.75$ ,  $\beta = 4.0$  (d).

The results of numerical simulations presented in [2] confirmed the statement given by item (i) which has been obtained with the aid of the variational method. However, since numerical simulations of the NSE with higher than one number of transverse variables are rather laborious and require large CPU times, thus no definite conclusions regarding the cases (ii), (iii) were obtained in [2]. Concerning (iv), the available numerical results suggested that self-trapped solution could arise in the configuration under discussion, i.e. described by Eqs. (1a, b), for properly chosen parameters of the system.

In this paper we continue those investigations hoping to shed more light on the above mentioned problems. First of all, based on our numerical simulations, we observed that catastrophic self-focusing of the long pulse, which takes place when its spatial width vanishes to zero while its temporal width remains larger than zero on the whole propagation distance, can be induced by the short pulse. We can conclude then, that the variational predictions concerning the problem represented by item (ii) were indeed correct. As an example of the parameters of the system, for which catastrophic self-focusing occurs simultaneously for both coupled pulses, let us take energy of the long pulse sufficiently large and energy of the short pulse below the threshold of catastrophic self-focusing, e.g.  $\kappa_1 = 1.88 < \kappa_{\text{thv}} = 1.885$  and  $\kappa_2 = 1.5$ . Similar situation occurs, e.g. in the case of two coupled (2+1)-dimensional NSE, namely it is possible to find such parameters of the system for which catastrophic self focusing cannot develop when the pulses propagate separately, but it occurs when they propagate simultaneously [12]. Note that this example explore, at the same time, the variational predictions given by (iii).

Let us return finally to the main problem of this paper, namely the possibility of formation of a soliton-like solution of two coupled NSEs, the (2+1)- and the (1+1)-dimensional. For this, let us examine first Fig. 2, where the evolution of the temporal and the spatial widths of both pulses has been displayed for two different parameters of the system:  $\kappa_1 = 1.0, \kappa_2 = 1.5$  and  $\kappa_1 = 1.0, \kappa_2 = 2.0$ . From Fig. 1, presenting the results of the variational method, and from Fig. 2, where the results of the numerical simulations are displayed, it is evident that for both parameters of the system considered here synchronous oscillations of the widths of the pulses occur. The amplitudes of the temporal oscillations are larger than the amplitudes of the spatial ones, the period of the oscillations seems to increase with increasing energy of the long pulse,  $\kappa_2$ . Moreover, as follows from the numerical results, the amplitudes of those oscillations decrease with the propagation distance  $\zeta$ . This process is much faster for the case considered in Figs. 2c, d than for the example illustrated in Figs. 2a, b. In the first case, for sufficiently large propagation distance,  $\zeta > 15$ , the amplitudes of the oscillations of the widths of both pulses vanish practically to zero. For the second case oscillations are still significant for  $\zeta = 20$  (the available numerical results do not allow us to determine whether they can vanish for a sufficiently large propagation distance; let us recall that for the (1+1)-dimensional NSE similar oscillations disappear for the propagation distance  $\zeta \approx 100$ ).

Let us examine also the spatial and the temporal cross-sections of the pulses, which are displayed in Fig. 3. It is evident that all of them, excluding the temporal width of the long pulse, can be well approximated by the sech functions.

To sum up, we have shown that self-trapped solutions of two coupled NSEs, the (2+1)- and the (1+1)-dimensional, can exist. They arise when the parameters of the

system are properly chosen (energy of the pulse whose evolution is described by the (2+1)-dimensional NSE is below the threshold of catastrophic self-focusing, energy of the pulse whose evolution is described by the (2+1)-dimensional NSE is above the threshold of soliton generation). The evolution of two coupled pulses is similar to the evolution of the pulse modelled by both the (2+1)-dimensional NSE with saturation-type nonlinearity (analogous oscillations) [13] or by the (1+1)-dimensional NSE (analogous oscillations and the cross-sections of the pulses). We can conclude then that the (1+1)-dimensional NSE acts to stabilize the (2+1)-dimensional NSE, therefore neither spatio-temporal spreading nor catastrophic self-focusing occur, while in the case of the single (2+1)-dimensional NSE one of these effects would certainly develop. From the physical point of view this stabilization can be interpreted in the following way: a pulse, whose dynamics is described by the (1+1)-dimensional NSE creates a waveguide in the medium and the other pulse is trapped in this waveguide.

Still, it remains an open question whether the discussed above self-trapped solutions are stable against small perturbations. We hope to give an answer to this question in the nearest future.

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