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On the propagation of vector ultra-short pulses

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Abstract

A two component vector generalization of the Schäfer-Wayne short pulse equation is derived. It describes propagation of ultra-short pulses in optical fibers with Kerr nonlinearity beyond the slowly varying envelope approximation and takes into account the effects of anisotropy and polarization. We show that in a special case this system gives rise to three different integrable two-component short pulse equations which represent the counterpart of the Manakov system in the case of ultra-short pulses.

1 Introduction

The nonlinear propagation of optical pulses in Kerr media is usually described by nonlinear Schrödinger equation which is an integrable system in (1+1) dimensions. The derivation of this equation is based on the slowly varying envelope approximation [1]. However, this approximation is not valid for the ultra-short pulses whose temporal extent is less than just a few periods of the corresponding optical wave (see e.g. [2]). There is an increasing number of experiments and applications which involve such ultra-short pulses in femto-second atto-second domain [3].

The problem of mathematical description of the propagation ultra-short pulses has been addressed from different points of view which range from the consideration of the full Maxwell-Bloch system to the studies of the nonlinear Schrödinger equation with higher order corrections (see e.g. [4–7]).

In [8] Schäfer and Wayne have derived an equation which describes ultra-short infrared pulses in silica optical fibers. In properly normalized units the equation takes the form [11]

$$u_{zt} = u + \frac{1}{6}(u^3)_{tt}, \tag{1.1}$$

where $u(z,t)$ represents the magnitude of the electric field. Following [11] we will refer to it as the *Short Pulse Equation* (SPE). SPE represents the opposite extreme from the slowly varying envelope approximation: as the pulse duration shortens, the description using the nonlinear Schrödinger equation becomes less accurate, while the SPE provides increasingly better approximation to the corresponding solution of the Maxwell equations. Sakovich & Sakovich [11] and Brunelli [12] have shown using different methods that this equation is integrable. Moreover, Sakovich & Sakovich [13] have constructed a first analytical solution of (1) which possibly represents ultra-short pulses.

However, the single-component SPE neglects the fact that all single mode optical fibers actually support two orthogonally polarized modes. It is only in perfectly isotropic fibers that the polarization modes are completely degenerate and the treatment in terms of the single-component equations is justified. In reality optical fibers are supporting two orthogonally polarized modes with different propagation constants, i.e the fibers are birefringent. The birefringence and its interplay with nonlinearity may have strong influence on propagation of optical pulses along the fiber. Moreover, the dynamics of polarization in specially fabricated anisotropic fibers and microstructured or photonic crystal fibers [14] is also interesting from the point of view of applications, e.g. to polarimetric sensors [15] or in soliton computing [16].

The standard description of optical pulses in birefringent fibers in the slowly varying envelope approximation is achieved by means of the pair of coupled nonlinear Schrödinger equations [17], also known as the vector nonlinear Schrödinger equation [18, 19]. A particular case of this system is the well known integrable Manakov system [19]. The vector nonlinear Schrödinger equation also appears in many other different contexts in nonlinear optics, see e.g. [20–22].

In this paper we consider the nonlinear propagation of ultra-short pulses with two orthogonally polarized components. In Section 2 we derive a system of two coupled short-pulse equations. This system plays the same role in the dynamics of ultra-short pulses as the vector non-linear Schrödinger equation does in the case of broader pulses which can be described by the slowly varying envelope approximation. In Section 3 we show that in a particular case this system gives rise to three different integrable two-component systems which can be viewed as an ultra-short pulse analogue of the Manakov system.

2 Vector short-pulse equation

We consider a propagation of two orthogonally polarized modes in the anisotropic fiber along the direction z

$$E = E_i \mathbf{e}_i = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2, \tag{2.1}$$

where $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1 = \mathbf{e}_2 \cdot \mathbf{e}_2, \mathbf{e}_1 \cdot \mathbf{e}_2 = 0$. The starting point is Maxwell's equation ($c = 1$) for the wave propagating in z-direction

$$E_{zz} - E_{tt} = P_{tt} \tag{2.2}$$

which is valid if the diffraction terms $\Delta_\perp E$ and the transverse inhomogeneities of the polarization $\nabla \mathrm{div} P$ are neglected. Here P is the polarization of the medium in response to the electric field. It has both linear and nonlinear contributions: $P = P^{ln} + P^{nl}$.

Assuming the medium is homogeneous and anisotropic and neglecting the spacial dispersion, the most general linear contribution to polarization is

$$P_i^{ln}(z,t) = \int_{-\infty}^{+\infty} d\tau \chi_{ij}^{(1)}(t-\tau) E_j(z,\tau) \tag{2.3}$$

which accounts for the retarded response of the medium if the causality is enforced by the condition $\chi_{ij}^{(1)}(t) = 0$ if $t < 0$. For the linear susceptibility we assume that the frequency range of the pulse under consideration and the pulse frequencies are much higher than the resonance frequencies. In this case the Fourier transformation of $\chi_{ij}^{(1)}(t-\tau)$ is given by

$$\tilde{\chi}_{ij}^{(1)}(\omega) \approx -\chi_{ij}\omega^{-2}. \tag{2.4}$$

Substituting this expression to (1) we obtain in the linear approximation the equation

$$(E_i)_{zz} - (E_i)_{tt} = \chi_{ij} E_j. \tag{2.5}$$

A rigorous derivaton of the isotropic version of (5) is found in Proposition 2.4 in [9].

Next, we turn to the nonlinear contribution to polarizability, P^{nl}. We restrict our attention to the centrosymmetric materials, so that there is no quadratic nonlinearity (c.f. [23]) and the lowest order nonlinearity is cubic:

$$P_i^{nl}(z,t) = \int d\tau_1 d\tau_2 d\tau_3 \, \chi_{ijkl}^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3) E_j(z,\tau_1) E_k(z,\tau_2) E_l(z,\tau_3). \tag{2.6}$$

We shall take into account only the instantaneous nonlinear response

$$\chi_{ijkl}^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3) = \chi_{ijkl}\, \delta(t-\tau_1)\delta(t-\tau_2)\delta(t-\tau_3). \tag{2.7}$$

Though the effects of delay in nonlinear response of the medium can be a part of the ultra-short pulse dynamics [6], the instantaneous contribution is expected to dominate in the case of very short, small amplitude pulses [8]. In this approximation Eq. (3) reduces to

$$(E_i)_{zz} - (E_i)_{tt} = \chi_{ij} E_j + \chi_{ijkl} \, (E_j E_k E_l)_{tt}. \tag{2.8}$$

The solutions of the linear part of this equation split into forward- and backward-propagating wave packets and the nonlinear term may generate interaction between them. However, in the case of very short pulses this interaction can be neglected. In order to incorporate the effects of the nonlinear and dispersive terms in (9) we make a multiple scale Ansatz [24]

$$E_i(z,t) = \epsilon U_i^{(0)}(\zeta, z_1, z_2, ...) + \epsilon^2 U_i^{(1)}(\zeta, z_1, z_2, ...) \tag{2.9}$$

with

$$\zeta := \frac{t-z}{\epsilon}, \quad z_n := \epsilon^n z. \tag{2.10}$$

At $z = 0$ this Ansatz reduces to

$$E(0,t) = \epsilon U_0(t/\epsilon) + \epsilon^2 U_1(t/\epsilon) + ...,$$

which represents a short pulse at small ϵ.

We insert (10) into (9) and find that the chosen form of the multiple scale Ansatz cancels the terms of the order $\mathcal{O}(\frac{1}{\epsilon})$ and that there are no terms $\mathcal{O}(\epsilon^0)$. Hence, in the leading nontrivial order $\mathcal{O}(\epsilon)$ we obtain

$$-2\partial_{z_1}\partial_\zeta U_i^{(0)} = \chi_{ij}U_j^{(0)} + \chi_{ijkl}\partial_{\zeta\zeta}(U_j^{(0)}U_k^{(0)}U_l^{(0)}). \tag{2.11}$$

This system describes the unidirectional propagation of ultra-short pulses with orthogonal polarization components in a general anisotropic fiber whose material is characterized by linear and third-order nonlinear susceptibility coefficients χ_{ij} and χ_{ijkl}.

Let us consider now the birefringent optical fibers characterized by anisotropic linear susceptibility and isotropic nonlinear susceptibility. Then the nonlinear polarization consistent with the underlying spatial symmetries has the form [25]

$$P_i = \gamma(E_1^2 + E_2^2)E_i \tag{2.12}$$

which implies

$$\gamma = \chi_{1111} = \chi_{2222} = \chi_{1122} + \chi_{1212} + \chi_{1221} = \chi_{2112} + \chi_{2121} + \chi_{2211}. \tag{2.13}$$

Assuming that the linear susceptibility is homogeneous along the z direction, we transform (12) to the eigenbasis of χ_{ij}. Denoting the modes along the eigendirections of the linear susceptibility as A and B and the corresponding eigenvalues of $\frac{1}{2}\chi_{ij}$ as a and b, we obtain the following system of coupled SPE-s

$$\begin{aligned} A_{z_1\zeta} &= aA + \frac{\gamma}{2}(A^3 + B^2A)_{\zeta\zeta}, \\ B_{z_1\zeta} &= bB + \frac{\gamma}{2}(B^3 + A^2B)_{\zeta\zeta}. \end{aligned} \tag{2.14}$$

3 The integrable interaction of ultra-short pulses

In the previous section we have obtained a two-component generalization of the short-pulse equation of Schäfer and Wayne [8]. The single-component SPE has been proved to be integrable by Sakovich & Sakovich [11] and Brunelli [12]. The proof by Sakovich & Sakovich is based on the explicit construction of the zero-curvature representation of SPE. In this section we construct an integrable system of two coupled SPEs by generalizing the results of Sakovich & Sakovich to the multicomponent case. For the sake of simplicity, in this section we denote the rescaled variables (z_1, ζ) as (z, t).

Let us consider a linear system

$$\Psi_t = T\Psi, \tag{3.1}$$

$$\Psi_z = Z\Psi, \tag{3.2}$$

with

$$T = \begin{pmatrix} \lambda & \lambda U_t \\ \lambda U_t & -\lambda \end{pmatrix} \tag{3.3}$$

and

$$Z = \begin{pmatrix} \frac{\lambda}{2}U^2 + \frac{1}{4\lambda} & \frac{\lambda}{6}(U^3)_{tt} - \frac{1}{2}U \\ \frac{\lambda}{6}(U^3)_{tt} + \frac{1}{2}U & -\frac{\lambda}{2}U^2 - \frac{1}{4\lambda} \end{pmatrix}, \tag{3.4}$$

where U is assumed to be a square matrix and λ is an arbitrary nonzero constant. The compatibility of (1) and (2), $\Psi_{zt} = \Psi_{tz}$, gives rise to the zero-curvature condition

$$T_z - Z_t + [T, Z] = 0. \tag{3.5}$$

Now, by direct calculation we obtain

$$T_z - Z_t = \begin{pmatrix} -\frac{\lambda}{2}(U^2)_z & \lambda U_{zt} - \frac{\lambda}{6}(U^3)_{ttt} + \frac{1}{2}U_t \\ \lambda U_{zt} - \frac{\lambda}{6}(U^3)_{ttt} - \frac{1}{2}U_t & \frac{\lambda}{2}(U^2)_t \end{pmatrix} \tag{3.6}$$

and

$$[T, Z] = \begin{pmatrix} \frac{\lambda^2}{6}V + \frac{\lambda}{2}(U^2)_t & \frac{\lambda^2}{6}W - \frac{1}{2}U_t - \lambda U \\ \frac{\lambda^2}{6}W + \frac{1}{2}U_t - \lambda U & \frac{\lambda^2}{6}V - \frac{\lambda}{2}(U^2)_t \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned} V &:= U_t(U^3)_t - (U^3)_t U_t, \\ W &:= \frac{1}{3}(U^3)_t - \frac{1}{2}U^2 U_t - \frac{1}{2}U_t U^2. \end{aligned} \tag{3.8}$$

Therefore, if $V = 0 = W$ the zero-curvature condition (5) is equivalent to

$$U_{zt} - U - \frac{1}{6}(U^3)_{tt} = 0 \tag{3.9}$$

which is a matrix generalization of SPE. The existence of the zero-curvature representation shows that this matrix generalization of SPE is integrable provided the conditions $V = 0$, $W = 0$ are fulfilled identically. These conditions restrict the choice of admissible matrix variables U.

Let us consider the case of 2×2 matrices

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If follows from (8) that the sufficient condition that $V = 0 = W$ is fulfilled can be taken in the form

$$UU_t - U_t U = 0.$$

In terms of the components of U we obtain

$$\begin{pmatrix} BC_t - B_tC & AB_t - A_tB + BD_t - B_tD \\ CA_t - C_tA + DC_t - D_tC & CB_t - BC_t \end{pmatrix} = 0. \tag{3.10}$$

This condition is fulfilled if, for example,

$$\text{(i)} \quad B = C = 0, \quad \text{i.e.} \quad U = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \tag{3.11}$$

Substituting (11) to (9) we obtain just a set of two uncoupled SPE-s for functions A and D.

The condition (10) can be also satisfied by taking

$$\text{(ii)} \quad B = C, \quad A = D, \quad \text{so that} \quad U = \begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{3.12}$$

The substitution of (12) to (9) yields a new system of two coupled SPE-s

$$\begin{aligned} A_{zt} &= A + \frac{1}{6}(A^3 + 3B^2A)_{tt}, \\ B_{zt} &= B + \frac{1}{6}(B^3 + 3A^2B)_{tt}. \end{aligned} \tag{3.13}$$

As it follows from the consideration in the previous section this equation describes an integrable interaction of two ultra-short pulses. It can be viewed as a short-pulse analogue of the Manakov equation [19]. The value of the cross-modulation coefficient $\beta = 3$ corresponding to the integrable system (13) is different from $\beta = 1$ for birefringent fibers made of nonlinearly isotropic material. Note, that the value $\beta = 3$ can in principle be realized in the materials with the point symmetry 432, $\bar{4}3m$ or $m3m$ [25]. However, in the non-centrosymmetric materials 432, $\bar{4}3m$ the effects of quadratic nonlinearities would dominate over those of cubic nonlinearity considered here.

Another possibility to satisfy (10) is to take

$$\text{(iii)} \quad B = -C, \quad A = D, \quad \text{so that} \quad U = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{3.14}$$

In this case the matrix system (9) yields

$$\begin{aligned} A_{zt} &= A + \frac{1}{6}(A^3 - 3B^2A)_{tt}, \\ B_{zt} &= B - \frac{1}{6}(B^3 - 3A^2B)_{tt}. \end{aligned} \tag{3.15}$$

The corresponding Kerr coefficients are compatible in principle with the symmetries of $\bar{1}$, $2/m$ and mmm materials [26].

The last possibility to satisfy (10) is to take

$$\text{(iv)} \quad A = D, \quad C = 0, \quad \text{so that} \quad U = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}. \tag{3.16}$$

Substituting (16) to (9) yields

$$A_{zt} = A + \frac{1}{6}(A^3)_{tt},$$
$$B_{zt} = B + \frac{1}{2}(A^2 B)_{tt}. \tag{3.17}$$

This system describes a propagation of small perturbation B on the background of the solution A. Note, that a nonlinear Schrödinger equation analogue of this system has been studied in the context of the so-called induced phase modulation [27].

4 Conclusions

We have extended the multiple scale analysis of the ultra-short pulse propagation in a one-dimensional non-resonant Kerr medium [8] by taking into account the polarization and anisotropy. It leads to a system of coupled SPEs. The two-component vector generalization of SPE describes propagation of two orthogonally polarized ultra-short pulses when the slowly varying approximation is not valid anymore.

The potential applicability of the vector SPE extends beyond the nonlinear fiber optics of ultra-short pulses. It can be used in all those situations where the coupled nonlinear Schrödinger equation has been proven to be useful, when the ultra-short pulses are used instead of the usual broader pulses described by the slowly varying envelope.

Using the zero-curvature representation of Sakovich & Sakovich [8] we have analized the integrability of the vector SPE by viewing it as a matrix generalization of the single component short pulse equation. This allowed us to construct the short pulse analogues of the integrable coupled nonlinear Schrödinger equations, i.e. the Manakov system [19]. We have briefly outlined the possible applicability of these equations to the propagation of ultra-short pulses in the fibers made of natural or artificial materials with specific symmetry groups.

The analysis of the solutions of the presented equations is beyond the scope of this paper. An analysis of the short-pulse equation from the point of view of the multisymplectic Hamiltonian formalism and the results of the numerical calculations will be presented in fothcomming publications, see e.g. [28].

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