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# MEASUREMENT OF THE WAVELENGTH WITH APPLICATION OF A DIFFRACTION GRATING AND A SPECTROMETER

## 1. Fundamentals

Electromagnetic waves are periodical changes of electric and magnetic fields propagated periodically in space. Vectors of electric E and magnetic B fields are mutually perpendicular and their magnitudes are proportional. Thus it is enough to choose one of them (for instance E) to describe the wave phenomena. An electromagnetic wave propagated along the X axis can be represented with a wave function:

$$E(x,t) = E_0 \sin(\omega t - kx) \tag{1a}$$

where: $E_0$ **denotes amplitude of the electric field**, argument of sine function, $(\omega t - kx)$ is called **wave phase**, $\omega$ – **angular frequency**, k – **wave number** related to wavelength $\lambda$ according to the pattern:

$$k = \frac{2\pi}{\lambda}. \tag{1b}$$

According to the formulas (1a) and (1b), if a wave passes the $x = \lambda$ distance, it changes its phase by $2\pi$ angle. As $2\pi$ is the period of the sine function, thus all the points with the phase difference of $2\pi$ will have the same values of the electric field E. One says in this case that electric field vibrations are **in phase** at these points.

The electromagnetic wave is a **transverse wave** which means that electric and magnetic fields vectors are always perpendicular to the direction of the wave propagation. In case of the wave given with the formula (1a), they will change only long the X axis – by contrast, they will be constant on XY planes perpendicular to X axis. All the points on the chosen YZ plane will have the identical phase. Such a wave is called a **plane wave**. Generally speaking, points in space featuring the same phase create **wavefront**. The **wavefront** of a plane wave is a plane.

### 1.1. Interference, diffraction and diffraction grating

**Interference** and **diffraction** are the main phenomena related to propagation and interaction of waves. Interference is a phenomenon in which a **countable** number of waves superpose in a chosen point in space which may lead to their strengthening (constructive interference) or weakening (destructive interference) – depending on the phase difference. The result of interference – i.e. interference image - can be seen when: (1) wave sources are monochromatic (they emit waves of the same wavelength) and (2) sources of interfering waves are coherent – i.e. the waves they emit maintain a phase difference that is constant in time.

Interference and diffraction are inseparably related to the **Huygens’ principle**. According to it, each point of the space a wave reaches can be treated as a source of a new, **secondary spherical wave**. These partial spherical waves propagate in all the directions superposing each other, creating a new wave. If there are no obstacles in the propagation spaces, in the result of overlapping **uncountable** quantity of secondary spherical waves, the primary wave is recovered and its front remains unchanged. If the propagating wave reaches an obstacle (which can be a slit), the interference of the secondary waves does not lead to recovery of the wavefront. The shape of the wavefront changes, the distribution of the electric field is changed which might lead to creation of directions where the wave is strengthened or weakened. This set of phenomena is called **diffraction**. One can easily notice that diffraction is a complex phenomenon covering both the Huygens’ principle and interference of uncountable amounts of spherical waves resulting in a change of a wavefront shape.

The interference image can be created with the application of a set if parallel slits which is called a diffraction grating. The main parameter characterizing a diffraction grating is the **distance between slits d** called **grating constant**. Illumination of a diffraction grating with a parallel light beam leads to creation of an interference image on the screen located behind the grater which will consist of a set of light and dark bands shown in figure 1a. The image is clearly visible if the mentioned above conditions are fulfilled and if the grating constant is comparable with the wavelength. For the visible light range of a wavelength from 400 to 700 nm the distance between the slits should be about 1 $\mu$m. It means that the beam of the width 2 mm illuminates 2000 slits.

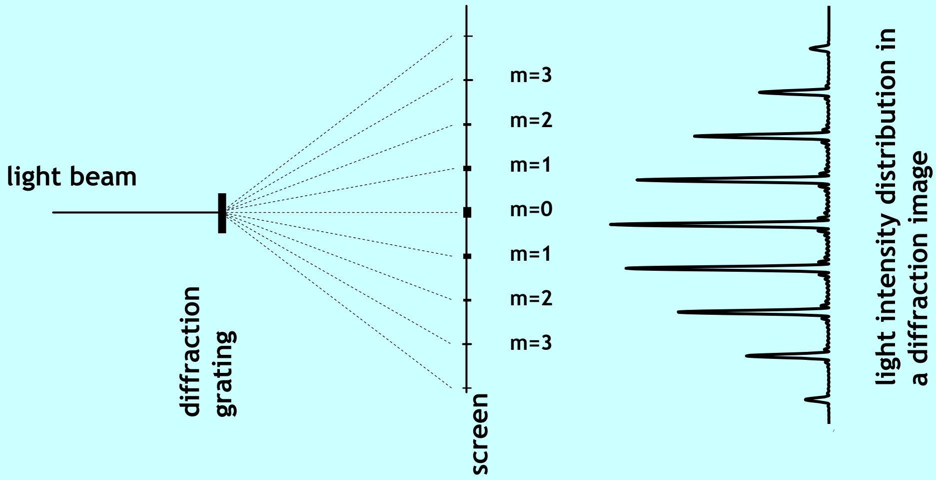


***Fig. 1a. Creation and distribution of light intensity in a diffraction image.***

The description of creation of such an image on the screen can be originated once again from the **Huygens' principle**. As it has already been mentioned, the principle says that every point of the space where the wave reaches can be treated as a source of a new **secondary spherical wave**. The spherical wave propagates in all the directions, and the observed wave is a superposition of all elementary spherical waves.

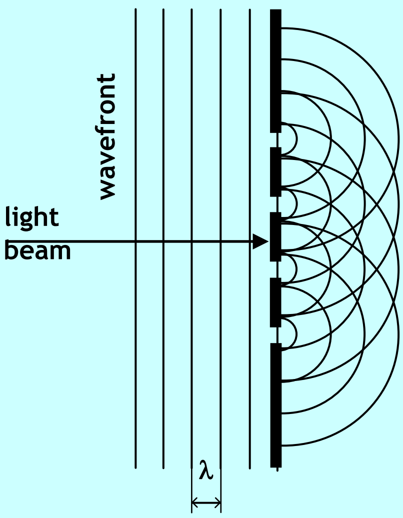


***Fig. 1b. Illustration of Huygens' principle.***

Now, let's assume that a plane wave strikes a diffraction grating of d constant with such narrow slits that they can be treated as the point sources of spherical waves. According to Huygens' principle each slit of diffraction grating is a source of a new spherical wave of identical initial phase (figure 1b). It means that spherical waves are propagated in the space behind the grating. The number of these waves is equal to the number of slits illuminated with the light beam. Each point of the space behind the grating is being reached by waves from **all** the sources and the interference phenomenon takes place. According to the definition given, interference is the phenomenon of overlapping of a countable number of waves that can lead to their **strengthening or weakening**, dependent on the phase difference. **Maximum of intensity will happen at points where the interfering waves are in phase i.e. their phase difference will be:**

$$\varphi = m \cdot 2\pi \quad \text{(where } m=0, \pm 1, \pm 2, \ldots\text{).} \qquad (2a)$$

Under assumption of equality of initial phases of all spherical waves created by the diffraction grating, the phase different at any point of the space P depends only on difference of **optical paths** (geometrical paths in vacuum) $\Delta$. The path difference that is equal to wavelength, $\Delta=\lambda$ corresponds phase difference $\varphi = 2\pi$. In general one can write this dependence as:

$$\frac{\varphi}{2\pi} = \frac{\Delta}{\lambda} \qquad \text{.......(2b)}$$

Comparing (2b) formula with (2a) one obtains a formula:

$$\Delta = m \cdot \lambda. \qquad (2c)$$

**Thus the strengthening (interference maximum) takes place when the optical path difference is equal to the multiple wavelength.**

**Minimum** intensity happens at points where the phase difference of interfering waves equals to odd multiple $\pi$**:**

$$\varphi = (2m+1)\,\pi \quad \text{(where } m=0, \pm 1, \pm 2, \ldots\text{),} \qquad (3a)$$

what corresponds to difference of optical paths:

$$\Delta = (2m+1) \cdot \frac{\lambda}{2}, \qquad m = 0, 1, 2, \ldots . \qquad (3b)$$

## 1.2. Creation of interference maxima in a diffraction grating

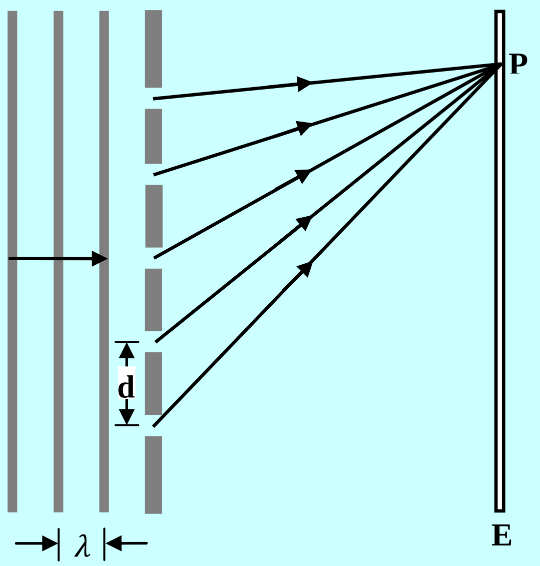


*Fig.2a An idealized diffraction grating, consisting of only five rulings, that produces an interference pattern on a distant viewing screen E.*

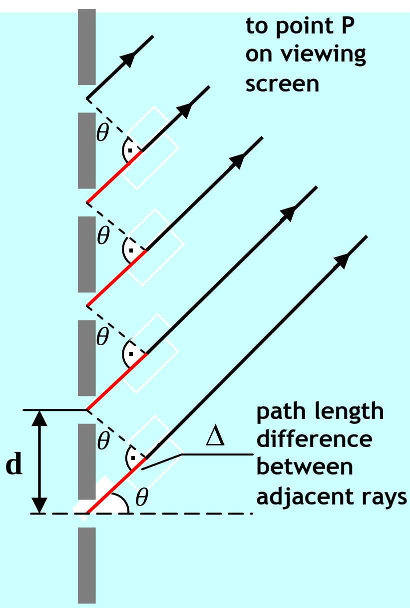


*Fig.2b Due to a long distance to the screen with respect to the slit's width, light beam leaving the slits are approximately parallel.*

As it is shown in Figure 2b, the optical path difference of two adjacent rays is $\Delta$ = **d**$\sin\theta$. **Interference maxima** in diffraction grating are observed according to the condition defined by the formula (2c) when:

$$\mathbf{d\sin\theta = m\lambda.} \tag{4}$$

$\theta$ angle in this formula denotes angle (measured to the normal axis of the grating) at which the maximum of m-th order is seen on the screen. The maxima location is marked in Fig 1a.

As it is seen in the above formula, the angles at which one observes interference maxima do not depend on the number of slits, but they depend on the distance between slits, **d**, and on the wavelength $\lambda$ of intrinsic light. Thus the diffraction grating can be used for decomposition of intrinsic wave on the component of various wavelengths.

***Fig. 3. Helium lamp spectrum.***

Through the measurement of $\theta$ angles and knowing the grating constant one can figure out the wavelength of the light source: :$\lambda$=d$\sin\theta$/m.

## 1.3. Intensity of light in the interference image

The resultant intensity i.e. the average power transferred by waves sent from N slits of diffraction grating can be expressed with the formula:

$$\mathbf{I = I_0 \frac{\sin^2(N\varphi/2)}{\sin^2(\varphi/2)}}\ . \tag{5}$$

where: $\mathbf{I_0}$ is intensity of wave originating from one source (slit) and is equal to the squared amplitude $E_0^2$, $\varphi$ - denotes phase difference of two various waves sent from adjacent slits of the diffraction grating.

Dependence of intensity I with respect to $\varphi$ angle (which is dependent on $\theta$ *angle), contains a variable factor* $\sin^2(N\varphi/2)$, modulated by a significantly less variable $\sin^2(\varphi/2)$ expression. Each of this factors and their quotient are presented in Fig. 4. The derivation of the formula can be found in the Appendix at the end of this manual.
The value of the formula (5) for $\varphi$ = 0, can be found with application of the approximation $\sin(N\varphi/2) \sim (N\varphi/2)$ and $\sin(\varphi/2) \sim (\varphi/2)$, transforming $\varphi \to 0$. One obtains then:

$$\mathbf{I = I_0 \frac{(N\varphi/2)^2}{(\varphi/2)^2} = N^2 I_0}$$

The identical result can be obtained for all angles fulfilling the condition: $\varphi = m\cdot 2\pi$. Thus the resultant intensity in principal maxima is $N^2$ times higher than the intensity from a single slit.

With the increase of $\varphi$ from the 0 value, the ratio of two squared sine functions in the formula (5) is decreasing and the first diffraction minimum will be obtained when the numerator of the formula (5) has the zero value, i.e. when $(N\varphi/2) = \pi$, which means that $\varphi = 2\pi/N$. The further increase of $\varphi$ phase leads to the increase of the resulting amplitude and creation of the side maximum. The side maxima occur for $\varphi$ angles for which the numerator of the formula (5) is equal to 1. However, they are far weaker than the principal maxima (Fig. 3).

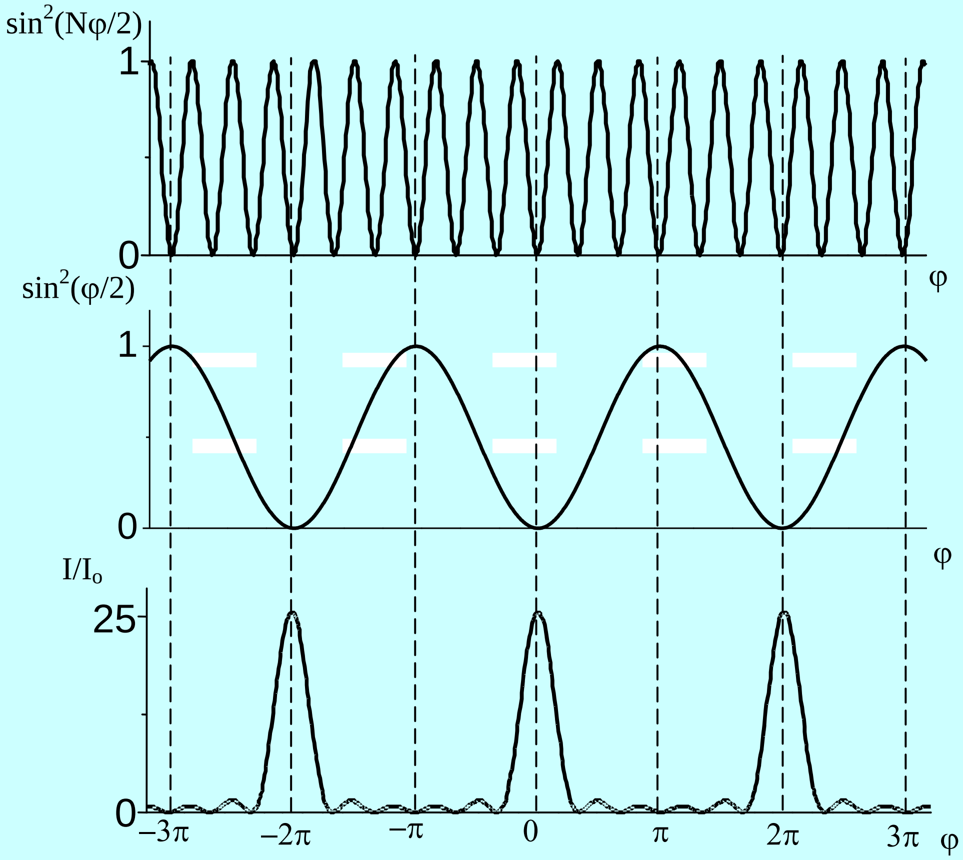


***Fig. 4. Interference image for five slits. The factors of Fig. 5 have been shown and their quotient. The principal maxima are separated by a number of weaker side maxima.***

## <u>1.4. Resolvability of a diffraction grating</u>

As it has already been mentioned, a diffraction grating can be applied to separate different wavelengths. The question is what it she minimum difference between wavelengths λ and λ' so that they can be distinguished with the application of diffraction grating. For this purpose, we introduce a concept of resolvability of a grating R, which is defined as:

$$R = \frac{\lambda}{\Delta\lambda} \quad , \tag{6}$$

where: λ – is one of two spectral lines and Δλ = λ'- λ is a wavelength difference between them.

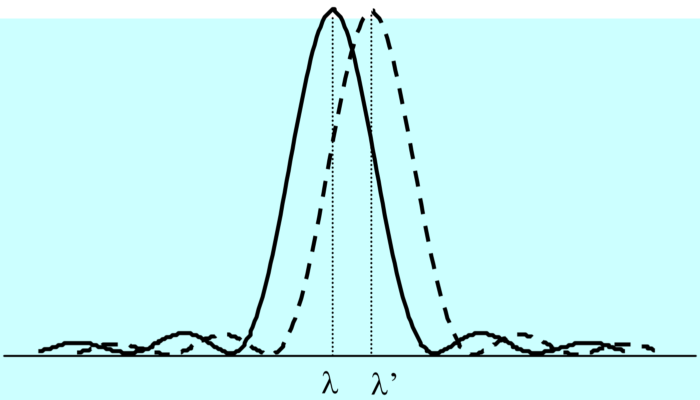


***Fig.5. Illustration of Rayleigh's criterion.***

A commonly used condition of separation of two waves of similar wavelengths is so-called **Rayleigh's criterion** according to which two principal maxima are distinguishable when the angular distance has such a value that the minimum of one line at the maximum of the second line (fig. 5). As we know the first diffraction minimum is located at the distance $\varphi$ = ($2\pi/N$) from the principal maximum (the numerator in the equation (5) is equal to 0). Such a phase difference corresponds to the difference of optical paths ($\lambda/N$). Thus the condition for the first minimum for the spectrum of the m-th order can written as:

$$d\sin\theta = m\lambda + \frac{\lambda}{N} \quad . \tag{7}$$

Simultaneously for a wavelength λ' one can obtain at this location a maximum intensity, so $d\sin\theta = m\lambda'$. Subtracting these formulas side by side we obtain after a transformation:

$$R = \frac{\lambda}{\Delta\lambda} = mN \tag{8}$$

where: $\Delta\lambda = \lambda' - \lambda$ , m-th order of the spectrum, N is a number of slits.

One can see that the higher number of slits and the higher order of spectrum the higher is the resolvability of the grating. One can prove this fact easily by observing the interference image with application of the spectrometer with diffraction grating that can be illuminated with – for example – neon lamp. The fringes in the spectrum of 2$^{nd}$ order are better separated than in the 1$^{st}$ order, but there is a certain difficulty in their observation as they have a weaker intensity compared by the 1$^{st}$ order fringes. How can you explain this?

**1.5. Influence of the finite width of a single slit on the interference image.**

Based on the considerations until now, one can see that all the principal maxima should have the same intensity. However, we cannot ignore the fact that we have obtained this result under assumption that we can neglect the phase differences between points within one slit. In practice, this condition is not fulfilled and one has to consider diffraction on a single slit. To obtain a formula for a single slit we can proceed in a similar way as in case of the diffraction grating. We separate the slit into M equal, very narrow stripes. With the limit $M \to \infty$, while maintaining the constant phase difference $\alpha = M\varphi$ between both edges of the slit, the angle $\varphi$ in the formula (5) is getting so small that the approximation $\sin(\alpha/M) \sim (\alpha/M)$ is getting valid. Then $I_0 = I_0' M^2$ – where $I_0'$ is intensity of light sent by one of stripes into which we divided the slit. The formula for the intensity of light diffracted on one slit is then:

$$I_{dyf.} = I_0 \frac{\sin^2(\alpha/2)}{(\alpha/2)^2} , \tag{9}$$

where: $\alpha$– denotes a phase difference between two rays sent from two edges of the slit, $I_0$ – intensity of light sent from one slit.

Thus the formula for intensity of light from one diffraction grating will be a composition of two formulas (5) and (9):

$$I = I_{dyf.} \frac{\sin^2(N\varphi/2)}{\sin^2(\varphi/2)} \quad . \tag{10}$$

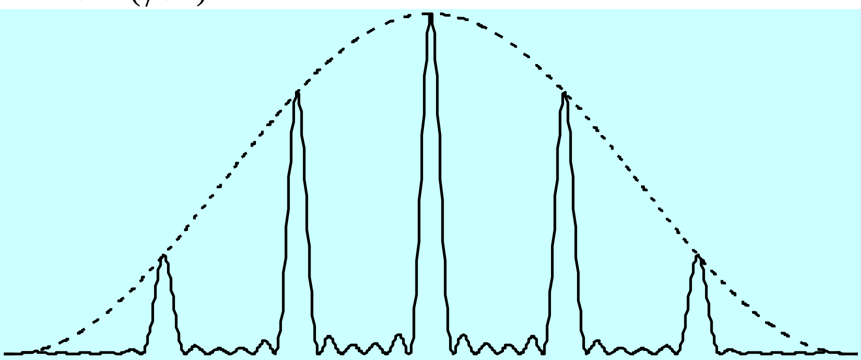

*Fig.6. Intensity distribution for a diffraction grating for which the slit width is a = (d/3), where d is the distance between slits.*

In Figure 6, the interference image for the diffraction grating with N=5 slits has been shown, taking into account diffraction on a single slit of width a = d/3, where d is a distance between slits. In this case, one can easily note that $\alpha = \varphi/3$, so the formula (9) is changing slower than (5), thus one obtains a gradual decrease of the brightness of further pars of spectrum. The distribution of intensities presented in Figure 6 was obtained under assumption of ideal slits of sharp parallel edges. Through the appropriate choice of slit shape one can find the form of modulating factor, $I_{dif.}$, in the formula (10). For example to make better visible the further orders of the spectrum, which have a better resolvability, .

## 2. Measurements

1. Turn the sodium lamp on and set the diffraction grating on the spectrometer table perpendicularly to the light beam exiting from the collimator.
2. Measure the angles at which the subsequent orders of the spectrum are seen at the left and right side with respect to the intrinsic beam. If diffraction angles measured and the left and right side differ by more than 6', adjust the position of the grating. The yellow fringe of the sodium light consists in practice of two very near lines of wavelengths of $\lambda_1$ = 589,6 nm and $\lambda_2$ =589,0 nm. Observe for which diffraction order the separated sodium doublet is visible.
3. Turn the lamp chosen by the supervisor and perform measurements of angles for the observed fringes.

## 3. Calculations

1. Based on the measurements performed with the sodium lamp, figure out the grating constant (formula 4) and its uncertainty. Consider the uncertainties of type A and B. Assume the wavelength of the sodium lamp as $\lambda_{Na}$= 589,3 nm.
2. Knowing the grating constant, figure out the wavelength of the light emitted by a second gas and calculate the combined standard uncertainties. Report the results correctly and compare them with the reference data.
3. Based on the measurement and observations performed in the point 2 in the section in *Measurements* section figure out the resolvability of the diffraction grating and calculate the number of fringes taking part in the interference (formula (8)).

## 4. Questions (a full list is available on the laboratory website)

1. When can the interference image be observed?
2. Explain the formula for the location of maxima of the interference image. (formula 4).
3. What is the resolvability of a diffraction grating and how to increase it?
4. Why the further orders of the spectrum are less visible?

## 5. References

1. D. Halliday, R. Resnick, J. Walker, Fundamentals of Physics, Wiley (2011), part.4 chapter 35,36.
2. J. Orear, Fundamental physics, Willey (1979) chapter 22.

# Appendix

## Intensity of light in the interference image

Let's analyze now, how the interference image will look like at points located between the principal maxima, for a grating consisting of N slits. For this purpose, we are going to use the graphical method called method of phasors. In this method, the amplitude of E field described with the formula (1a) is represented as a vector of a magnitude $E_0$ and the phase – as the angle $\alpha$ between the vector and the X.

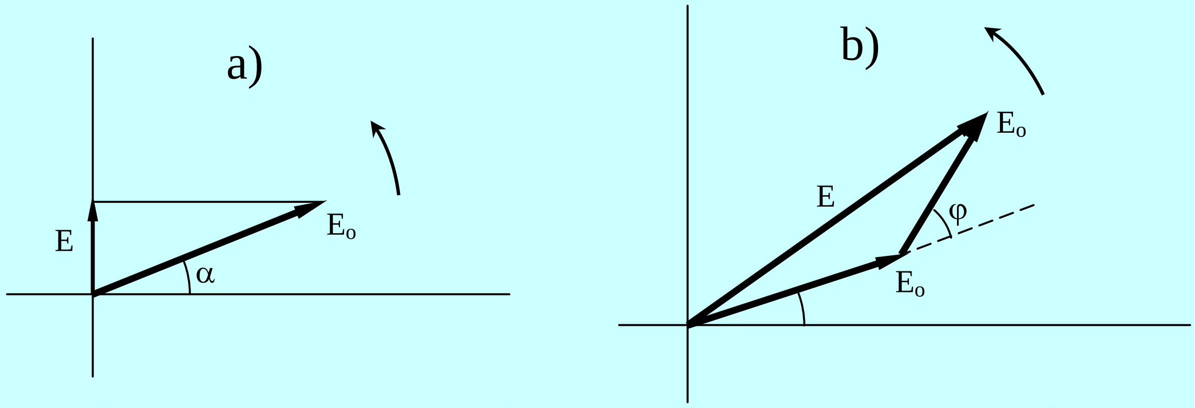


***Fig.1D** a) Vector representation of the formula (1a): $E_0$ - wave amplitude, a = (ωt - kx) - phase, $E = E_0 sina$. Vector rotates counterclockwise. b) Vector addition of two waves, φ - phase difference, $E_w$ - resulting amplitude.*

As the phase is changing with time, this vector is rotating counterclockwise. As the phase difference of two waves resulting from adjacent slit is φ the vector diagram of perturbations will contain N vectors of equal magnitudes E and of the angle between adjacent vectors equal φ.

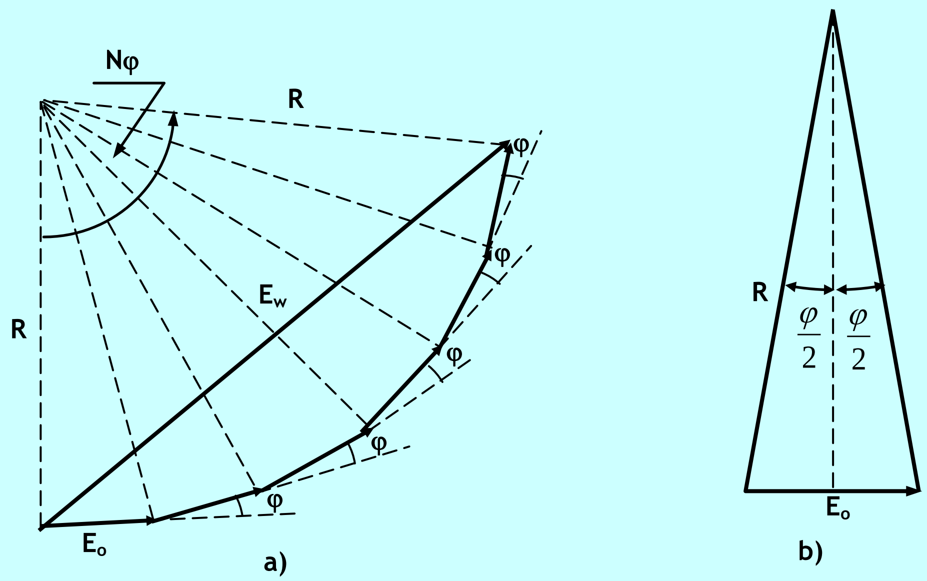


***Fig.2D.** Graphical addition wave functions originating from N parallel slits for which phase difference between adjacent slits is φ. The figure presents the situation of N = 5 slits.*

As it is shown in Fig. 2D, ends of these vectors are located on the circle whose radius R is given with the formula:

$$\frac{1}{2}E_0 = R\sin\frac{\varphi}{2} \ . \tag{1D}$$

The resulting amplitude $E_w$ is the base of isosceles triangle of sides equal to R and the vertex angle Nφ. Thus:

$$E_w = 2R\sin\frac{N\varphi}{2} \, . \tag{2D}$$

Combining these two formulas, w obtain a formula for the resulting amplitude:

$$E_w = E_0\frac{\sin(N\varphi/2)}{\sin(\varphi/2)} \ . \tag{3D}$$

Resulting amplitude i.e. the average power carried by the wave is proportional to the square of the resulting amplitude $E_w$ and is equal to:

$$I = I_0\frac{\sin^2(N\varphi/2)}{\sin^2(\varphi/2)} \ . \tag{4D}$$