Evaluation of uncertainty in measurements

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Introduction

- The aim of the measurement is to determine the measured value. Thus, the measurement begins with specifying the quantity to be measured, the method used for measurement (e.g. comparative, differential, etc.) and the measurement procedure (set of steps described in detail and applied while measuring with the selected measuring method).
- In general, the result of a measurement is only an approximation or estimate of the value of the specific quantity subject to measurement, that is, the **measurand**. Thus, the result of measurement is complete only when accompanied by a quantitative statement of its uncertainty.
- International Standard Organization (ISO) prepared „*Guide to the Expression of Uncertainty in Measurement*", which is definitive document describing norms and procedures in the measurements uncertainty evaluation. Based on the international ISO standard, Polish norm „*Wyrażanie niepewności pomiaru. Przewodnik"* was accepted in the 1999.

Sources of uncertainty in a measurement

- incomplete definition of the measurand;
- imperfect realization of the definition of the measurand;
- nonrepresentative sampling — the sample measured may not represent the defined measurand;
- inadequate knowledge of the effects of environmental conditions on the measurement or imperfect measurement of environmental conditions;
- personal bias in reading analogue instruments;
- finite instrument resolution or discrimination threshold;
- inexact values of measurement standards and reference materials;
- inexact values of constants and other parameters obtained from external sources and used in the data-reduction algorithm;
- approximations and assumptions incorporated in the measurement method and procedure;
- variations in repeated observations of the measurand under apparently identical conditions.

Types of measurements

- **Direct measurement** – measured quantity can be directly compared with the external standard, or the measurement is made using a single instrument giving result straightaway
  - series of measurements
  - gross error
- **Indirect measurement** – measuring one or more physical quantities to determine quantity dependent on them

Basic definitions (1)

- ***Measurement uncertainty*** -
  parameter associated with the result of measurement characterizing dispersion of the values attributed to the measured quantity

- **S*tandard uncertainty u(x)*** –
  the uncertainty of measurement expressed as a standard deviation.

Uncertainty can be reported in three different ways:

u, u(x) or *u*(*acceleration*), where quantity *x* can be expressed also in words (in the example *x* is *acceleration*).

Please note, that ***u* is a number**, not a function.

Direct measurements

Basic definitions (2)

- ***Type A evaluation of uncertainty*** – the evaluation of uncertainty by the statistical analysis of series of observations.
- Result of a series of measurements: mean value $$x \equiv \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
  - Assumptions:
    - Distribution function is symmetrical – probability for results smaller as well as bigger than mean value are the same
    - The bigger deviation from the mean value the lower probability
  - Result: for bigger number of measurements observed distribution of data points is similar to **Gauss function**
- Example of a Type A evaluation of uncertainty: the **standard deviation** of a series of independent observations can be calculated, or **least squares method** can be applied to fit the data with a curve and determine its parameters and their standard uncertainties.

Gauss distribution

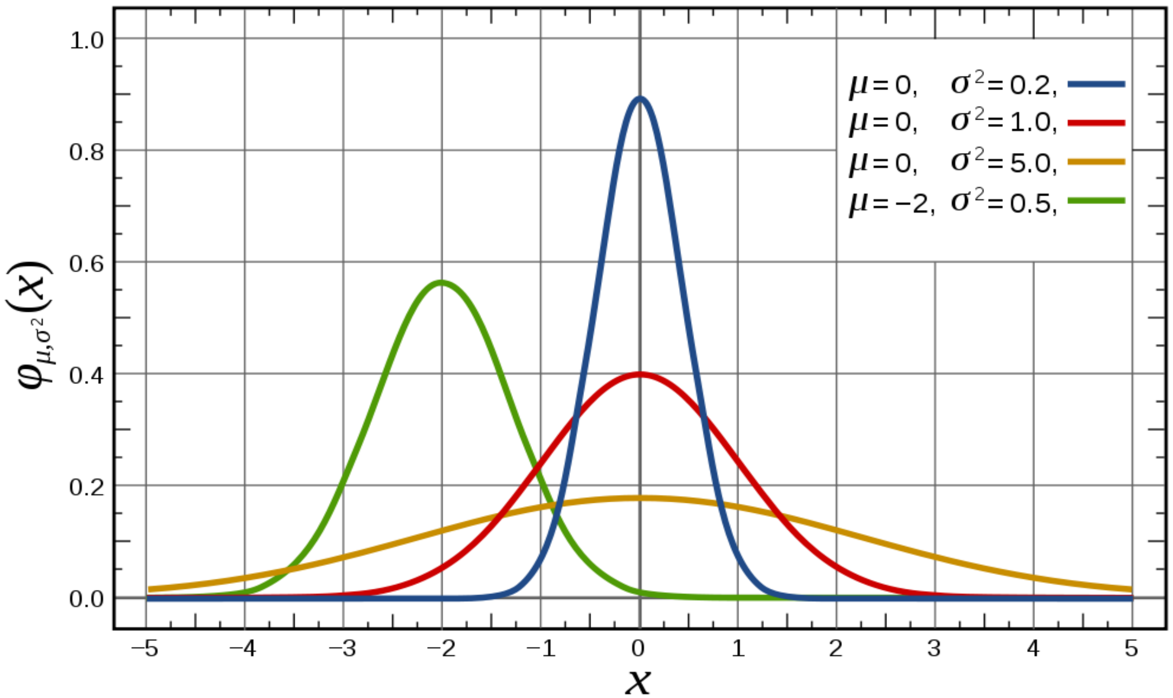
Distribution for continous variable x:

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\mu - x)^2}{2\sigma^2}\right)$$

μ – expected value

σ – standard deviation

$$\int_{-\infty}^{+\infty} \varphi(x)dx = 1$$



$$\int_{-\sigma}^{+\sigma} \varphi(x)dx = 0.683$$

$$\int_{-3\sigma}^{+3\sigma} \varphi(x)dx = 0.997$$

$$\int_{-2\sigma}^{+2\sigma} \varphi(x)dx = 0.954$$

Gauss distribution for finite number of points: **expected value is equal to mean value, standard deviation is equal to standard deviation of a mean value**

Type A standard uncertainty for a series of measurements is equal to standard deviation of a mean value

$$u(x) = \sigma = \sqrt{s_{\bar{x}}^2} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Basic definitions (3)

- **Type B evaluation of uncertainty** – the evaluation of uncertainty by means other than the statistical analysis of series of observations, thus using method other than in type A.

  Type B evaluation of standard uncertainty is usually based on scientific judgment based on experience and general knowledge, and is a skill that can be learned with practice.

- Assumption: uniform distribution – probability is constant in the whole interval determined by measurement and calibration uncertainty
  - calibration uncertainty (due to measurement device Δx)
  - investigator uncertainty (due to investigator's experimental skills Δx_e)

$$u(x) = \frac{\Delta x}{\sqrt{3}} = \sqrt{\frac{(\Delta x)^2}{3}}$$

- Combination of uncertainties

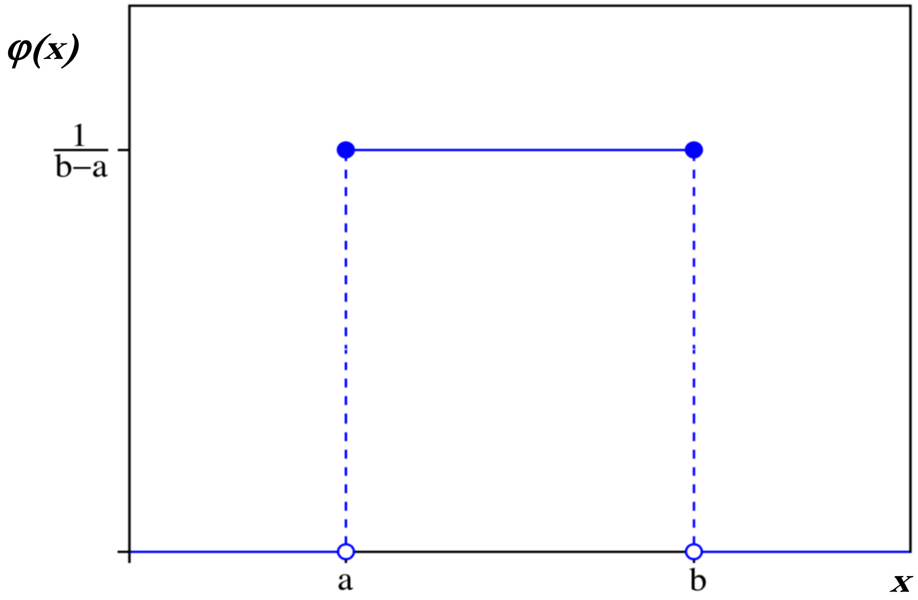
$$u(x) = \sqrt{s_{\bar{x}}^2 + \frac{(\Delta x)^2}{3} + \frac{(\Delta x_e)^2}{3}}$$

Uniform distribution

- Probability density in the interval a to b is constant and different from zero and equal to zero outside this interval
- Density probability function for uniform distribution:

$$\varphi(x) = \frac{1}{2\sigma\sqrt{3}} \qquad -\sigma\sqrt{3} \le x-\mu \le \sigma\sqrt{3}$$

$$\varphi(x) = 0 \qquad \textit{outside this range}$$



Type B standard uncertainty is equal to standard deviation

- Expected value: $\mu = \dfrac{a+b}{2}$
- Variance: $\sigma^2 = \dfrac{(b-a)^2}{12}$

$a = -\Delta x$

$b = \Delta x$

$$u(x) = \sqrt{\sigma^2} = \frac{\Delta x}{\sqrt{3}} = \sqrt{\frac{(\Delta x)^2}{3}}$$

Type B standard uncertainty (1) – mechanical devices

- Rulers, micrometers, calipers
- Thermometer, baromether
- Stopper
- Analoge devices

calibration uncertainty Δx:

half of the scale interval

$$u(x) = \frac{\Delta x}{\sqrt{3}}$$

Type B standard uncertainty (2) – analogue devices

Measurement range – maximal value to be measured for the set range.

Class of the instrument describes the precision of the measurement device in converting measured signal into value presented on a scale. Class describes uncertainty in the percentage of the measurement range.

Calibration uncertainty:

$$\Delta x = \frac{class \cdot range}{100}$$

$$u(x) = \frac{\Delta x}{\sqrt{3}}$$

Investigator uncertainty:

Δx_e can be estimated only by the investigator

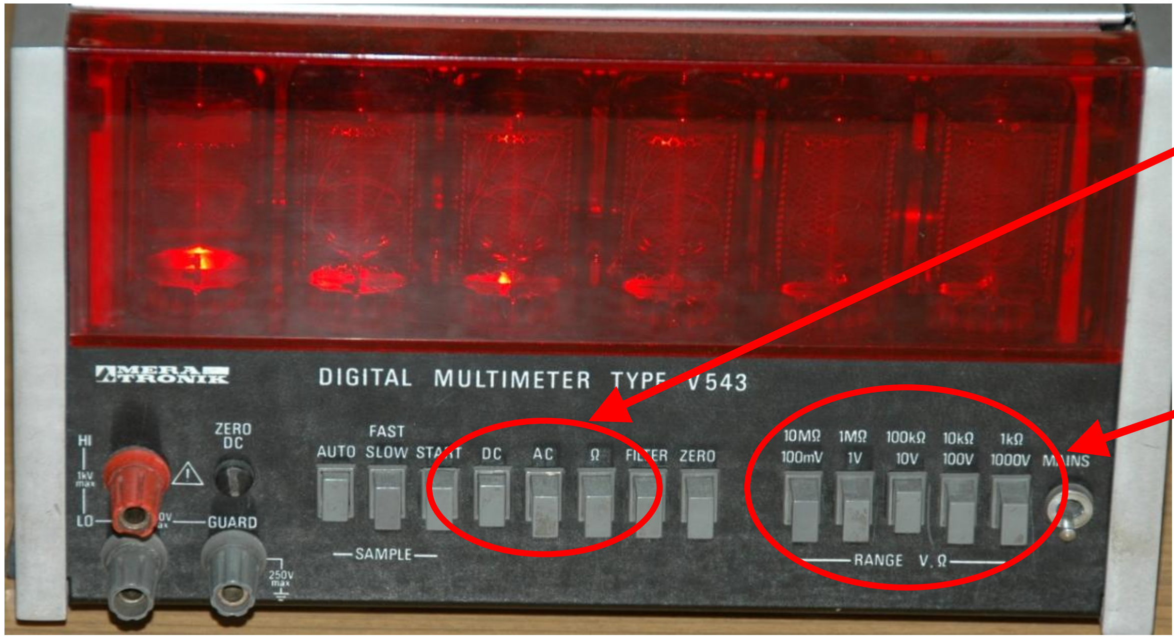
$$u(x) = \frac{\Delta x_e}{\sqrt{3}}$$

Type B standard uncertainty (3) – digital devices

- Measurement uncertainty for digital devices:
  - x – measured value
  - z – measurement range
  - c_1, c_2 – device constants e.g. $c_1 = 0.1\%$, $c_2 = 0.01\%$

$$\Delta x = c_1 x + c_2 z$$

$$u(x) = \frac{\Delta x}{\sqrt{3}}$$



Availabe functions

Measurement range

Uncertainty evaluation – direct measurements summary

- Perform measurement (single or series)

- Type A uncertainty
  - Measurement result – mean value
  - Standard uncertainty – standard deviation of the mean value

$$x \equiv \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$u(x) = \sqrt{s_{\bar{x}}^2} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- Type B uncertainty
  - Calibration uncertainty Δx
  - Investigator uncrtainty Δx_e

$$u(x) = \frac{\Delta x}{\sqrt{3}} = \sqrt{\frac{(\Delta x)^2}{3}}$$

- Combination of uncertainties

$$u(x) = \sqrt{s_{\bar{x}}^2 + \frac{(\Delta x)^2}{3} + \frac{(\Delta x_e)^2}{3}}$$

Indirect measurements

Basic definitions (4)

- ***Combined standard uncertainty $u_c(x)$*** – standard uncertainty of the value x calculated based on measurements of other quantities uncertainty propagation rule
  - Measurements of correlated quantities
  - Measurements of uncorrelated quantities

In the Physics Laboratory all measurements are uncorellated

Uncertainty evaluation – indirect measurement summary

- Measure k quantities x_i directly (single or series) ⇨ $z = f(x_1, x_2, \ldots, x_k)$
- Calculate mean value $\bar{x}_i$ and standard uncertainty $u(x_i)$ for every quantity using Type A or Type B evaluation method ⇨ $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ ⇨ $u(x_1), u(x_2), \ldots, u(x_k)$
- Calculate final value of studied quantity z ⇨ $z = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$
- Calculate combined uncertainty $u_c(z)$ (uncertainty propagation law) ⇨

$$u_c(z) = \sqrt{\sum_{j=1}^{k} \left(\frac{\partial f(x_j)}{\partial x_j} \right)^2 u^2(x_j)}$$

- Example for **two quantities**

$$u_c(z) = \sqrt{\left(\frac{\partial f(x, y)}{\partial x} \right)^2 u^2(x) + \left(\frac{\partial f(x, y)}{\partial y} \right)^2 u^2(y)}$$

Basic definitions (5)

- ***Expanded uncertainty U(x)*** or ***U_c(x)*** – the measure of uncertainty that defines interval about the measurement result, that may be expected to encompass a large fraction of the distribution
  - Standard uncertainty *u(x)* defines interval about the measured value, where the true value exist with probability:
    - 68% for Type A uncertainty
    - 58% for Type B uncertainty
  - Expanded uncertainty:
    - Allows to compare results from different laboratories
    - Allows to compare results with reference database or theoretical value
    - Useful for commercial purposes
    - Required for industry, health and security regulations

Basic definitions (6)

- ***Coverage factor*** *k* – number used to multiply standard uncertainty to calculate expanded uncertainty
  Typically k varies from 2 to 3.
  In the most cases in the Physics Laboratory $k = 2$ should be used.
  - Expanded uncertainty $U(x)$ defines interval about the measured value, where the true value exist with probability for $k = 2$:
    - 95% for Type A uncertainty
    - 100% for Type B uncertainty (100% also for k=1.73!)

$$U(x) = k \cdot u(x)$$

Reporting measurement results

Reporting measurement results (1)

- **Uncertainty is presented with accuracy (rounded) to two significant digits**
- **The measurement result (the most probable value) is presented with an accuracy specified by the uncertainty, which means that the last digit of the measurement result and the measurements uncertainty must be at the same decimal place.**
- **Rounding of uncertainties and measurement results follows the mathematical rules of rounding**

  - **Standard uncertainty**
    $t = 21.364$ s. $u(t) = 0.023$ s
    $\boldsymbol{t} = \mathbf{21.364(23)}$ **s, recommended notation**
    $t = 21.364(0.023)$ s

  - **Expanded uncertainty**
    $t = 21.364$ s. $U(t) = 0.046$ s ($k = 2$) $n = 11$ ⟵ not required
    $\boldsymbol{t} = \mathbf{(21.364 \pm 0.046)}$ **s, recommended notation**

Reporting measurement results (2) – examples

Measurement	**Proper Reporting**
$a = 321.735$ m/s; $u(a) = 0.24678$ m/s	$a = 321.74$ m/s; $u(a) = 0.25$ m/s $a = 321.74(0.25)$ m/s $a = 321.74(25)$ m/s
$b = 321785$ m; $u(b) = 1330$ m	$b = 321800$ m; $u(b) = 1300$ m $b = 321800(1300)$ m $b = 321.8(1.3)\cdot 10^3$ m $b = 321.8(13)$ km
$C = 0.0002210045$ F; $u_c(C) = 0.00000056$ F	C=0.00022100 F; $u_c(C)$=0.00000056 F $C = 221.00(0.56)\cdot 10^{-6}$ F $C = 221.00(56)\cdot 10^{-6}$ F $C = 221.00(56)$ μF
$T = 373.4213$ K; $u(T) = 2.3456$ K	$T = 373.4$ K; $u(T) = 2.3$ K $T = 373.4(23)$ K **$U(T)$ = 4.7 K** **T = (373.4 ± 4.7) K**

Hypothesis verification

Linear function hypothesis

- Graphical
- Least squares method
- Statistical tests

Graphical test

- The most simple
- Plot theoretical model function. It should cross uncertainties bars for more then **2/3** of experimental data points
- If not – hypothesis should be rejected

Least squares method (1)

- **Goal:** to verify the theoretical model dependence between measured quantities is valid
- **Assumption:** every model can be converted into linear type function **$y = a + b x$**
- **Method:** least squares – to find the line for which the sum of squared deviations of experimental points from this line is the smallest – to find line which is the closest to all experimental points
- **Results:** **a**, **b** and **uncertainty u(a)** and **uncertainty u(b)** (Type A standard uncertainty)

Least squares method (2)

y* = *ax* + *b

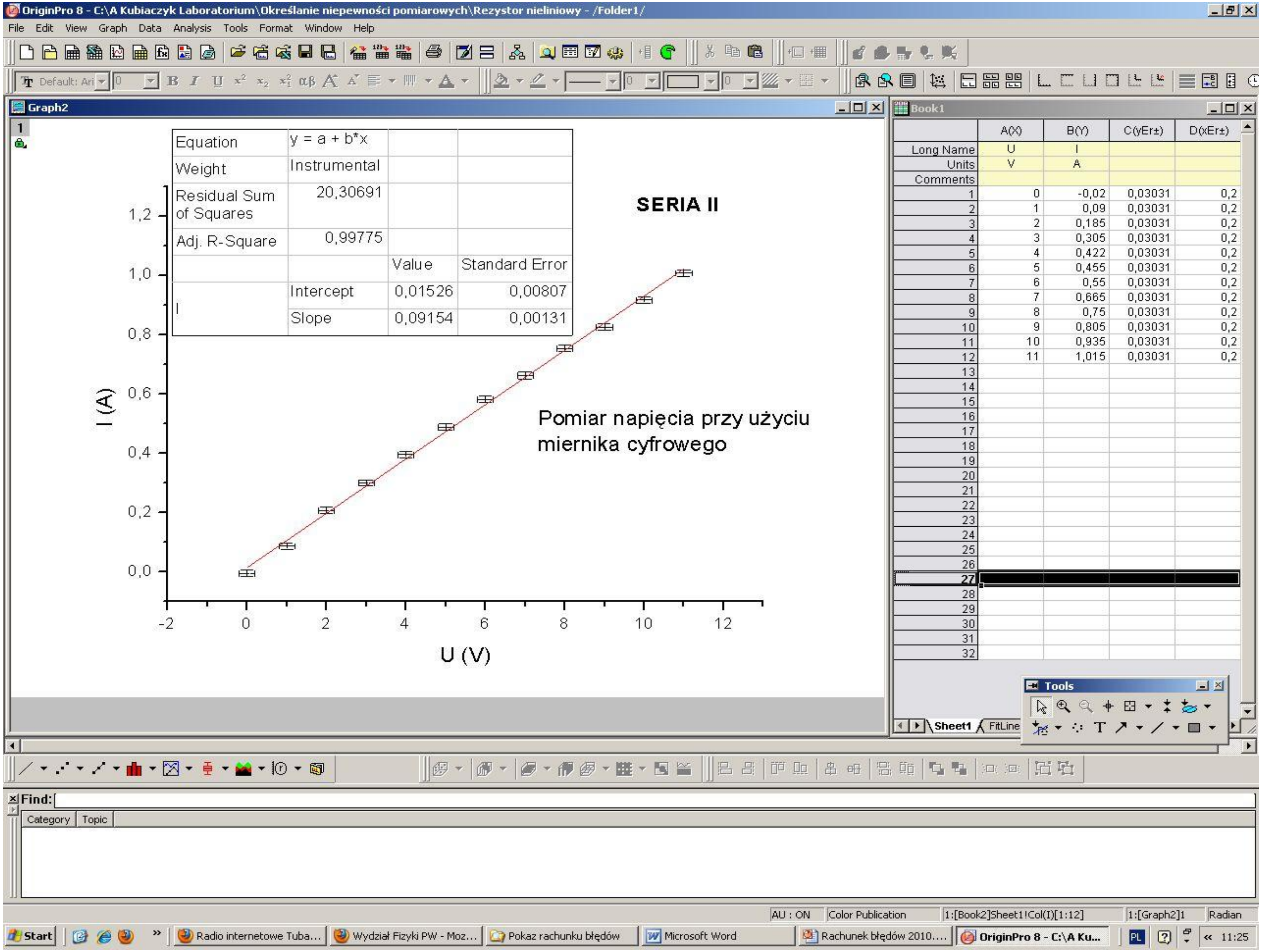
$$\tilde{x}_i = x_i - \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\bar{a} = \frac{\sum_{i=1}^{n} \tilde{x}_i y_i}{\sum_{i=1}^{n} \tilde{x}_i^2} \qquad \bar{b} = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{\bar{a}}{n}\sum_{i=1}^{n} x_i$$

$$\tilde{d}_i = y_i - \bar{a}\tilde{x}_i - \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$s_{\bar{a}} = \sqrt{\frac{1}{n-2}\frac{\sum_{i=1}^{n} \tilde{d}_i^2}{\sum_{i=1}^{n} \tilde{x}_i^2}} \qquad s_{\bar{b}} = s_{\bar{a}}\sqrt{\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i^2 + \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2}$$

Least squares method(3)



Test χ^2

- Test function χ^2
  - Definition
  - Statistical weight
  - Linear type function

$$\chi^2 = \sum_{i=1}^{n} w_i (y_i - y(x_i))^2$$

$$w_i = [u(y_i)]^{-2}$$

$$\chi^2 = \sum_{i=1}^{n} w_i (y_i - B(x_i) - A)^2$$

- **Significance value α** – probability of hypothesis rejection
  - Value in the range of 1 to 0
  - Determined by the investigator (typically **0.05**)
  - Depends on **number of degrees of freedom** (number of measurement points minus number of calculated parameters)

- **Critical value $\chi^2_{critical}$** (listed in the table for every significance value and number of degrees of freedom)

- Test function
  - **$\chi^2 \leq \chi^2_{critical}$ – there is no arguments to reject hypothesis**
  - **$\chi^2 > \chi^2_{kcritical}$ – hypothesis should be rejected**