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SIMPLE HARMONIC VIBRATIONS: REVERSIBLE PENDULUM AND TORSION PENDULUM

1. Fundamentals

One of the most widespread phenomena in nature is the phenomenon of vibrations. The main feature of oscillations is their periodicity. We can distinguish between two basic types of oscillatory movements:

a) when the same sequence of identical system states is repeated regularly at equal intervals (Fig. 1a) - non-extinguishing vibration.
b) When similar sequences of states of the system are repeated periodically, the value of the maximum deflection from the equilibrium position decreases (Fig. 1b) - fading vibrations.

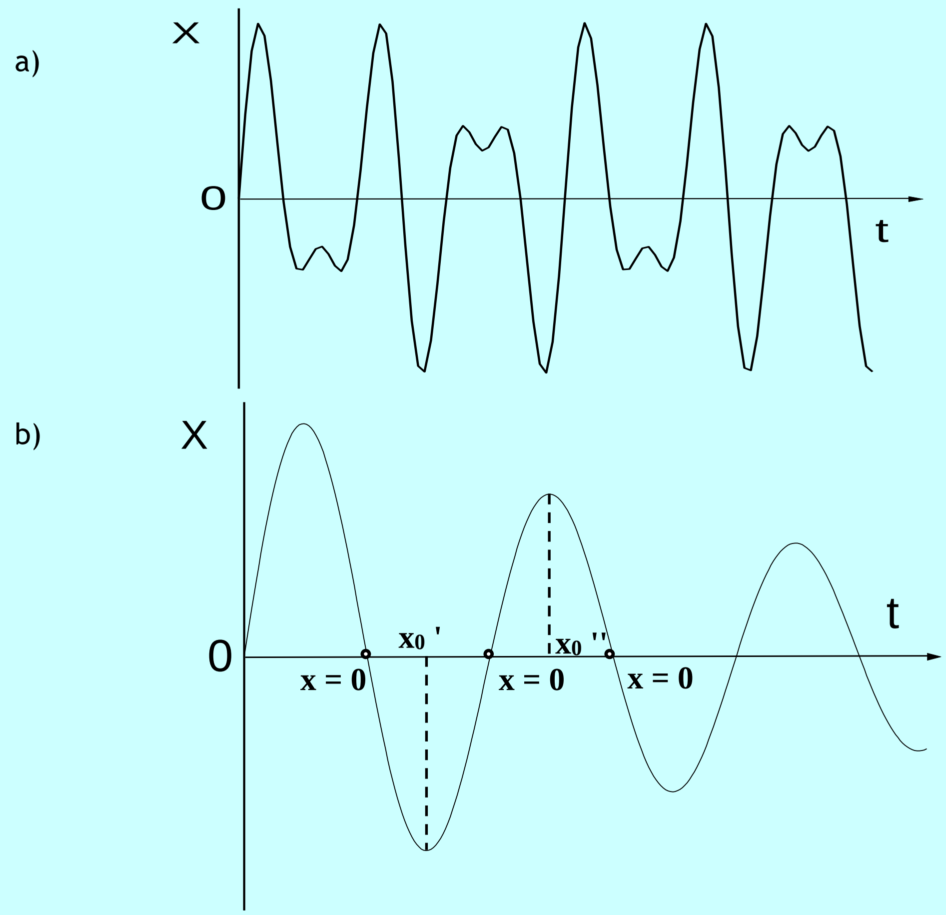

Fig.1 a) non extinguishable vibrations, b) fading vibration.

1.1. Harmonic movement

Among the numerous types of non-extinguishable vibrations, the simplest is **harmonic motion**. Let us assume that a body moved out of equilibrium is subjected to a force that causes the body to return to this state, i.e., it is directed to the equilibrium position, and this force is proportional to the deflection from this position. We may write this force as:

$$\vec{F} = -k\vec{x}\,, \tag{1}$$

where x is the deviation of the body from its equilibrium position, k is the coefficient of proportionality. Newton's second law of dynamics for a body of mass m has then the following form:

$$ma = -kx \tag{2}$$

Knowing that acceleration is the second derivative of position after time, we can rewrite equation (2) as:

$$m\frac{d^2x}{dt^2} = -kx \tag{3}$$

Dividing the above equation by *m* into both sides and substituting $\frac{k}{m} = \omega^2$, we obtain:

$$\frac{d^2x}{dt^2} = -\omega^2 x\,. \tag{4}$$

The resulting equation is called **the harmonic oscillator equation**, whose general solution is the function:

$$x(t) = A\sin(\omega t + \phi)\,, \tag{5}$$

where x - denotes the position of the body at time t (distance from the origin of the coordinate system assumed at the equilibrium position). *A* is called the oscillation amplitude (the system's maximum deviation from the equilibrium position), the argument of the sine function ($\omega t+\phi$) - **oscillation phase**, ϕ - **phase shift** or **initial phase**, and ω – **circular frequency**. The initial amplitude and the phase do not depend on the properties of the system but are determined by the so-called "initial conditions," i.e., the state of the system at time t = 0. Circular frequency ω depends on the system properties and does not depend on the oscillation amplitude.

We will now determine **the period of oscillation - T** in harmonic motion, i.e., the shortest time after which the deflection, velocity, and acceleration of motion take the same value. For this condition to be fulfilled, the phase of motion must change by 2π:

$A\sin(\omega t + \phi + 2\pi) = A\sin[\omega(t+T)+\phi]$, from $\omega T = 2\pi$, so:

$$T = \frac{2\pi}{\omega} \tag{6}$$

In general, a body can perform several vibrational motions simultaneously. The motion moves then are the resultant of all the component motions. If we consider only one oscillatory motion, we can choose the beginning of time counting so that the initial phase ϕ = 0 (at time t = 0, the deflection is zero), so $x(t) = A\sin(\omega t)$.

1.2. Physical gravity pendulum

A rigid body placed in the gravitational field suspended on a fixed horizontal axis that does not pass through its center of gravity forms a so-called **gravitational pendulum** (fig.2). When deflected from its equilibrium position, it oscillates about this position. Each of its points moves in a curve. Suppose the length connecting the solid's center of gravity - S - with the axis of rotation O is deflected by the angle α from a vertical line passing through the fixing point. In that case, a moment of gravity acts on the solid:

$$M = -mgd \cdot \sin\alpha \tag{7}$$

Where d - is the distance from the axis of rotation to the center of gravity. A minus sign means that this momentum produces a rotation opposite to the direction the angle α is measured. Using the Taylor series expansion of the sine function for small angles $\sin\alpha = \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{5} - \dots$, (angle α expressed in arc measure) and taking into account only the first term of the series, we obtain the equation:

$$M = -mgd\alpha = -D\alpha \qquad (8)$$

D = *mgd* is called **the steering moment** - the maximum value that the moment of force can take when trying to return a body to its equilibrium position.

The second law of dynamics for rotary motion has the form:

$$I\vec{\varepsilon} = \vec{M}\,, \qquad (9)$$

where $\varepsilon = \frac{d^2\alpha}{dt^2}$ is the angular acceleration of the body, and *I* is the moment of body inertia about the given axis of rotation ($I = \sum_i^n m_i r_i^2$ for a system of material points m_i, whose distances from the axis of rotation are respectively r_i; $I = \int r^2 dm$ for continuous mass distribution). That is, for a pendulum tilted by a small angle:

$$I\frac{d^2\alpha}{dt^2} = -D\alpha\,, \qquad (9a)$$

so:

$$\frac{d^2\alpha}{dt^2} = -\frac{D}{I}\alpha\,. \qquad (10)$$

We obtain an equation analogous to equation (5), so its solution will have the form:

$$\alpha = \alpha_0 \sin(\omega t + \phi)\,, \qquad (11)$$

where frequency $\omega = \sqrt{\frac{D}{I}}$, and the period of oscillation of the physical pendulum *T* is:

$$T = 2\pi\sqrt{\frac{I}{D}}\,. \qquad (12)$$

Substituting *D* = *mgd*, we get:

$$T = 2\pi\sqrt{\frac{I}{mgd}}\,. \qquad (12a)$$

We shall now introduce the concept of the reduced length of a physical pendulum.

The reduced length *L* of a physical pendulum is equal to the length of a mathematical pendulum, which has the same period of oscillation as the physical pendulum:

$$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{mgd}} . \qquad (13)$$

From comparing the expressions under the square roots, we obtain:

$$L = \frac{I}{md} . \qquad (14)$$

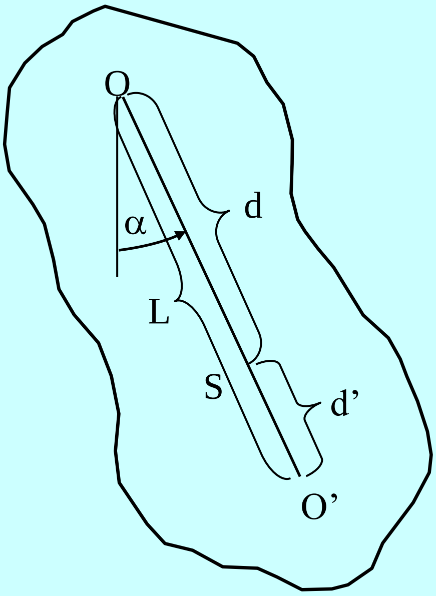

Fig. 2 Physical pendulum.

Steiner's theorem states that the moment of inertia *I* of a solid relative to any axis is equal to the moment of inertia I_0 of the solid relative to the axis passing through its **center of mass** (and parallel to the given axis), increased by the product of the mass of the solid by the square of the distance between the axes: $I = I_0 + md^2$. Therefore equation (14) can be written in the form:

$$L = d + \frac{I_0}{md} , \qquad (15)$$

so:

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\left(d + \frac{I_0}{md}\right)\frac{1}{g}} . \qquad (16)$$

The point *O'* away by *L* from the axis of rotation *O* is called the **center of gravitational oscillation of a physical pendulum**.

We will show that if we pass the axis of rotation parallel to the original axis through point *O'*, the period of vibrations with respect to this new axis will be equal to the period with respect to the original axis passing through point *O*.

An inverted pendulum (with its axis of rotation at the center of oscillation *O'*) has a period of vibration *T'* expressed by the formula:

$$T' = 2\pi\sqrt{\frac{I_0 + md'^2}{mgd'}} = 2\pi\sqrt{\frac{1}{g}\left(\frac{I_0}{md'} + d'\right)}. \quad \textbf{(16a)}$$

Figure 2 shows that L-d=d', and from the equation (15): $L - d = \frac{I_0}{md}$, so $d' = \frac{I_0}{md}$. After some calculations, we obtain: $\frac{I_0}{md'} = d$, so we get:

$$T' = 2\pi\sqrt{\frac{1}{g}(d + d')} = 2\pi\sqrt{\frac{L}{g}} = T \quad \textbf{(16b)}$$

This fact is used to determine the **gravitational acceleration** using a physical pendulum of special construction, the so-called **reversible pendulum**.

1.3. Physical torsion pendulum

In a **gravitational pendulum**, the guiding moment is produced by the force of gravity. In a **torsion pendulum**, it is caused by the elastic force from a twisted rod or other elastic body.

After the deformation of an elastic body by an angle α from the equilibrium position, it vibrates under the influence of a torsional moment: M' = - Dα, which returns the body always to its equilibrium position. As in the case of the gravitational pendulum, the proportionality coefficient D is called **the guiding moment**. Thus the equation of motion has the same form as for the gravitational pendulum (equation 10), and therefore the period of oscillation is expressed by the same formula: $T = 2\pi\sqrt{\frac{I}{D}}$. The magnitude of *D* is defined here by the physical properties of the tested system.

Consider the case where forces were acting on a body cause it to deform elastically (the deformation disappears when the deforming force F ceases).

Depending on the angle between the force vector and the surface of the deformed body, we distinguish between normal forces F_n acting perpendicular to the surface and tangential forces to the surface, F_s. These are the forces we will be dealing with in our exercise.

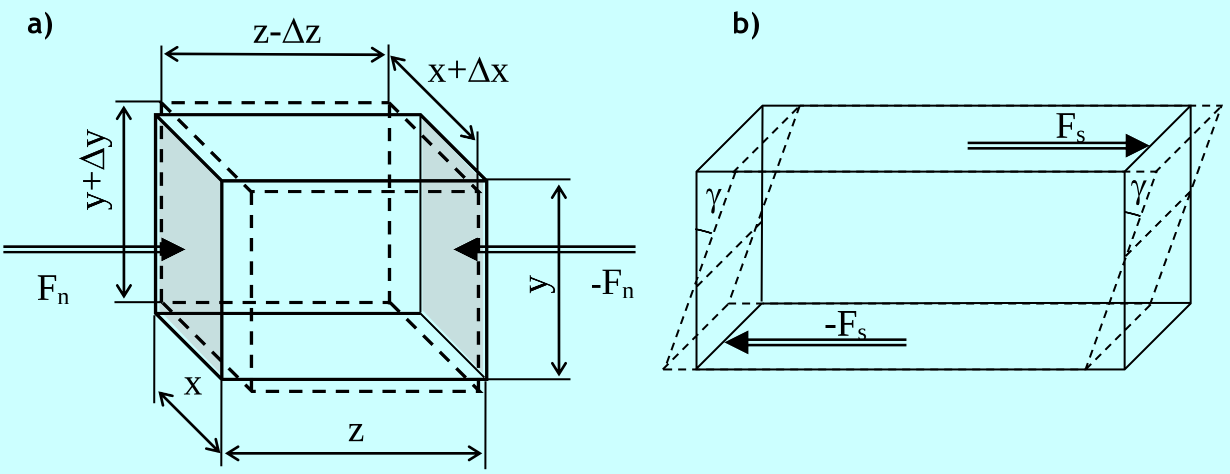

Fig.3 Deformation of a cuboid under the influence of forces: a) normal, b) tangential.

Tangential stress τ - is the ratio of the tangential force F_s to the surface S on which this force acts. The effect of such stress is called simple **shear**.

$$\tau = \frac{F_S}{S} \tag{17}$$

The deformation is then measured using the so-called **shear angle** γ, i.e., the angle formed by the original plane with the rotated plane due to shearing (Figures 3b and 4b). **Between the quantities τ i γ, there is a relationship known as Hooke's law which takes the form:**

$$\tau = G \cdot \gamma \tag{18}$$

The factor G called the **modulus of rigidity,** or the **shear modulus** has the dimension $Nm^{-2}\ rad^{-1} = Pa \cdot rad^{-1}$. It characterizes the elastic properties of a material. The higher it is, the more difficult it is to change the body's shape. Its values vary from $1{,}5 \cdot 10^{6}\ Pa \cdot rad^{-1}$ for rubber, up to approx. $8{,}5 \cdot 10^{10}\ Pa \cdot rad^{-1}$ for steel.

In this exercise, we determine G using harmonic vibrations of a metal bar (rod) under the influence of elastic forces. Each element of the tested bar, twisted by an external force, undergoes a simple shear deformation. As a reaction to this force, an elastic force appears in the bar forcing it to return to an equilibrium state and, consequently, causing vibrations.

The second law of dynamics for rotary motion $I\varepsilon = M$, we can write for this case as:

$$I\frac{d^2\alpha}{dt^2} = -\frac{\pi G r^4 \alpha}{2L} = -D\alpha, \tag{19}$$

where (see derivation in Appendix):

$$D = \frac{\pi G r^4}{2L}. \tag{20}$$

This equation is analogous to equation (9a), i.e., the period of oscillation can be expressed by the formula (12):

$$T = 2\pi\sqrt{\frac{I}{D}} = 2\pi\sqrt{\frac{2LI}{\pi G r^4}}. \tag{21}$$

By transforming this expression, we find the value of the modulus of elasticity:

$$G = \frac{8\pi LI}{r^4 T^2}. \tag{22}$$

Recall that L - is the length of the rod, r - its radius, I - the moment of inertia of the mass vibrated about the axis through the axis of the rod, T - the period of vibration.

2. Experiment

2.1. Determination of gravitational acceleration using a reversible pendulum.

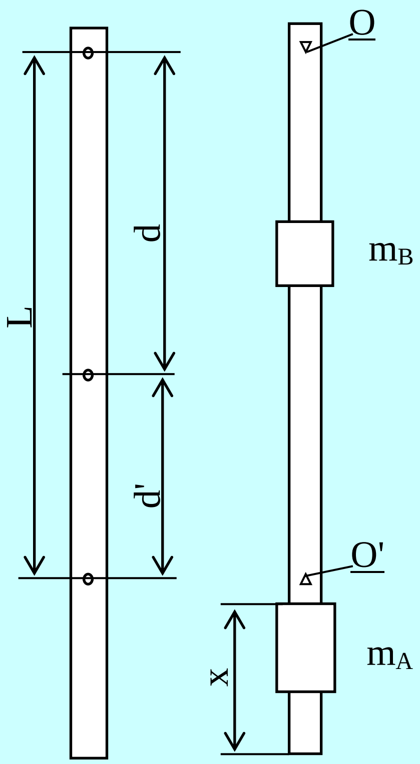

Fig.4 The reversible pendulum.

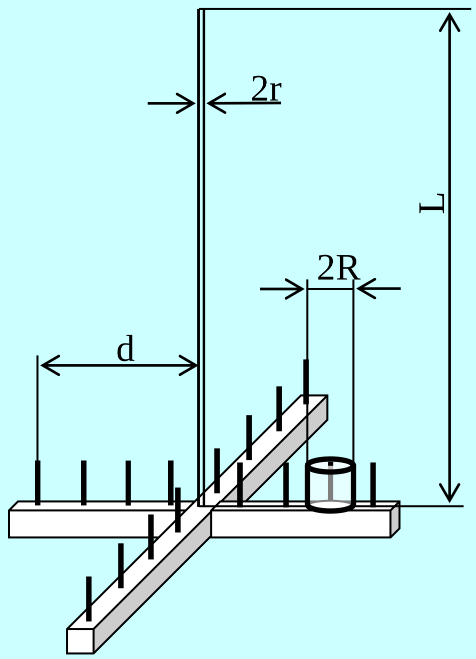

Fig.5. The torsion pendulum.

The reversible pendulum consists of a metal rod with two sliding weights m_A and m_B (fig.4), and two blades (axes of rotation) 0 and 0', which positions can also be changed. By changing the position of the weights, we can make the periods of oscillations on the axis 0 and 0' equal to each other. Then the distance between these axes will become the reduced length L of the considered physical pendulum. Knowing the reduced length and the period of oscillation T will allow us to calculate the value of the gravitational acceleration g from the formula (16a):

$$g = \frac{4\pi^2 L}{T^2} \qquad (23)$$

2.2. Determination of elastic modulus using a torsion pendulum.

The elastic modulus can be determined experimentally using the simple instrument shown in Fig. 5.

The test bar of length L is loaded with a vibrator in the form of a cross, on which weights can be placed. Torsion of the vibrator by a small angle causes elastic forces to develop in the bar, which induces harmonic vibrations of the whole system.

All quantities present in formula (22) can be easily measured except for the moment of inertia I. It would be very complicated to determine the moment of inertia of a solid such as a vibrator. We shall overcome this difficulty in the following way: In the first phase of the experiment, we set the vibrator in motion unloaded, or place on it weights giving a "preload" (to be considered as part of the mass of the unloaded vibrator) and find the period of vibration of such a system:

$$T_1 = 2\pi\sqrt{\frac{I}{D}}\,. \qquad \textbf{(24)}$$

We then place additional weights on the vibrator, whose moment of inertia about the axis passing through their center of mass can be easily determined, and measure the new period of vibration T_2:

$$T_2 = 2\pi\sqrt{\frac{I + I_z}{D}}\,, \qquad \textbf{(25)}$$

I_z - is the moment of inertia of the additional weights.

By squaring the last two equations, subtracting them from each other and considering equation (22), we obtain:

$$T_2^2 - T_1^2 = 4\pi^2\frac{I_z}{D} = \frac{8\pi I_z L}{Gr^4} \qquad \textbf{(26)}$$

and finally:

$$G = \frac{8\pi L I_z}{r^4(T_2^2 - T_1^2)}\,. \qquad \textbf{(27)}$$

In the particular case where this additional load consists of homogeneous cylinders having a moment of inertia about an axis through their center of gravity and parallel to the axis of the bar, which is $I_0 = \frac{1}{2}mR^2$ (m - is the mass of the cylinder, R - is its radius), and if we place these cylinders in the distance d from the axis of the bar, then, according to Steiner's theorem, the value of $I_z = n(I_0 + md^2) = n(\frac{1}{2}mR^2 + md^2)$, where *d* - is the mean distance of the center of the loading cylinder from the vibrator centerline, and n is the number of weights. The stiffness modulus G is determined from formula (27) by substituting into it the expression for I_z:

$$G = \frac{8n\pi L\left(\frac{1}{2}\overline{R}^2 + \overline{d}^2\right)\overline{m}}{\overline{r}^4(T_2^2 - T_1^2)}\,. \qquad \textbf{(28)}$$

3. Measurements

3.1. Reversible pendulum

1. Place the weight m_A in the position closest to the 0' axis. (Do not change the position of the B weight.)
2. Start the pendulum and measure the time taken for twenty oscillations about the 0 axes.
3. Determine the period of oscillation T_0. Invert the pendulum, measure the time of 20 oscillations about the 0' axis and determine the period of oscillation T_0'.
4. Move the weight m_A by 2 cm, find the periods of oscillation T0 and T0' about the 0 and 0' axes again. Each time measure the distance of the moving weight A from the 0' axis and proceed until the weight m_A is at the end of the pendulum.
5. After measuring the periods of oscillation of the pendulum suspended on 0'-T_0' and 0-T_0 axes as a function of the position of the moving weight - x draw a graph $T_0 = T_0(x)$ and $T_0' = T_0'(x)$ (dependence of the periods of oscillation of the pendulum on the distance of the moving weight from the selected axis of rotation).
6. Find the point of intersection of the curves $T_0(x)$ and $T_0'(x)$ – point (x_0, T).
7. If you find that the curves on the graph do not intersect, ask the assistant to reposition the weight m_B , and the experiment must be repeated from the beginning.
8. If the curves intersect, check by adjusting the weight m_A at point x_0, then $T_0 = T_0'$. If it turns out that the periods are not exactly equal for this setting, move m_A by about 1 cm in either direction. Repeat the measurements. Change the position of the weight until the periods within the uncertainties are equal.
9. Measure the distance L between the axes (reduced length).
10. The found value of T and L are substituted into the formula (23), from which the value of gravitational acceleration can be calculated.
11. Calculate the combined uncertainty u_c of the measured quantity. Calculate the expanded uncertainty U_c.
12. Compare the determined quantity with the table value and evaluate the correctness of the applied measurement method.

Measurement results table

L =				*n* =								
x												
t_0												
$T_0 = \frac{t_0}{n}$												
t_0'												
$T_0' = \frac{t_0'}{n}$												

3.2. Torsion pendulum

1. Using a micrometer screw, measure the diameter of the test bar several times at various points (2*r*).
2. Measure the length of the test rod (*L*).
3. Set the vibrator in motion, unloaded or preloaded. Measure the time t_1 of the twenty vibration periods.
4. Measure the diameters (2R) of *n* additional weights.
5. Weigh *n* additional weights.
6. Measure the distance between the pins on which the weights are placed (2*d*), (fig.5).

7. After placing the weights on the pins, vibrate the vibrator again. Measure the time t_2 of the twenty periods of vibration.
8. After taking appropriate measurements, determine the value of T_1 and T_2 and calculate the average values $\overline{r}, \overline{R}, \overline{d}, \overline{m}$.
9. Determine G from the formula (28).
10. Determine the standard uncertainties of all quantities measured directly. For each uncertainty state, whether the uncertainty is determined by the Type A or Type B method.
11. Determine the composite uncertainty and the expanded uncertainty of quantity G. Correctly record the results.
12. Compare the determined value of G with the table value (whether found quantity falls within the uncertainty range).

4. Questions

1. What conditions must be met for a body to move in harmonic motion?
2. Draw the time dependence of acceleration and velocity in harmonic motion. Do they change in phase with the tilt?
3. Think about the purpose of changing the position of weights in a reversible pendulum (with a fixed distance between axes).
4. Can the vibrations of a physical pendulum be observed in weightlessness? A torsion pendulum?

5. References

1. J. Orear – Fizyka.
2. H. Szydłowski – Pracownia fizyczna.

APPENDIX – Relation between stiffness modulus and moment of force

Derive the mathematical relationship between the stiffness modulus G and the moment of force acting on a twisted bar. Consider a cylindrical member with inside radius *r*', thickness *dr*', and length *L*, equal to the length of the whole bar (*L*>>*r*') (fig.4). The tangential stress in this case is:

$$\tau = G \cdot \gamma = G\frac{s}{L},$$

where s - is an element of the arc. But $\frac{s}{r'} = \alpha$, so:

$$\tau = G\frac{r'}{L}\alpha .$$

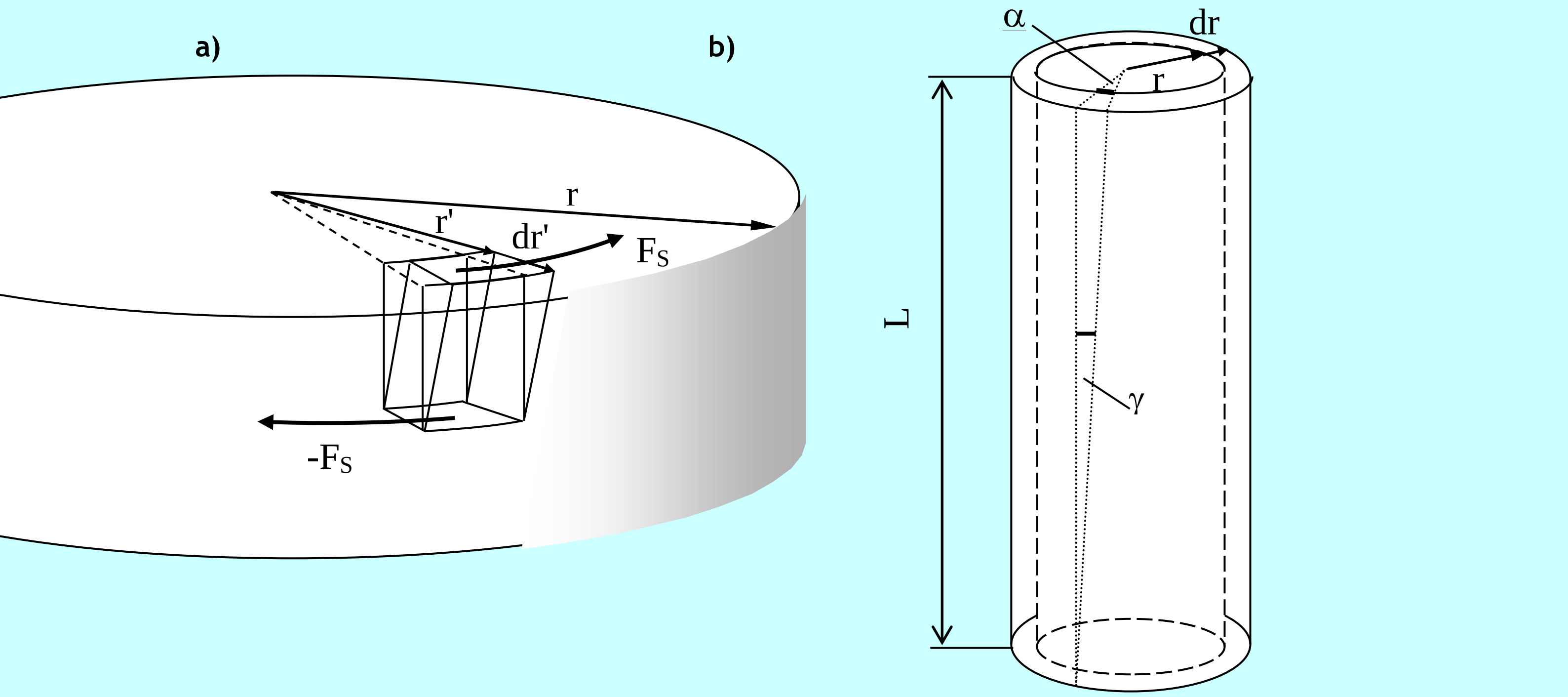

Fig.6 a) Deformation of the torsion bar members.
b) Deformation of the cylindrical layer of a torsion bar.

The area *ds* of the cross-section of the ring bounded by the perimeters of radii *r*' i *r'*+ *dr*' is $2\pi r'dr'$. The value of the tangential force acting on such a ring can be determined using formula (17) and the preceding:

$$dF_s = \tau dS = G\frac{r'}{L}\alpha \cdot 2\pi r'dr'$$

while the moment of force is equal to:

$$dM = dF_s r' = \frac{2\pi}{L}G\alpha r'^3\, dr'$$

Integrating this expression from 0 to r gives the value of the moment of force acting on the entire cross-sectional area of the bar:

$$M = \int_0^r \frac{2\pi}{L}G\alpha r'^3\, dr' = \frac{\pi G r^4}{2L}\alpha$$